Bull. Korean Math. Soc. **53** (2016), No. 6, pp. 1823–1830
http://dx.doi.org/10.4134/BKMS.b151005
pISSN: 1015-8634 / eISSN: 2234-3016

RANGE INCLUSION OF TWO SAME TYPE CONCRETE OPERATORS

Takahiko Nakazi

Abstract. Let H and K be two Hilbert spaces, and let A and B be two bounded linear operators from H to K. We are interested in Range$B^* \supseteq$ RangeA^*. It is well known that this is equivalent to the inequality $A^*A \geq \varepsilon B^*B$ for a positive constant ε. We study conditions in terms of symbols when A and B are singular integral operators, Hankel operators or Toeplitz operators, etc.

1. Introduction

Let Γ be the unit circle in the complex plane $\mathbb{C}$ and m the normalized Lebesgue measure on Γ. For $1 \leq p \leq \infty$, L^p denotes the Lebesgue space and H^p denotes the Hardy space.

For a closed subspace N in L^2, P_N denotes the orthogonal projection from L^2 to N. We will write $P_N = P$ for $N = H^2$.

For ϕ in L^∞, $M_\phi f = \phi f$ $(f \in L^2)$. A Hankel operator H_ϕ and a Toeplitz operator T_ϕ are defined as the following,

$$H_\phi f = (I - P)M_\phi f \quad (f \in H^2)$$

and

$$T_\phi f = PM_\phi f \quad (f \in H^2).$$

For a and b in L^∞, put

$$S_{a,b} f = (M_a P + M_b(I - P))f \quad (f \in L^2)$$

then $S_{a,b}$ is called a singular integral operator. Let N be a closed subspace in H^2 such that $T_z^* N \subset N$. For ϕ in L^∞, a truncated Toeplitz operator S_ϕ is defined as the following,

$$S_\phi f = P_N M_\phi f \quad (f \in N).$$

Received December 7, 2015.
2010 *Mathematics Subject Classification.* Primary 47B35.
Key words and phrases. singular integral operator, Hankel operator, Toeplitz operator, range inclusion.

In this paper, we are interested in when Range$B^* \supseteq$ RangeA^*. It is known as a theorem of Douglas [2] that Range$B^* \supseteq$ RangeA^* if and only if $A^*A \geq \varepsilon B^*B$ for some positive constant ε. $\overline{\text{range}}A$ denotes the clousure of Range$A$.

B. A. Lotto [5] studied when $A = H_\psi$ and $B = T_\phi$ or $A = T_\phi$ and $B = H_\psi$. That is, he studied when A and B are different kinds of operators. In this paper, we study when A and B are same kinds of operators, that is, when both A and B are the following operators: (1) singular integral operators, (2) the adjoints of singular integral operators, (3) Hankel operators, (4) Toeplitz operators, (5) truncated Toeplitz operators. The case (3) was essentially studied by C. Gu [3].

When f is a nonzero function in $H^p(1 \leq p \leq \infty)$, if $|f| = 1$ *a.e.* on Γ, then f is called an inner function. If $|f| = |g|$ and g is in H^p, then $g = qf$ for some inner function q, then f is called an outer function. If $f/|f| = g/|g|$ and g is in H^p, then $g = cf$ for some positive constant c, then f is called a strongly outer function.

2. Adjoint of singular integral operator

T. Yamamoto [10] gives a necessary and sufficient condition for $S^*_{a,b}S_{a,b} \geq S^*_{c,d}S_{c,d}$ different from Lemma 3.

Lemma 1. *The following* (1), (2) *and* (3) *are equivalent.*

(1) $S^*_{a,b}S_{a,b} \geq S^*_{c,d}S_{c,d}$.

(2) *For* $f_1 \in H^2$ *and* $f_2 \in \bar{z}\bar{H}^2$,

$$\int |af_1 + bf_2|^2 d\theta/2\pi \geq \int |cf_1 + df_2|^2 d\theta/2\pi.$$

(3) *For* $f_1 \in H^2$ *and* $f_2 \in \bar{z}\bar{H}^2$

$$|\int f_1\bar{f}_2(a\bar{b} - c\bar{d})d\theta/2\pi|^2 \leq \int |f_1|^2(|a|^2 - |c|^2)d\theta/2\pi \int |f_2|^2(|b|^2 - |d|^2)d\theta/2\pi$$

and $|a|^2 - |c|^2 \geq 0$ *and* $|b|^2 - |d|^2 \geq 0$.

