

## AN ADAPTIVE PRIMAL-DUAL FULL-NEWTON STEP INFEASIBLE INTERIOR-POINT ALGORITHM FOR LINEAR OPTIMIZATION

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**ABSTRACT.** In this paper, we improve the full-Newton step infeasible interior-point algorithm proposed by Mansouri et al. [6]. The algorithm takes only one full-Newton step in a major iteration. To perform this step, the algorithm adopts the largest logical value for the barrier update parameter  $\theta$ . This value is adapted with the value of proximity function  $\delta$  related to  $(x, y, s)$  in current iteration of the algorithm. We derive a suitable interval to change the parameter  $\theta$  from iteration to iteration. This leads to more flexibilities in the algorithm, compared to the situation that  $\theta$  takes a default fixed value.

### 1. Introduction

After Karmarkar's pioneer work for LO [3], interior-point polynomial algorithms have been investigated by many researchers. Based on the starting point, two types of interior-point methods (IPMs) exist; feasible and infeasible IPMs (IIPMs). Feasible IPMs start from a strictly feasible point for the problem at hand, while IIPMs start from an arbitrary positive point. Due to the difficulty of finding a feasible starting point for various problems, the use of IIPMs is unavoidable. Roos [8] proposed the first full-Newton step infeasible interior-point algorithm for LO. This algorithm starts from strictly feasible iterates on the central path of the intermediate problems produced by suitable perturbations in LO problem. Then, it uses so-called feasibility steps that serves to generate strictly feasible iterates for the next perturbed problems. After accomplishing a few centering steps for the new perturbed problem, it obtains strictly feasible iterates close enough to central path of the new perturbed problems. This algorithm have extensively extended to other optimization problems, e.g., [2, 7, 11–13]. Subsequently, some authors have tried to improve the Roos's infeasible algorithm, see for example [4, 5]. Some improvements have done in order to reduce or remove the centering steps [1, 6, 9, 14].

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Recently, Mansouri et al. [6] proposed an IIPM for solving LO problems with a reformulation of the central path. Their algorithm need not to perform any centering step to reach the closeness of the iterates to their  $\mu$ -centers. To get this target, they select a (rather small) fixed default barrier update parameter  $\theta$  in their algorithm to return the iterates to their  $\tau$ -neighborhood of the central path without doing any centering step. This value for  $\theta$  seems to be undesirable for the practical purposes. In this paper, we modify the system defining the central path in such a way that, we keep the utility of elimination of the centering steps along with the variable values for  $\theta$ . In fact, in each main iteration, the algorithm adopts the largest possible value for  $\theta$ , adaptively with respect to the value of the proximity function at the same iteration and then updates the central path-parameters and performs the full-Newton step. This feature will enable us to choose some larger values for  $\theta$  and speed up the algorithm, significantly.

The paper is organized as follows: in the rest of this section, we introduce the primal-dual LO problems and their central paths. In Section 2, we introduce the perturbed problems pertaining to the original primal-dual pair. We also determine our new search directions and describe our algorithm in this section. In Section 3, we analyze our algorithm and derive a suitable interval to change the barrier update parameter  $\theta$ . Section 4 contains some numerical results. Finally, we present some concluding remarks in Section 5.

The following notations are used in this paper.  $\mathbb{R}_+^n$  ( $\mathbb{R}_{++}^n$ ) is the nonnegative (positive) orthant of  $\mathbb{R}^n$ . The 2-norm is denoted by  $\|\cdot\|$ . If  $x, s \in \mathbb{R}^n$ , then  $xs$  denotes the componentwise product of the vectors  $x$  and  $s$ .  $\min(x)$  means the smallest component of the vector  $x$ .  $e$  denotes the all ones vector in  $\mathbb{R}^n$ .

