MULTIPLE SOLUTIONS FOR EQUATIONS OF $p(x)$-LAPLACE TYPE WITH NONLINEAR NEUMANN BOUNDARY CONDITION

Yun-Ho Kim and Kisoeb Park

Abstract. In this paper, we are concerned with the nonlinear elliptic equations of the $p(x)$-Laplace type

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p(x)-2}u = \lambda f(x,u) & \text{in} \quad \Omega \\ a(x,\nabla u)\frac{\partial u}{\partial n} = \lambda\theta g(x,u) & \text{on} \quad \partial\Omega, \end{cases}$$

which is subject to nonlinear Neumann boundary condition. Here the function $a(x,v)$ is of type $|v|^{p(x)-2}v$ with continuous function $p: \overline{\Omega} \to (1,\infty)$ and the functions f, g satisfy a Carathéodory condition. The main purpose of this paper is to establish the existence of at least three solutions for the above problem by applying three critical points theory due to Ricceri. Furthermore, we localize three critical points interval for the given problem as applications of the theorem introduced by Arcoya and Carmona.

1. Introduction

Recently, the study of differential equations and variational problems involving $p(x)$-growth conditions have been extensively investigated and received much attention because they can be presented as models for many physical phenomena which arouse in the study of elastic mechanics, electro-rheological fluid dynamics and image processing, *etc*. We refer the readers to [20, 25] and the references therein. In the case of $p(x)$ which becomes p-Laplacian when $p(x) \equiv p$ (a constant), there are a bunch of papers, for instance, [1, 14, 17, 19, 21] and the references therein.

In the present paper, we are concerned with multiple solutions for equations of $p(x)$-Laplace type with nonlinear Neumann boundary condition

(N) $\qquad \begin{cases} -\operatorname{div}(a(x, \nabla u)) + |u|^{p(x)-2}u = \lambda f(x, u) & \text{in} \quad \Omega \\ a(x, \nabla u)\frac{\partial u}{\partial n} = \lambda\theta g(x, u) & \text{on} \quad \partial\Omega, \end{cases}$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$, the function $a(x, v)$ is of type $|v|^{p(x)-2}v$ with continuous function $p : \overline{\Omega} \to (1, \infty)$, the functions f, g satisfy a Carathéodory condition, and λ, θ are real parameters. The existence of nontrivial solutions to nonlinear elliptic boundary value problems has been extensively studied by many researchers; see [7, 8, 13, 26, 27, 28, 31] and references therein. Motivated by the pioneer work of A. Ambrosetti and P. Rabinowitz in [1], J. Yao [28] showed the existence of nontrivial solutions for the inhomogeneous and nonlinear Neumann boundary value problems involving the $p(x)$-Laplacian; see [7] for $p(x)$-Laplace type operator. The purpose of this paper is to establish the existence of at least three solutions for problem (N) as applications of an abstract three critical points theorem [3] which is the extension of the famous result of B. Ricceri [23]. The study about the existence of at least three solutions for elliptic equations has been an interesting topic; see [2, 6, 4, 9, 18, 21, 23, 24]. J. Liu and X. Shi [18] treated the existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x), q(x))$-Laplacian. Some existence and multiplicity results for nonlinear elliptic equations of the $p(x)$-Laplacian in the whole space $\mathbb{R}^N$ have been established in [2]. It is well known that B. Ricceri's theorems in [21, 22, 23] gave no accurate information on the location and size of an interval of the parameter λ in $\mathbb{R}$ for the existence of at least three critical points. The authors in [9] localized the interval for the existence of three solutions for equations of p-Laplace type with various boundary conditions (for example, homogeneous Dirichlet and inhomogeneous Robin problems) which were motivated by the study of D. Arcoya and J. Carmona [3]. Recently, J.-H. Bae, Y.-H. Kim, and C. Zhang [4] established the existence of at least three solutions for equations of $p(x)$-Laplace type and localized a three critical points interval for this problem which was based on the works of [6, 9]. In this respect, we determine precisely the intervals of λ's for which problem (N) admits only the trivial solution and for which problem (N) admits at least two nontrivial solutions by applying the three critical points theorems given in [3].