Proof. By well known routine calculations, it is easy to see the equivalences. □

Lemma 2. *Let* ϕ, W_1 *and* W_2 *be in* L^∞ *where* $W_j \geq 0$ $(j = 1, 2)$. *Then the following* (1) *and* (2) *are equivalent.*

(1) *For* $f_1 \in H^2$ *and* $f_2 \in \bar{z}\bar{H}^2$

$$|\int f_1\bar{f}_2\phi d\theta/2\pi|^2 \leq \int |f_1|^2 W_1 d\theta/2\pi \int |f_2|^2 W_2 d\theta/2\pi.$$

(2) *There exists* k *in* H^∞ *such that* $|\phi - k|^2 \leq W_1W_2$.

Proof. (1)$\Longrightarrow$(2). The Cotlar-Sadosky' lifting theorem [1] shows (2).

(2)$\Longrightarrow$(1). When $f_1 \in H^2$ and $f_2 \in \bar{z}\bar{H}^2$, $f_1\bar{f}_2 \in zH^1$ and so

$$|\int f_1\bar{f}_2\phi d\theta/2\pi| = |\int f_1\bar{f}_2(\phi - k)d\theta/2\pi|$$

$$\leq \int |f_1|W_1^{1/2}|f_2|W_2^{1/2}d\theta/2\pi$$
$$\leq \int |f_1|^2W_1 d\theta/2\pi \int |f_2|^2 W_2 d\theta/2\pi$$

because $|\phi - k|^2 \leq W_1W_2$. □

Lemma 3. *$S^*_{a,b}S_{a,b} \geq S^*_{c,d}S_{c,d}$ if and only if $|a| \geq |c|, |b| \geq |d|$ and there exists k in H^∞ such that*

$$|(a\bar{b} - c\bar{d}) + k|^2 \leq (|a|^2 - |c|^2)(|b|^2 - |d|^2).$$

Proof. By the equivalence (1) and (3) of Lemma 1, when $\phi = a\bar{b} - c\bar{d}$, $W_1 = |a|^2 - |c|^2$ and $W_2 = |b|^2 - |d|^2$, $S^*_{a,b}S_{a,b} \geq S^*_{c,d}S_{c,d}$ if and only if for $f_1 \in H^2$ and $f_2 \in \bar{z}\bar{H}^2$

$$|\int f_1\bar{f}_2\phi d\theta/2\pi|^2 \leq \int |f_1|^2W_1 d\theta/2\pi \int |f_2|^2W_2 d\theta/2\pi.$$

Hence by Lemma 2 there exists k in H^∞ such that $|\phi - k|^2 \leq W_1W_2$. Therefore $S^*_{a,b}S_{a,b} \geq S^*_{c,d}S_{c,d}$ if and only if there exists k in H^∞ such that

$$|(a\bar{b} - c\bar{d}) - k|^2 \leq (|a|^2 - |c|^2)(|b|^2 - |d|^2). \qquad \square$$

Theorem 1. *$RangeS^*_{a,b} \supseteq RangeS^*_{c,d}$ if and only if there exists a positive constant ε such that $|a| \geq \varepsilon|c|, |b| \geq \varepsilon|d|$, and there exists k in H^∞ such that*

$$|(a\bar{b} - \varepsilon^2c\bar{d}) + k|^2 \leq (|a|^2 - \varepsilon^2|c|^2)(|b|^2 - \varepsilon^2|d|^2).$$

Proof. By the Douglas theorem, it is clear from Lemma 3. □

3. Singular integral operator

K. Takahashi [9] gives a necessary and sufficient condition for that there exists a positive constant γ such that $\gamma(M_aPM_{\bar{a}}) \geq (M_cPM_{\bar{c}})$. He applied it to an interpolation theorem on the unit circle. We use this to study Range$S_{a,b} \supseteq$ Range$S_{c,d}$. The proof of following lemma is similar to [8]. In this section, put $Q = 1 - P$