### 1.1. The LO problem and its central path

Consider the primal-dual LO problems

$$(P) \quad \min\{x^T c \mid Ax = b, x \geq 0\},$$

and

$$(D) \quad \max\{b^T y \mid A^T y + s = c, s \geq 0\},$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  and  $c \in \mathbb{R}^n$ . Assuming  $A$  is a full rank matrix, because of K. K. T optimality conditions, finding simultaneous optimal solutions for both above problems is equivalent with to find a nonnegative solution  $(x, s)$  of the following system

$$(1) \quad Ax = b, A^T y + s = c, xs = 0.$$

The basic idea underlying primal-dual IPMs is to replace the complementarity condition  $xs = 0$  by the nonlinear equation  $xs = \mu e$ , where  $\mu > 0$ . Let the interior-point condition (IPC) be satisfied for the primal and dual problems, i.e., there exists  $(x, y, s)$ , with  $x, s > 0$ , satisfying the feasibility conditions of (P) and (D). Then, the new obtained system after the mentioned replacement,

has a unique solution for a fixed  $\mu > 0$ , called the  $\mu$ -center or *analytic center* (Sonnevend [10]). The set of  $\mu$ -centers for  $\mu > 0$  forms a well-behaved curve, called *central path* which plays an important role in convergence analysis of the IPMs. The central path converges to the primal-dual optimal solution as  $\mu \rightarrow 0$ .

Similar to [6], we replace the right hand side of the nonlinear equation  $xs = \mu e$  with  $\mu v$  in which  $v$  is the variance vector corresponding to  $x$  and  $s$ . Define

$$(2) \quad v := \sqrt{\frac{xs}{\mu}}.$$

So, in this paper, we consider the system

$$(3) \quad Ax = b, \quad A^T y + s = c, \quad xs = \mu v,$$

for finding a primal-dual optimal solution of  $(P)$  and  $(D)$ .

### 2. Infeasible full-Newton step IPM

In this section, we present an infeasible-start interior-point algorithm that generates a primal-dual  $\varepsilon$ -solution of  $(P)$  and  $(D)$ .

#### 2.1. The perturbed problems and their central paths

As usual for IIPMs, we suppose that there exists a primal-dual optimal solution  $(x^*, y^*, s^*)$  for LO. Then, the algorithm starts with the initial infeasible point

$$(4) \quad (x^0, y^0, s^0) = (\zeta e, 0, \zeta e) \quad \text{and} \quad \|x^* + s^*\| \leq \zeta$$

for some  $\zeta > 0$ . We denote the initial value of the residuals as  $r_b^0$  and  $r_c^0$ :

$$(5) \quad r_b^0 = b - Ax^0, \quad r_c^0 = c - A^T y^0 - s^0.$$

For any  $\nu$  with  $0 < \nu \leq 1$ , we consider the perturbed problems  $(P_\nu)$  and  $(D_\nu)$  defined by:

$$(P_\nu) \quad \min\{(c - \nu r_c^0)^T x \mid Ax = b - \nu r_b^0, \quad x \geq 0\},$$

$$(D_\nu) \quad \max\{(b - \nu r_b^0)^T y \mid A^T y + s = c - \nu r_c^0, \quad s \geq 0\}.$$

Note that if  $\nu = \nu^0 := 1$ , then  $(x, y, s) = (x^0, y^0, s^0)$  yields a primal-dual strictly feasible solution of  $(P_1)$  and  $(D_1)$ . We conclude that the perturbed problems  $(P_1)$  and  $(D_1)$  satisfy the IPC.

**Lemma 2.1** (Lemma 3.1 in [8]). *The original problems  $(P)$  and  $(D)$  are feasible if and only if the perturbed problems  $(P_\nu)$  and  $(D_\nu)$  satisfy the IPC.*

We assume that  $(P)$  and  $(D)$  are feasible. Letting  $0 < \nu \leq 1$ , Lemma 2.1 implies that the problems  $(P_\nu)$  and  $(D_\nu)$  satisfy the IPC, and hence their central path exists. This means that the system

$$(6) \quad \begin{aligned} Ax &= b - \nu r_b^0, \quad x \geq 0, \\ A^T y + s &= c - \nu r_c^0, \quad s \geq 0, \\ xs &= \mu v, \end{aligned}$$

has unique solutions  $x(\mu)$  and  $(y(\mu), s(\mu))$  for every  $\mu > 0$ . In the sequel, these solutions are referred as  $\mu$ -centers of the perturbed problems  $(P_\nu)$  and  $(D_\nu)$ . Note that since  $x^0 s^0 = \mu^0 e$ , then  $x^0$  and  $(y^0, s^0)$  are the  $\mu^0$ -centers of the perturbed problems  $(P_1)$  and  $(D_1)$ , respectively. Since  $\nu^0 = 1$  and due to the fact that both of  $\mu$  and  $\nu$  will be updated by the same factor, in the sequel, we note that the parameters  $\mu$  and  $\nu$  always satisfy the relation  $\mu = \nu \mu^0 = \nu \zeta^2$ .