This paper is organized as follows. We first state some basic results for the variable exponent Lebesgue-Sobolev spaces and present some properties of integral operators associated with the problem (N). Second, we observe multiple solutions for equations of $p(x)$-Laplace type with nonlinear Neumann boundary condition using abstract three critical points theory introduced by B. Ricceri [23]. And finally we determine precisely the intervals of λ's for which

problem (N) admits only the trivial solution and for which problem (N) has at least two nontrivial solutions.

2. Preliminaries

We introduce some definitions and basic properties of the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\Omega)$.

We set

$$C_+(\overline{\Omega}) = \left\{ h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 1 \right\}$$

and, for any $h \in C_+(\overline{\Omega})$, we denote

$$h_+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h_- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function,} \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $X := W^{1,p(\cdot)}(\Omega)$ is defined by

$$X = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

with the norm

$$(1) \qquad \|u\|_X = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Throughout this paper, we assume that a function $p : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω if there is a constant C_0 such that

$$|p(x) - p(y)| \le \frac{C_0}{-\log|x - y|}$$

for every $x, y \in \Omega$ with $|x - y| \le 1/2$.

Lemma 2.1 ([15]). *The space $L^{p(\cdot)}(\Omega)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \le 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Lemma 2.2 ([15]). *Denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} \, dx$$

for all $u \in L^{p(\cdot)}(\Omega)$. Then

(1) $\rho(u) > 1 \ (= 1; \ < 1)$ *if and only if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \ (= 1; \ < 1)$, *respectively;*

(2) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, *then* $\|u\|_{L^{p(\cdot)}(\Omega)}^{p_-} \le \rho(u) \le \|u\|_{L^{p(\cdot)}(\Omega)}^{p_+}$;

(3) *if* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, *then* $\|u\|_{L^{p(\cdot)}(\Omega)}^{p_+} \le \rho(u) \le \|u\|_{L^{p(\cdot)}(\Omega)}^{p_-}$.

Remark 2.3. Denote

$$\rho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) \, dx$$

for all $u \in X$. Then

(1) $\rho(u) > 1 \ (= 1; \ < 1)$ if and only if $\|u\|_X > 1 \ (= 1; \ < 1)$, respectively;
(2) if $\|u\|_X > 1$, then $\|u\|_X^{p_-} \le \rho(u) \le \|u\|_X^{p_+}$;
(3) if $\|u\|_X < 1$, then $\|u\|_X^{p_+} \le \rho(u) \le \|u\|_X^{p_-}$.

Lemma 2.4 ([10])**.** *Let* $q \in L^\infty(\Omega)$ *be such that* $1 \le p(x)q(x) \le \infty$ *for almost all* $x \in \Omega$. *If* $u \in L^{q(\cdot)}(\Omega)$ *with* $u \neq 0$, *then*

(1) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} > 1$, *then* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)}^{q_-} \le \| |u|^{q(x)} \|_{L^{p(\cdot)}(\Omega)} \le \|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)}^{q_+}$;

(2) *if* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)} < 1$, *then* $\|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)}^{q_+} \le \| |u|^{q(x)} \|_{L^{p(\cdot)}(\Omega)} \le \|u\|_{L^{p(\cdot)q(\cdot)}(\Omega)}^{q_-}$.

Lemma 2.5 ([12])**.** *Let* Ω *be an open, bounded set with Lipschitz boundary and let* $p \in C_+(\overline{\Omega})$ *with* $1 < p_- \le p_+ < \infty$. *If* $q \in C(\overline{\Omega}))$ *satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

then there is a continuous embedding

$$X \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the embedding is compact if $\inf_{x \in \Omega} (p^*(x) - q(x)) > 0$ *for all* $x \in \overline{\Omega}$.

Lemma 2.6 ([11])**.** *Let* $\Omega \subset \mathbb{R}^N, N \ge 2$ *be a bounded domain with smooth boundary. Suppose that* $p \in C_+(\overline{\Omega})$ *and* $r \in C(\partial\Omega)$ *satisfy the condition*

$$1 \le r(x) < p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x) \end{cases}$$

for all $x \in \partial\Omega$. *Then the embedding* $X \hookrightarrow L^{r(\cdot)}(\partial\Omega)$ *is compact and continuous.*

3. Existence of multiple solutions

Now we shall give the proof of the existence of at least three solutions for problem (N), by applying the abstract three critical points theory and the basic properties of the spaces $L^{p(\cdot)}(\Omega)$ and X.