Lemma 4. *$Range(PM_\phi)^* \supseteq Range(PM_\psi)^*$ if and only if there exists a function f in H^∞ such that $\psi = \bar{f}\phi$.*

Proof. The 'if' part is clear. In fact, $(PT_\psi)^*PM_\psi = (T_{\bar{f}}PM_\phi)^*(T_{\bar{f}}PM_\phi) = (PM_\phi)^*(T_fT_f^*)PM_\phi \leq \|f\|^2_\infty(PM_\phi)^*PM_\phi$. By the Douglas theorem,

$$\text{Range}(PM_\phi)^* \supseteq \text{Range}(PM_\psi)^*.$$

We will show the 'only if' part. By the Douglas theorem, there exists an operator on X on H^2 such that $XPM_\phi = PM_\psi$. Hence $\overline{\text{Range}}PM_\phi$ and $\overline{\text{Range}}PM_\psi$ are invariant for T^*_z because $T^*_zPM_\phi L^2 = PM_{\bar{z}}PM_\phi L^2 = PM_{\bar{z}}(I-Q)M_\phi L^2 = PM_{\bar{z}}M_\phi L^2 = PM_\phi L^2$.

$$T^*_zXPM_\phi = T^*_zPM_\psi = PM_\psi M_{\bar{z}} = XPM_\phi M_{\bar{z}} = XT^*_zPM_\phi.$$

Hence $[T_z^* \mid \overline{\mathrm{Range}} PM_\psi][X \mid \overline{\mathrm{Range}} PM_\phi] = [X \mid \overline{\mathrm{Range}} PM_\phi][T_z^* \mid \overline{\mathrm{Range}} PM_\phi]$. Then by the Nagy-Foias lifting theorem [5] there exists an operator Y on H^2 such that $T_z^* Y = Y T_z^*$ and $Y \mid \overline{\mathrm{Range}} PM_\phi = X \mid \overline{\mathrm{Range}} PM_\phi$, that is, there exists a function $f \in H^\infty$ such that $T_f^* PM_\phi = PM_\psi$. Hence $PM_{\bar{f}\phi} = PM_\psi$ and so $PM_{\bar{f}\phi - \psi} P = 0$. Thus $\bar{f}\phi = \psi$. □

Theorem 2. *Suppose $b = d$. Then Range$S_{a,b} \supseteq$ Range$S_{c,d}$ if and only if there exists a function f in H^∞ such that $c = fa$.*

Proof. By hypothesis,

$$S_{a,b} = (PM_{\bar{a}})^* + (QM_{\bar{b}})^* \text{ and } S_{c,d} = (PM_{\bar{c}})^* + (QM_{\bar{b}})^*.$$

Hence Range$S_{a,b} \supseteq$ Range$S_{c,d}$ if and only if Range$(PM_{\bar{a}})^* \supseteq$ Range$(PM_{\bar{c}})^*$. Now Lemma 4 shows the theorem. □

4. Hankel operator

The following theorem is essentially known in [3, Corollary 2] or [3, the proof of Lemma 4].

Theorem 3. *Let ϕ, ψ be in L^∞.*

(1) *Range$H_\psi^* \supseteq$ RangeH_ϕ^* if and only if there exists a function k in H^∞ such that $k\psi - \phi$ belongs to H^∞.*

(2) *Range$H_\psi \supseteq$ RangeH_ϕ if and only if there exists a contraction h in H^∞ such that $h\bar{\psi} - \bar{\phi}$ belongs to H^∞.*

Proof. (1) If Range$H_\psi^* \supseteq$ RangeH_ϕ^*, then by the Douglas theorem there exists a positive constant ε such that $H_\psi^* H_\psi \geq H_{\varepsilon\phi}^* H_{\varepsilon\phi}$. By [3, Corollary 2], there exists a contraction k_0 in H^∞ such that $k_0\psi - \varepsilon\phi$ belongs to H^∞. Put $k = k_0/\varepsilon$. Conversely if $k \in H^\infty$ and $k\psi - \phi \in H^\infty$, put $k_0 = k/\|k\|_\infty$ and $\varepsilon = 1/\|k\|_\infty$ where we may assume $k \neq 0$. [3, Corollary 2] shows $H_\psi^* H_\psi \geq H_{\varepsilon\phi}^* H_{\varepsilon\phi}$. Now apply the Douglas theorem

(2) It is a corollary of (1). Or it can be proved directly. □

5. Toeplitz operator

When we assume Ker$T_{\bar{\phi}} \neq \{0\}$, we can prove a general result. However when we do not assume Ker$T_{\bar{\phi}} \neq \{0\}$, we have to consider only a few special case. In general a strongly outer function is outer but the converse is not true.