## 2.2. Modified search directions

Let  $0 < \theta < 1$ , and  $\nu^+ := (1 - \theta)\nu$ . According to the definitions of  $(P_\nu)$  and  $(D_\nu)$ , the feasibility equations for  $(P_{\nu^+})$  and  $(D_{\nu^+})$  are as follow:

$$(7) \quad Ax = b - \nu^+ r_b^0, \quad A^T y + s = c - \nu^+ r_c^0, \quad (x, s) \geq 0.$$

To get iterates that are feasible for  $(P_{\nu^+})$  and  $(D_{\nu^+})$ , we need search directions  $\Delta x$  and  $\Delta s$  such that  $x + \Delta x$ ,  $y + \Delta y$  and  $s + \Delta s$  satisfy (7). Since  $x$  and  $(y, s)$  are feasible for  $(P_\nu)$  and  $(D_\nu)$ , respectively, it follows that  $\Delta x$ ,  $\Delta y$  and  $\Delta s$  should satisfy

$$A\Delta x = \theta \nu r_b^0, \quad A^T \Delta y + \Delta s = \theta \nu r_c^0.$$

Therefore, the following system is used to define displacements  $\Delta x$ ,  $\Delta y$  and  $\Delta s$ :

$$(8) \quad \begin{aligned} A\Delta x &= \theta \nu r_b^0, \\ A^T \Delta y + \Delta s &= \theta \nu r_c^0, \\ s\Delta x + x\Delta s &= \mu v - xs. \end{aligned}$$

In this paper, we also replace  $\mu$  at the right hand side of the third equation with  $\mu^+ := (1 - \theta)\mu$  and consider the system

$$(9) \quad \begin{aligned} A\Delta x &= \theta \nu r_b^0, \\ A^T \Delta y + \Delta s &= \theta \nu r_c^0, \\ s\Delta x + x\Delta s &= \mu^+ v - xs. \end{aligned}$$

Now, defining the scaled displacements

$$(10) \quad d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s},$$

we get the following system for the scaled search directions  $d_x$  and  $d_s$ :

$$(11) \quad \begin{aligned} \bar{A}d_x &= \theta \nu r_b^0, \\ \bar{A}^T \frac{\Delta y}{\mu} + d_s &= \theta \nu v s^{-1} r_c^0, \\ d_x + d_s &= (1 - \theta)e - v, \end{aligned}$$

where  $\bar{A} = AV^{-1}X$ ,  $V = \text{diag}(v)$  and  $X = \text{diag}(x)$ . The new iterates after performing the full-Newton step are given by

$$(12) \quad (x^+, y^+, s^+) := (x + \Delta x, y + \Delta y, s + \Delta s).$$

**2.3. Description of the algorithm**

The infeasible-start algorithm is given in Figure 1. In this algorithm, we use the proximity function

$$(13) \quad \delta(v) := \delta(x, s; \mu) = \|e - v\|,$$

to measure the distance between the new iterates and the  $\mu$ -centers. In the sequel we briefly denote  $\delta(v)$  by  $\delta$ . The initial data to launch the algorithm is denoted in Figure 1. So, initially we have  $\delta(x, s; \mu) = 0$ . Suppose that for some  $\mu \in (0, \mu^0)$  we have  $(x, s)$  satisfying the feasibility condition (6) for  $\nu = \frac{\mu}{\mu^0}$ , and  $\delta(v) \leq \tau$ , for some threshold parameter  $\tau \in (0, 1)$ . We select the largest value of  $\theta$ , adapted with the recent value for  $\delta$ , according to the given inequality in the while-loop of the algorithm in Figure 1. Then, we reduce  $\mu$  to  $\mu^+ = (1 - \theta)\mu$  and  $\nu$  to  $\nu^+ = (1 - \theta)\nu$  and find new iterates  $(x^+, y^+, s^+)$  that satisfy (6), with  $\mu$  replaced by  $\mu^+$  and  $\nu$  by  $\nu^+ = \frac{\mu^+}{\mu^0}$ . This process will be repeated until the norm of the residuals  $r_b^0$  and  $r_c^0$ , and the value of  $x^T s$  are less than the accuracy parameter  $\varepsilon$ .