Definition. We say that $u \in X$ is a weak solution of the problem (N) if

$$\int_\Omega a(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega |u|^{p(x)-2}uv \, dx = \lambda \int_\Omega f(x,u)v \, dx + \lambda\theta \int_{\partial\Omega} g(x,u)v \, dS$$

for all $v \in X$, where dS is the measure on the boundary.

We assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $J_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $J_0 = J_0(x, v)$, that is, $a(x, v) = \frac{d}{dv} J_0(x, v)$. Suppose that a and J_0 satisfy the following assumptions:

(J1) The equality
$$J_0(x, \mathbf{0}) = 0$$
holds for all $x \in \Omega$.

(J2) There are a function $b \in L^{p'(\cdot)}(\Omega)$ and a nonnegative constant b_1 such that
$$|a(x, v)| \le b(x) + b_1 |v|^{p(x)-1}$$
holds for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$.

(J3) $J_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $v \in \mathbb{R}^N$, where $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$.

(J4) There exists a positive constant c_* such that the relations
$$c_* |v|^{p(x)} \le a(x, v) \cdot v \quad \text{and} \quad c_* |v|^{p(x)} \le p_+ J_0(x, v)$$
hold for all $x \in \Omega$ and $v \in \mathbb{R}^N$.

Let us define the functional $J : X \to \mathbb{R}$ by

$$(2) \qquad J(u) = \int_\Omega J_0(x, \nabla u)\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx.$$

We define an operator $J' : X \to X^*$ by

$$(3) \qquad \langle J'(u), \varphi \rangle = \int_\Omega \langle a(x, \nabla u), \nabla \varphi \rangle\, dx + \int_\Omega |u|^{p(x)-2} u \varphi\, dx$$

for any $\varphi \in X$ where $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^* and the Euclidean scalar product on $\mathbb{R}^N$, respectively.

The fact that the operator J' is a mapping of type (S_+) plays an important role in obtaining our main results. The proof is essentially the same as the one in [16]; see also [17].

Lemma 3.1. *Assume that (J1)–(J4) hold. Then the functional $J : X \to \mathbb{R}$ is convex and weakly lower semicontinuous on X. Moreover, the operator J' is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in X as $n \to \infty$ and $\limsup_{n\to\infty} \langle J'(u_n) - J'(u), u_n - u \rangle \le 0$, then $u_n \to u$ in X as $n \to \infty$.*

Corollary 3.2. *Assume that (J1)–(J4) hold. Then the operator $J' : X \to X^*$ is strictly monotone, coercive and hemicontinuous on X. Furthermore, the operator J' is homeomorphism onto X^*.*

Proof. It is immediate that the operator J' is strictly monotone, coercive and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(J')^{-1}$ exists (see Theorem 26.A in [30]). If we apply Lemma 3.1, then the

proof of continuity of the inverse operator $(J')^{-1}$ is similar to that in the case of a constant exponent and is omitted here. $\qquad\square$

In order to deal with our main results, we need the following assumptions for f and g. Denoting $F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$, then we assume that

(H1) $p \in C_+(\overline{\Omega})$ and $1 < p_- \le p_+ < p^*(x)$ for all $x \in \Omega$.

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $\rho_1, \sigma_1 \in L^\infty(\Omega)$ such that

$$|f(x,s)| \le \rho_1(x) + \sigma_1(x)\,|s|^{\gamma_1(x)-1}$$

for all $(x,s) \in \Omega \times \mathbb{R}$, where $\gamma_1 \in C_+(\overline{\Omega})$ and $(\gamma_1)_+ < p_-$.

(F2) There exist a real number s_0 and a positive constant r_0 so small that

$$\int_{B_N(x_0,r_0)} F(x,|s_0|)\,dx > 0 \quad \text{and} \quad F(x,t) \ge 0$$

for almost all $x \in B_N(x_0,r_0) \setminus B_N(x_0,\sigma r_0)$ with $\sigma \in (0,1)$ and for all $0 \le t \le |s_0|$, where $B_N(x_0,r_0) = \{x \in \Omega : |x - x_0| \le r_0\} \subset \Omega$.