Lemma 5. *If* Ker$T_{\bar{\phi}} \neq \{0\}$, *then $\bar{\phi} = \bar{z}\bar{q}g$ where q is inner and g^2 is strongly outer, and* Ker$T_{\bar{\phi}} = (H^2 \ominus zqH^2)g$.

Proof. This is a theorem of E. Hayashi [4]. □

Theorem 4. *Suppose RangeT_ϕ is not dense in H^2. If Range$T_\phi \supseteq$ RangeT_ψ, then $\phi = ptg/\overline{tg}$ and $\psi = qg/\bar{g}$ where p, q and g satisfy the following condition:*

(i) *p and q are inner and $pt \in H^2 \ominus qzH^2$.*

(ii) *g^2 and $(tg)^2$ are strongly outer.*

The converse is valid when Range T_ϕ is closed, and both g and t are bounded.

Proof. If Range$T_\phi \supseteq$ RangeT_ψ, then Ker$T_{\bar\psi} \supseteq$ Ker$T_{\bar\phi}$. Since RangeT_ϕ is not dense, Ker$T_{\bar\phi} \neq \{0\}$. By Lemma 4, Ker$T_{\bar\phi} = (H^2 \ominus zpH^2)f$ and Ker$T_{\bar\psi} = (H^2 \ominus zqH^2)g$, and $\bar\phi = \bar z\bar p\bar f/f$ and $\bar\psi = \bar z\bar q\bar g/g$ where p and q are inner, and f^2 and g^2 are strongly outer. Since Ker$T_{\bar\psi} \supseteq$ Ker$T_{\bar\phi}, pf = sg$ for some $s = pt$ and t is outer and $f = tg$ for some $t \in H^2 \ominus zqH^2$. This shows the first part the theorem. The converse is clear. □

Theorem 5. *Let ϕ and ψ be in L^∞. Then the following* (1) *and* (2) *hold.*

(1) *When ϕ and ψ are nonzero functions in H^∞, Range$T_\phi \supseteq$ RangeT_ψ if and only if there exists k in H^∞ such that $\psi = k\phi$.*

(2) *When $\bar\phi$ and $\bar\psi$ are nonzero functions in H^∞, if there exists m in H^∞ such that $\psi = \bar m\phi$, then Range$T_\phi \supseteq$ RangeT_ψ.*

Proof. (1) By the Douglas theorem, it is enough to show $T_\phi^*T_\phi \geq \varepsilon^2 T_\psi^* T_\psi$ for some $\varepsilon > 0$ if and only if $|\phi| \geq \varepsilon|\psi|$. But it is clear because $T_\phi^*T_\phi = T_{|\phi|^2}$ and $T_\psi^*T_\psi = T_{|\psi|^2}$ and $T_{|\phi|^2-\varepsilon^2|\psi|^2} \geq 0$. This shows (1)

(2) It is clear because $T_\psi H^2 = T_\phi T_{\bar m} H^2 \subseteq T_\phi H^2$. □

The converse of (2) of Theorem 5 does not hold. In fact, if $\bar\phi$ and $\bar\psi$ are inner, then $T_\phi H^2 = T_\psi H^2 = H^2$. Thus we should consider the converse when $\bar\phi$ and $\bar\psi$ are outer. However if $\bar\phi = 1 + z$ and $\bar\psi = 1 - z$, then we can see $T_\phi H^2 = T_\psi H^2 \underset{\neq}{\subset} H^2$.

When $T_\phi H^2 = H^2$, we can tell nothing about RangeT_ψ. Now we consider some special case such that $T_\phi H^2 \neq H^2$ and $T_\phi H^2$ is dense in H^2.