**3. Analysis of the algorithm**

**3.1. Analysis of the full-Newton step and the effect of a  $\mu$ -update**

In this subsection, we analyze the full-Newton step taken in the algorithm. Firstly, we get a sufficient condition for the strict feasibility of the full-Newton step.

**Lemma 3.1.** *The new iterates  $(x^+, s^+)$  are strictly feasible if and only if  $(1 - \theta)v + d_x d_s > 0$ .*

*Proof.* The proof is similar to the proof of Lemma 4.3 in [15] and therefore is omitted. □

The following result can immediately be achieved from the above lemma.

**Corollary 3.2.** *The new iterates  $(x^+, s^+)$  are certainly strictly feasible if*

$$\|d_x d_s\|_\infty < (1 - \theta) \min(v).$$

By the next lemma, we obtain some bounds for the components of the vector  $v$  in terms of  $\delta$ .

**Lemma 3.3.** *One has*

$$1 - \delta \leq v_i \leq 1 + \delta, \quad 1 \leq i \leq n.$$

*Proof.* Since  $|1 - v_i| \leq \|e - v\| = \delta$ , the result easily follows. □

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**Figure 1: The Algorithm**


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**Input:**

Accuracy parameter  $\varepsilon > 0$ ;  
 threshold parameter  $\tau \in (0, 1)$ ;  
 parameter  $\zeta$  corresponds to (4);

**begin**

$x := \zeta e$ ;  $y = 0$ ;  $s := \zeta e$ ;  $\mu := \zeta^2$ ;  $\nu := 1$ ;

**while**  $\max(x^T s, \|r_b\|, \|r_c\|) \geq \varepsilon$  **do**

**begin**

Choose the largest value for  $\theta$  satisfying

$$\gamma^2 + (\gamma + \delta + \theta\sqrt{n})^2 \leq 2(1 - \theta)(\tau(2 - \tau) - \delta),$$

where  $\gamma$  is defined in (25);

$$\mu := (1 - \theta)\mu;$$

$$\nu := (1 - \theta)\nu;$$

$$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s);$$

Calculate  $\delta$  via (13);

**end****end**


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Using (10), we may write

$$(14) \quad \begin{aligned} x^+ &= x + \Delta x = x \left( e + \frac{\Delta x}{x} \right) = x \left( e + \frac{d_x}{v} \right), \\ s^+ &= s + \Delta s = s \left( e + \frac{\Delta s}{s} \right) = s \left( e + \frac{d_s}{v} \right). \end{aligned}$$

From (14) and the second equation in (11), we derive

$$(15) \quad \begin{aligned} x^+ s^+ &= \mu(v + d_x)(v + d_s) = v^2 + v(d_x + d_s) + d_x d_s \\ &= \mu((1 - \theta)v + d_x d_s). \end{aligned}$$

Since in each iteration of the algorithm, the value of  $\mu$  reduces with the factor  $(1 - \theta)$ , the next lemma shows that after each full-Newton step, the value of  $x^T s$  will be reduced by this factor.