(F3) $\limsup_{s\to 0} \left(\operatorname{ess\,sup}_{x\in\Omega} \frac{|F(x,s)|}{|s|^{\kappa(x)}} \right) < +\infty$, where $\kappa \in C_+(\overline{\Omega})$ is such that $p_+ < \kappa_- \le \kappa(x) < p^*(x)$ for all $x \in \Omega$.

(G1) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and there exist two nonnegative functions $\rho_2, \sigma_2 \in L^\infty(\partial\Omega)$

$$|g(x,s)| \le \rho_2(x) + \sigma_2(x)\,|s|^{\gamma_2(x)-1}$$

for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $\gamma_2 \in C_+(\partial\Omega)$ and $(\gamma_2)_+ < p_-$.

Define the functionals $\Psi, H : X \to \mathbb{R}$ by

$$(4) \qquad \Psi(u) = -\int_\Omega F(x,u)\,dx, \quad H(u) = -\int_{\partial\Omega} G(x,u)\,dS.$$

Then it is easy to check that $\Psi, H \in C^1(X,\mathbb{R})$ and their Fréchet derivatives are

$$(5) \quad \langle \Psi'(u), v \rangle = -\int_\Omega f(x,u)v\,dx \quad \text{and} \quad \langle H'(u), v \rangle = -\int_{\partial\Omega} g(x,u)v\,dS$$

for any $u, v \in X$.

Lemma 3.3 ([23]). *Let X be a reflexive real Banach space; $I \subset \mathbb{R}$ an interval; $J : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1-functional whose derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a C^1-functional with compact derivative. In addition, let J be bounded on each bounded subset of X. Assume that*

$$\lim_{\|u\|_X \to \infty} (J(u) + \lambda\Psi(u)) = +\infty$$

for all $\lambda \in I$ and there exists $\rho \in \mathbb{R}$ such that

$$(6) \qquad \sup_{\lambda \in I} \inf_{u \in X} \left(J(u) + \lambda(\Psi(u) + \rho) \right) < \inf_{u \in X} \sup_{\lambda \in I} \left(J(u) + \lambda(\Psi(u) + \rho) \right).$$

Then there exist a nonempty open set $\Lambda \subset I$ and a positive real number $R > 0$ with the following property: for every $\lambda \in \Lambda$ and every C^1-functional $H : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation

$$J'(u) + \lambda\left(\Psi'(u) + \theta H'(u) \right) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 3.4. *Assume that* (H1), (J1)–(J4), (F1), *and* (G1) *hold. Then*

$$\lim_{\|u\|_X \to \infty} \left\{ J(u) + \lambda(\Psi(u) + \theta H(u)) \right\} = +\infty$$

for all $\lambda, \theta \in \mathbb{R}$.

Proof. For $\|u\|_X$ large enough and for all $\lambda, \theta \in \mathbb{R}$, it follows from Lemmas 2.5 and 2.6 that

$$J(u) + \lambda(\Psi(u) + \theta H(u))$$

$$= \int_\Omega J_0(x, \nabla u)\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx - \lambda \int_\Omega F(x, u)\, dx - \lambda\theta \int_{\partial\Omega} G(x, u)\, dS$$

$$\geq \frac{c_*}{p_+} \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx$$

$$\quad - |\lambda| \int_\Omega |\rho_1(x)|\, |u|\, dx - |\lambda| \int_\Omega \frac{1}{\gamma_1(x)} |\sigma_1(x)|\, |u|^{\gamma_1(x)}\, dx$$

$$\quad - |\lambda|\,|\theta| \int_{\partial\Omega} |\rho_2(x)|\, |u|\, dS - |\lambda|\,|\theta| \int_{\partial\Omega} \frac{1}{\gamma_2(x)} |\sigma_2(x)|\, |u|^{\gamma_2(x)}\, dS$$

$$\geq \frac{c_*}{p_+} \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{1}{p_+} \int_\Omega |u|^{p(x)}\, dx$$