Theorem 6. *Let E and F be measurable sets on Γ.*

(1) *If Range$T_{\chi_E} \supseteq$ RangeT_{χ_F}, then $m(F \cup E) = 1$ or $E = F$. Moreover if $F = E^c$, then $m(E) = 1$.*

(2) *If $\phi = \chi_E$ and $\bar\psi$ is a nonzero function in H^∞, then Range$T_\phi \not\supseteq$ RangeT_ψ.*

(3) *If $\phi = \chi_E$ and ψ is a nonzero function in H^∞, then Range$T_\phi \not\supseteq$ RangeT_ψ.*

Proof. (1) For a measurable set G, put $h_G = T_{\chi_G} h$ where $h \in H^2$. Then $\chi_G h = h_G + \overline{h^G}$ where $h_G \in H^2$ and $\overline{h^G} \in \bar z\bar H^2$. Suppose Range$T_{\chi_E} \supseteq$ RangeT_{χ_F}. For $f \in H^2$, if $\chi_F f = f_F + \overline{f^F}$, then there exists e in H^2 such that $e_E = f_F$ and $\chi_E e = e_E + \overline{e^E}$. Hence $\chi_F f - \chi_E e = \overline{f^F} - \overline{e^E}$. If $m(E \cup F) < 1$, then $\chi_F = \chi_E$. Since for any g there exists f such that $T_{\chi_E} f = T_{\chi_F} g$, $T_{\chi_E} f = g - T_{\chi_E} g$ and so $g = T_{\chi_E}(f + g)$. This shows $T_{\chi_E} H^2 = H^2$ and so $m(E) = 1$.

(2) Since $\psi \not\equiv 0$, there exists a nonnegative integer n such that $\bar\psi = z^n \ell$ for $\ell \in H^2$ with $\ell(0) \neq 0$. If $T_\phi H^2 \supseteq T_\psi H^2$, then $T_\phi H^2 \ni \bar\ell(0)$ and so there exists $f \in H^2$ and $g \in zH^2$ such that $\chi_E f = \bar\ell(0) + \bar g \in \bar H^2$. This contradiction shows $T_\phi H^2 \not\supseteq T_\psi H^2$.

(3) If $T_\phi H^2 \supseteq T_\psi H^2$, then there exist $g \in H^2$ and $h \in zH^2$ such that $\phi g = \psi + \bar h$. Hence $\phi z g = z\psi + \overline{h(0)} + \bar h_1$ where we may assume $h(0) \neq 0$.

Since $T_\phi H^2 \ni z\psi$, there exist $g_1 \in H^2$ and $h_2 \in zH^2$ such that $\phi g_1 = z\psi + \bar{h}_2$. Therefore $\phi(zg - g_1) = \overline{h(0)} + \overline{(h_1 - h_2)}$. This implies $T_\phi H^2 \ni 1$ and so $\phi g' = 1 + \bar{h}'$ for some $g' \in H^2$ and $h' \in zH^2$. This shows $\chi_E \bar{g}'$ belongs to H^2. This contradiction shows $T_\phi H^2 \not\supseteq T_\psi H^2$. □

6. Truncated Toeplitz operator

Let q be a nonconstant inner function and $N = N_q = H^2 \ominus qH^2$. For a function ϕ in L^∞, $S_\phi = P_N T_\phi \mid N$ is called a truncated Toeplitz operator.

Lemma 6. *If ϕ is a nonzero function in H^∞, then* $\mathrm{Ker} S_\phi^* = H^2 \ominus Q_\phi H^2$ *and* $\overline{\mathrm{Range}} S_\phi = Q_\phi(H^2 \ominus q_\phi H^2)$ *where Q_ϕ and q_ϕ are inner functions such that $q = Q_\phi q_\phi$.*

Proof. Since $S_z S_\phi = S_\phi S_z$, $\mathrm{Ker} S_\phi^*$ is invariant under S_z^* and hence invariant under T_z^*. By the Beurling theorem, $\mathrm{Ker} S_\phi^* = H^2 \ominus Q_\phi H^2$ for some inner function Q_ϕ where $Q_\phi H^2 \supseteq qH^2$. Therefore $N_q = (H^2 \ominus Q_\phi H^2) \oplus Q_\phi(H^2 \ominus q_\phi H^2)$ where $q_\phi = q\bar{Q}_\phi$. This implies the lemma. □

For a function ϕ in H^∞ let Q_ϕ and q_ϕ be two inner functions such that $q = Q_\phi q_\phi$ in Lemma 6.