**Lemma 3.4.** *One has*

$$(x^+)^T s^+ \leq \frac{n\mu((1 + \delta)^2 + (1 - \theta)^2)}{2}.$$

*Proof.* Define

$$(16) \quad p := d_x + d_s = (1 - \theta)e - v, \quad q := d_x - d_s,$$

with  $v$  as defined in (2). Using (16), we obtain

$$(17) \quad d_x d_s = \frac{p^2 - q^2}{4}, \quad (1 - \theta)v = vp + v^2.$$

Substituting (17) in (15), we have

$$(18) \quad x^+ s^+ = \mu \left( v^2 + vp + \frac{p^2 - q^2}{4} \right).$$

Using (16) and (17), we also obtain

$$(19) \quad vp = \frac{1}{2} \left( (1 - \theta)^2 e - v^2 - p^2 \right).$$

From (18) and (19), using Lemma 3.3, we derive

$$\begin{aligned} (x^+)^T s^+ &= \mu e^T (x^+ s^+) \\ &= \frac{\mu}{2} e^T \left( v^2 + (1 - \theta)^2 e - \frac{p^2 + q^2}{2} \right) \\ &\leq \frac{\mu}{2} e^T (v^2 + (1 - \theta)^2 e) \\ &= \frac{\mu}{2} (e^T v^2 + n(1 - \theta)^2) \\ &\leq \frac{n\mu}{2} \left( (1 + \delta)^2 + (1 - \theta)^2 \right), \end{aligned}$$

which completes the proof. □

Let  $\omega = \|d_x\|^2 + \|d_s\|^2$ . Then, we easily obtain

$$(20) \quad \|d_x d_s\|_\infty \leq \|d_x d_s\| \leq \frac{\omega}{2},$$

which is used in the next theorem to investigate the effect of the simultaneous full-Newton step and the  $\mu$ -update on the proximity function.

**Lemma 3.5.** *Let  $\delta(v^+) = \delta(x^+, s^+; \mu^+)$  and  $v^+$  be the variance vector of the iterates  $(x^+, s^+)$  with respect to  $\mu^+$ , i.e.,*

$$v^+ = \sqrt{\frac{x^+ s^+}{\mu^+}}.$$

Then, we have

$$\delta(v^+) \leq \frac{(1 - \theta)\delta + \frac{\omega}{2}}{1 - \theta + \sqrt{(1 - \theta)^2(1 - \delta) - (1 - \theta)\frac{\omega}{2}}}.$$

*Proof.* From Lemma 3.3, we have

$$(21) \quad \begin{aligned} \min((1 - \theta)v + d_x d_s) &\geq (1 - \theta) \min(v) - \|d_x d_s\|_\infty \\ &\geq (1 - \theta)(1 - \delta) - \|d_x d_s\|_\infty. \end{aligned}$$

We may write

$$\begin{aligned}
 \delta(v^+)^2 &= \left\| e - \sqrt{\frac{x^+s^+}{\mu^+}} \right\|^2 = \frac{1}{1-\theta} \left\| \sqrt{1-\theta}e - \sqrt{\frac{x^+s^+}{\mu}} \right\|^2 \\
 &= \frac{1}{1-\theta} \sum_{i=1}^n \left( \sqrt{1-\theta} - \left( \sqrt{\frac{x^+s^+}{\mu}} \right)_i \right)^2 \\
 &= \frac{1}{1-\theta} \sum_{i=1}^n \left( \frac{1-\theta - \left( \frac{x^+s^+}{\mu} \right)_i}{\sqrt{1-\theta} + \left( \sqrt{\frac{x^+s^+}{\mu}} \right)_i} \right)^2 \\
 &\leq \frac{1}{1-\theta} \sum_{i=1}^n \left( \frac{1-\theta - \left( \frac{x^+s^+}{\mu} \right)_i}{\sqrt{1-\theta} + \min \left( \sqrt{\frac{x^+s^+}{\mu}} \right)} \right)^2 \\
 &= \frac{1}{1-\theta} \sum_{i=1}^n \left( \frac{1-\theta - ((1-\theta)v + d_x d_s)_i}{\sqrt{1-\theta} + \min \left( \sqrt{(1-\theta)v + d_x d_s} \right)} \right)^2 \\
 &= \frac{1}{1-\theta} \sum_{i=1}^n \left( \frac{(1-\theta)(e-v)_i - (d_x d_s)_i}{\sqrt{1-\theta} + \min \left( \sqrt{(1-\theta)v + d_x d_s} \right)} \right)^2 \\
 &= \frac{\|(1-\theta)(e-v) - (d_x d_s)\|^2}{(1-\theta) \left( \sqrt{1-\theta} + \min \left( \sqrt{(1-\theta)v + d_x d_s} \right) \right)^2} \\
 &\leq \frac{((1-\theta)\delta + \|d_x d_s\|)^2}{(1-\theta) \left( \sqrt{1-\theta} + \sqrt{(1-\theta)(1-\delta)} - \|d_x d_s\|_\infty \right)^2},
 \end{aligned}$$

where the fifth equality follows from (15) and the last inequality is due to (21). Using (20) in the last inequality above follows the lemma.  $\square$

After doing the full-Newton step towards the  $\mu^+$ -centers, we need  $\delta(v^+) \leq \tau$ , which means that the new iterates are also in the  $\tau$ -neighborhood of the central path. Based on the result of Lemma 3.5, this certainly holds, if

$$\frac{(1-\theta)\delta + \frac{\omega}{2}}{1-\theta + \sqrt{(1-\theta)^2(1-\delta)} - (1-\theta)\frac{\omega}{2}} \leq \tau.$$

Solving this inequality with respect to  $\omega$ , we obtain

$$(22) \quad \omega \leq 2(1-\theta) (\tau(2-\tau) - \delta).$$

In the following, we show that (22) ensures also the strict feasibility of the iterates. By Lemma 3.3, we have  $\min(v) \geq 1 - \delta$ . Therefore, from Lemma 3.2



and noticing (20), we find that the iterates are strictly feasible, whenever

$$(23) \quad \omega \leq 2(1 - \theta)(1 - \delta).$$

Assuming (22), then (23) is satisfied, if

$$\tau(2 - \tau) - \delta \leq 1 - \delta.$$

But, this inequality leads to the true inequality  $(\tau - 1)^2 \geq 0$ . So, (22) fulfills also the strict feasibility of the iterates. We call (22) the condition for adaptive updating of the algorithm. Now, we proceed by considering  $\omega$  in more details.

### 3.2. Upper bound for $\omega$

In this subsection, we first recall some important results from [5] and [8], and then we obtain an upper bound for  $\omega$ . Consider the system (11) which defines the scaled displacements  $d_x$  and  $d_s$ . Let  $\mathfrak{L}$  denotes the null space of the matrix  $\bar{A}$ . So,  $\mathfrak{L} := \{\xi \in \mathbb{R}^n : \bar{A}\xi = 0\}$ . Obviously, the affine space  $\{\xi \in \mathbb{R}^n : \bar{A}\xi = \theta\nu r_b^0\}$  equals to  $d_x + \mathfrak{L}$ . The row space of  $\bar{A}$  equals to the orthogonal complement  $\mathfrak{L}^\perp$  of  $\mathfrak{L}$ , and  $d_s \in \theta\nu v s^{-1} r_c^0 + \mathfrak{L}^\perp$ . The following lemma gives an upper bound for  $\omega$ .

**Lemma 3.6.** *Let  $q$  be the (unique) point in the intersection of the affine spaces  $d_x + \mathfrak{L}$  and  $d_s + \mathfrak{L}^\perp$ . Then*

$$\omega \leq \|q\|^2 + (\|q\| + \delta + \theta\sqrt{n})^2.$$

*Proof.* The proof of this lemma is similar to that of Lemma 4.6 in [8]. □

Now, from Lemma 2.4 in [5], we have the following bound for the quantity  $\|q\|$ .

**Lemma 3.7** (Lemma 2.4 in [5]). *One has*

$$\|q\| \leq \frac{\theta(\|x\|_1 + \|s\|_1)}{\zeta \min(v)}.$$

**Lemma 3.8.** *Let  $(x, y, s)$  be feasible for the perturbed problems  $(P_\nu)$  and  $(D_\nu)$ ,  $(x^0, y^0, s^0)$  as defined in (4) and  $(x^*, y^*, s^*)$  be a primal-dual optimal solution. Then*

$$\|x\|_1 + \|s\|_1 \leq \frac{n\zeta(2 + (1 + \delta)^2 + (1 - \theta)^2)}{2}.$$