Theorem 7. *Let ϕ and ψ be in H^∞.*

(1) *If there exists a function k in H^∞ such that $k\phi - \psi$ belongs to qH^∞, then $\mathrm{Range}S_\phi^* \supseteq \mathrm{Range}S_\psi^*$.*

(2) *If $\mathrm{Range}S_\phi^*$ is dense and $\mathrm{Range}S_\phi^* \supseteq \mathrm{Range}S_\psi^*$, then there exists a function k in H^∞ such that $k\phi - \psi$ belongs to qH^∞.*

(3) *There exist two inner functions q_ϕ and q_ψ such that* $\overline{\mathrm{Range}} S_\phi^* = H^2 \ominus q_\phi H^2$ *and* $\overline{\mathrm{Range}} S_\psi^* = H^2 \ominus q_\psi H^2$ *where $\overline{q_\phi} q$ and $\overline{q_\psi} q$ are in H^∞. If Range S_ϕ^* is closed, then Range $S_\phi^* \supseteq$ Range S_ψ^* if and only if $\overline{q_\psi} q_\phi$ belongs to H^∞.*

Proof. (1) Since $S_{k\phi - \psi} = 0$, $S_k S_\phi = S_{k\phi} = S_\psi$ and so $\mathrm{Range}S_\phi^* \supseteq \mathrm{Range}S_\psi^*$.

(2) $\mathrm{Range}S_\phi^* \supseteq \mathrm{Range}S_\psi^*$ if and only if there exists a bounded linear operator B such that $BS_\phi = S_\psi$. Then $BS_z S_\phi = BS_\phi S_z = S_\psi S_z = S_z S_\psi = S_z B S_\phi$. Since S_ϕ has a dense range, $BS_z = S_z B$ and so by a theorem of Sarason $B = S_k$ for some $k \in H^\infty$. Hence $S_{k\phi} = S_\psi$ and so $k\phi - \psi \in qH^\infty$.

(3) Since $S_z^* S_\phi^* = S_\phi^* S_z^*$ and $S_z^* S_\psi^* = S_\psi^* S_z^*$, there exist inner functions q_ϕ and q_ψ such that $\overline{\mathrm{Range}} S_\phi^* = H^2 \ominus q_\phi H^2$ and $\overline{\mathrm{Range}} S_\psi^* = H^2 \ominus q_\psi H^2$ where $\overline{q_\phi} q$ and $\overline{q_\psi} q$ are in H^∞. Suppose Range S_ϕ^* is closed. If Range $S_\phi^* \supseteq$ Range S_ψ^*, then $H^2 \ominus q_\phi H^2 \supseteq H^2 \ominus q_\psi H^2$. Conversely if $H^2 \ominus q_\phi H^2 \supseteq H^2 \ominus q_\psi H^2$, then Range $S_\phi^* \supseteq H^2 \ominus q_\psi H^2 \supseteq \overline{\mathrm{Range}} S_\psi^*$. □

Theorem 8. *Let ϕ and ψ be in H^∞.*

(1) *Suppose q is a Blaschke product with simple zeros in D. Then, $\mathrm{Range}S_\phi \supseteq \mathrm{Range}S_\psi$ if and only if there exists a positive constant γ such that $\gamma|\phi(a)| \geq |\psi(a)|$ if $q(a) = 0$ and $a \in D$.*

(2) *If* $\mathrm{Range}S_\phi \supseteq \mathrm{Range}S_\psi$*, then* $Q_\psi = q_0 Q_\phi$ *and* $q_\phi = q_0 q_\psi$ *for some inner function* q_0*.*

(3) *When* $\mathrm{Range}S_\phi$ *is closed,* $\mathrm{Range}S_\phi \supseteq \mathrm{Range}S_\psi$ *if and only if* $Q_\psi = q_0 Q_\phi$ *and* $q_\phi = q_0 q_\psi$ *for some inner function* q_0*.*

Proof. (1) Range $S_\phi \supseteq \mathrm{Range}S_\psi$ if and only if $\gamma^2 S_\phi S_\phi^* \geq S_\psi S_\psi^*$ for some positive constant γ. If K_a is a reproducing kernel for $a \in D$ and $q(a) = 0$, then $\gamma^2 \langle S_\phi S_\phi^* K_a, K_a\rangle \geq \langle S_\psi S_\psi^* K_a, K_a\rangle$. Hence $\gamma|\phi(a)| \geq |\psi(a)|$ if $q(a) = 0$. The linear span of K_a with $q(a) = 0$ is dense in N and so we can prove the converse.