*Proof.* By the assumptions of the lemma and the definitions of the perturbed problems, we may write

$$\begin{aligned} A(x - \nu x^0 - (1 - \nu)x^*) &= 0, \\ A^T(y - \nu y^0 - (1 - \nu)y^*) + (s - \nu s^0 - (1 - \nu)s^*) &= 0. \end{aligned}$$

Therefore, we find that  $(x - \nu x^0 - (1 - \nu)x^*)^T(s - \nu s^0 - (1 - \nu)s^*) = 0$ . By direct expanding the last equality, using the facts that  $(x^*)^T s^* = 0$  and  $x^T s^* + s^T x^* \geq 0$ , we derive

$$(24) \quad \nu(x^T s^0 + s^T x^0) \leq (\nu^2 n \mu^0 + \nu(1 - \nu))((x^*)^T s^0 + (x^0)^T s^*) + x^T s.$$

Considering  $x^0 = s^0 = \zeta e$ ,  $\mu^0 = \zeta^2$ , and  $\nu = \frac{\mu}{\mu^0}$ , we may write (24) equivalently as follows

$$\zeta(e^T x + e^T s) \leq n\nu\zeta^2 + (1 - \nu)\zeta e^T(x^* + s^*) + \zeta^2 \frac{x^T s}{\mu}.$$

Using Lemma 3.4 and noticing  $\|x^* + s^*\|_\infty \leq \zeta$ , the following inequality is obtained

$$\begin{aligned} \zeta(\|x\|_1 + \|s\|_1) &\leq n\nu\zeta^2 + (1 - \nu)n\zeta \|x^* + s^*\|_\infty + \frac{n\zeta^2}{2}((1 + \delta)^2 + (1 - \theta)^2) \\ &\leq \frac{n\zeta^2}{2}(2 + (1 + \delta)^2 + (1 - \theta)^2), \end{aligned}$$

which proves the lemma. □

Substituting the result of Lemma 3.8 in Lemma 3.7, using Lemma 3.3, we get the following

$$(25) \quad \|q\| \leq \frac{n\theta(2 + (1 - \theta)^2 + (1 - \delta)^2)}{2(1 - \delta)} := \gamma.$$

Now, substituting (25) in Lemma 3.6, we have

$$(26) \quad \omega \leq \gamma^2 + (\gamma + \delta + \theta\sqrt{n})^2.$$

### 3.3. Value for $\theta$

At this place, we return to the adaptive updating condition (22). Based on the above arguments, which lead to the inequality (26), we find that the adaptive updating condition (22) is satisfied, whenever

$$(27) \quad \gamma^2 + (\gamma + \delta + \theta\sqrt{n})^2 \leq 2(1 - \theta)(\tau(2 - \tau) - \delta).$$

The algorithm passes this condition to adopt the largest adapted (with the recent value of  $\delta$ ) possible value for  $\theta$  and then updates the central path-parameters  $\mu$  and  $\nu$  according to this value of  $\theta$ .

Let us, we choose the default value  $\tau := \frac{1}{5}$ . Then, (27) reduces to

$$(28) \quad \gamma^2 + (\gamma + \delta + \theta\sqrt{n})^2 \leq 2(1 - \theta)\left(\frac{9}{25} - \delta\right).$$

Considering the minimum possible values for  $\delta$ , i.e.,  $\delta = 0$ , then (28) reduces to

$$(29) \quad \left(\frac{n\theta(3 + (1 - \theta)^2)}{2}\right)^2 + \left(\frac{n\theta(3 + (1 - \theta)^2)}{2} + \theta\sqrt{n}\right)^2 \leq \frac{18}{25}(1 - \theta).$$

By some elementary calculations, we find that this inequality is satisfied, whenever  $\theta = \frac{1}{11n}$  for  $n \geq 2$ . When  $\delta$  takes the maximum value,  $\delta = \tau$ , then (28) reduces to

$$(30) \quad \left( \frac{5n\theta}{8} \left( \frac{86}{25} + (1-\theta)^2 \right) \right)^2 + \left( \frac{5n\theta}{8} \left( \frac{86}{25} + (1-\theta)^2 \right) + \frac{1}{5} + \theta\sqrt{n} \right)^2 \leq \frac{8}{25}(1-\theta).$$

Again, by some elementary calculations, this inequality is satisfied, whenever  $\theta = \frac{1}{4n}$  for  $n \geq 2$ . Hence, when we are using adaptive updates, the actual value of  $\theta$  varies from iteration to iteration, but it always lies in the interval  $\left[ \frac{1}{11n}, \frac{1}{4n} \right]$ .

Each main iteration of the algorithm consists of at most one so-called inner iteration with the necessity of computing a full-Newton step search direction. Since in each iteration both the value of  $x^T s$  and the norms of the residuals are reduced by the factor  $1 - \theta$ , the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max \left\{ (x^0)^T s^0, \|r_b^0\|, \|r_c^0\| \right\}}{\varepsilon}.$$

So, we may state without further proof, the main result of the paper.

**Theorem 3.9.** *If the problems (P) and (D) have optimal solution  $(x^*, y^*, s^*)$  such that  $\|x^* + s^*\| \leq \zeta$  for some  $\zeta > 0$ , then after at most*

$$\frac{1}{\theta} \log \frac{\max \left\{ n\zeta^2, \|r_b^0\|, \|r_c^0\| \right\}}{\varepsilon}$$

*iterations, the algorithm finds an  $\varepsilon$ -solution of LO. Here,  $\theta \in \left[ \frac{1}{11n}, \frac{1}{4n} \right]$ .*

#### 4. Numerical results

In this section we solve some LO problems from NETLIB by the original Roos's infeasible algorithm [8], and by short updating algorithm [6], as well as by the adaptive updating algorithm in Figure 1. For all these cases, the initialization parameter  $\zeta$  is calculated as described in Section 2 correspond to some obtained optimal solutions of the problems by the feasible interior-point algorithm in [8]. The accuracy parameter  $\varepsilon$  is set to  $10^{-4}$ . For the Roos's infeasible algorithm [8], the parameter  $\theta$  and the threshold parameter  $\tau$  is set to the obtained values by the theoretical analysis as is given in [8]. Table 1 compares the number of iterations to solve the given problems via these algorithms. In this table, Algorithm 1 means the Roos's original infeasible algorithm [8]; Algorithm 2 is the short updating algorithm in [6], and Algorithm 3 is the presented our algorithm in Figure 1. Noticing the fourth and the fifth columns of the table shows that eliminating the centering steps along with short updates, reduces the number of iterations in the original Roos's

TABLE 1. The number of iterations

Problem	$m$	$n$	Algorithm 1	Algorithm 2	Algorithm 3
blend	74	114	390	121	52
share1b	117	253	586	183	103
share2b	96	162	435	144	83
adlittle	56	138	427	141	72
scsd1	77	760	781	263	130
sc105	105	163	479	161	89
agg	488	615	753	247	112
scagr7	129	185	479	153	93

infeasible algorithm nearly up to 60%. Comparing the fifth and the sixth columns, it is clear that the adaptive updating scheme is practically decreased the number of required iterations. Since for both algorithms 2 and 3 the work in every iteration is almost the same, this shows that the adaptive updating strategy reduces the number of iterations, significantly. Comparing also the fourth and the last columns of the table illustrates that our algorithm gives a huge reduction in the number of iterations in the original Roos's infeasible algorithm [8].

## 5. Conclusions

In this paper, we presented an IIPM for solving LO, based on modified search directions such that only one full-Newton step is needed in each main iteration. We also improved the IIPM presented for LO in [6]. The algorithm in [6] takes a default value as  $\theta = \frac{1}{15n}$  for the barrier update parameter. This value for  $\theta$  is rather small and leads to possibly large number of iterations in practical purposes. In this paper, we allowed the value of  $\theta$  changes in an appropriate interval, iteration to iteration, adaptively with respect to the value of the proximity function taken by the current iterates. Note that, in this paper, the original infeasible algorithm of Roos [8] is much improved; not only the centering steps in which have been removed and the algorithm performs only one full-Newton step in a major iteration, but also to perform the step, the algorithm uses an adaptive updating scheme which may lead to larger updates of the central path-parameters. We have solved some LO problems from NETLIB, which reveals that our presented algorithm accelerates practically the original infeasible Roos's algorithm [8] and the short updating algorithm in [6].

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