(2) If $\mathrm{Range}S_\phi \supseteq \mathrm{Range}S_\psi$, then $\mathrm{Ker}S_\phi^* \subseteq \mathrm{Ker}S_\psi^*$ and so by Lemma 6 $Q_\phi H^2 \supseteq Q_\psi H^2$. Hence $q = Q_\phi q_\phi = Q_\psi q_\psi$ and $q_0 = Q_\psi \bar{Q}_\phi$. Therefore $Q_\psi = q_0 Q_\phi$ and $q_\phi = q_0 q_\psi$.

(3) The 'only if' part follows from (2). Conversely if $Q_\psi = q_0 Q_\phi$ and $q_\phi = q_0 q_\psi$, then

$$\begin{aligned} Q_\phi(H^2 \ominus q_\phi H^2) &= Q_\phi\{(H^2 \ominus q_0 H^2) + q_0(H^2 \ominus q_\psi H^2)\} \\ &= Q_\phi(H^2 \ominus q_0 H^2) + Q_\psi(H^2 \ominus q_\psi H^2) \\ &\supseteq Q_\psi(H^2 \ominus q_\psi H^2). \end{aligned}$$

Now by Lemma 6 $\mathrm{Range}S_\phi = \overline{\mathrm{Range}}S_\phi \supseteq \overline{\mathrm{Range}}S_\psi \supseteq \mathrm{Range}S_\psi$. □

In Theorems 4, 7 and 8, the following Example will be interesting. In two inner function q_1 and q_2, if these do not have common inner divisors except unimodular constants, then we write $q_1 \wedge q_2 = 1$.

Example. (1) Let ϕ be a unimodular and dist $(\phi, H^\infty) < 1$. Then Range T_ϕ is closed.

(2) Let Q be an inner function and $q = Qq_0$ where q_0 is inner. Suppose $\phi = Qh$ where h is an invertible outer function in H^∞. Then $\mathrm{Range}S_\phi$ is closed.

(3) Let Q be an inner function and $q = Qq_0$ where q_0 is inner. If $\phi = Qh$ where h is outer, then $\mathrm{Range}S_\phi^*$ is closed.

(4) Let Q be an inner function and $Q \wedge q = 1$. If $\phi = Q$, then $\mathrm{Range}S_\phi^*$ is dense in N_q.

Proof. (1) It is well known.

If $q = Qq_0$, then

$$\begin{aligned} N_q &= (H^2 \ominus q_0 H^2) \oplus q_0(H^2 \ominus QH^2) \\ &= (H^2 \ominus QH^2) \oplus Q(H^2 \ominus q_0 H^2). \end{aligned}$$

(2) Range S_ϕ = Range S_Q because h is an invertible outer function in H^∞. By the above remark, $S_Q N_q = Q(H^2 \ominus q_0 H^2)$ and so $\mathrm{Range}S_\phi$ is closed.

(3) Since $\bar{Q}(H^2 \ominus QH^2) \subseteq \bar{z}\bar{H}^2$, by the above remark, $S_Q^* N_q = H^2 \ominus q_0 H^2$ and so $\mathrm{Range}S_\phi^*$ is closed.

(4) It is enough to prove $\mathrm{Ker} S_\phi = \{0\}$. If $S_\phi f = 0$, then $Qf \in qH^2$ and $f \perp qH^2$ because $f \in N_q$. This contradicts $Q \wedge q = 1$ because we may assume f is outer. □

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Takahiko Nakazi
Professor Emeritus
Hokkaido University
Sapporo 060-0810, Japan
E-mail address: tnakazi70@gmail.com