$$\quad - |\lambda|\, \|\rho_1\|_{L^\infty(\Omega)} \|u\|_{L^1(\Omega)} - \frac{|\lambda|}{(\gamma_1)_-} \|\sigma_1\|_{L^\infty(\Omega)} \big\| |u|^{\gamma_1(\cdot)} \big\|_{L^1(\Omega)}$$

$$\quad - |\lambda|\,|\theta|\, \|\rho_2\|_{L^\infty(\partial\Omega)} \|u\|_{L^1(\partial\Omega)} - \frac{|\lambda|\,|\theta|}{(\gamma_2)_-} \|\sigma_2\|_{L^\infty(\partial\Omega)} \big\| |u|^{\gamma_2(\cdot)} \big\|_{L^1(\partial\Omega)}$$

$$\geq \frac{\min\{c_*, 1\}}{p_+} \left(\int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx \right)$$

$$\quad - |\lambda|\, C_1 \|u\|_X - \frac{|\lambda|\, C_2}{(\gamma_1)_-} \|u\|_{L^{\gamma_1(\cdot)}(\Omega)}^{(\gamma_1)_+} - |\lambda|\,|\theta|\, C_3 \|u\|_X - \frac{|\lambda|\,|\theta|\, C_4}{(\gamma_2)_-} \|u\|_{L^{\gamma_2(\cdot)}(\partial\Omega)}^{(\gamma_2)_+}$$

$$\geq \frac{\min\{c_*, 1\}}{p_+} \|u\|_X^{p_-} - |\lambda|\, C_1 \|u\|_X - \frac{|\lambda|\, C_5}{(\gamma_1)_-} \|u\|_X^{(\gamma_1)_+}$$

$$\quad - |\lambda|\,|\theta|\, C_3 \|u\|_X - \frac{|\lambda|\,|\theta|\, C_6}{(\gamma_2)_-} \|u\|_X^{(\gamma_2)_+}$$

for some positive constants $C_i(i = 1, \ldots, 6)$. Since $p_- > (\gamma_1)_+ > 1$ and $p_- > (\gamma_2)_+ > 1$, we deduce

$$\lim_{\|u\|_X \to \infty} \{J(u) + \lambda(\Psi(u) + \theta H(u))\} = +\infty$$

for all $\lambda, \theta \in \mathbb{R}$. $\square$

The following lemma is crucial to obtain the assumption (6) of Lemma 3.3.

Lemma 3.5 ([21]). *Let X be a nonempty set and J, Ψ two real functionals on X. Assume that there are $\mu > 0$ and $u_0, u_1 \in X$ such that*

$$J(u_0) = -\Psi(u_0) = 0, \quad J(u_1) > \mu,$$

(7) $$\sup_{u \in J^{-1}((-\infty, \mu])} -\Psi(u) < \mu \frac{-\Psi(u_1)}{J(u_1)}.$$

Then, for each ρ satisfying

$$\sup_{u \in J^{-1}((-\infty, \mu])} -\Psi(u) < \rho < \mu \frac{-\Psi(u_1)}{J(u_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} (J(u) + \lambda(\rho + \Psi(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (J(u) + \lambda(\rho + \Psi(u))).$$

The following consequence for the existence of at least three solutions for problem (N) can be established by applying Lemma 3.3.

Theorem 3.6. *Assume that* (J1)–(J4), (H1), (F1)–(F3), *and* (G1) *hold. Then there exist a nonempty open set $\Lambda \subset [0, +\infty)$ and a positive real number $R > 0$ with the following property: for every $\lambda \in \Lambda$ and every Carathéodory function g satisfying the condition* (G1), *there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation* (N) *has at least three solutions in X whose norms are less then R.*

Proof. By Lemma 3.1, the functional $J : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous C^1-functional. Moreover, it is bounded on each bounded subset of X. Using Corollary 3.2, the operator J' is homeomorphism onto X^*, that is, there exists a continuous inverse operator $(J')^{-1} : X^* \to X$. Moreover, with the aid of Lemmas 2.5 and 2.6, the modification of the proof of Proposition 3.1 in [7] yields that the operators $\Psi', H' : X \to X^*$ are compact. Applying Lemma 3.4 when the parameter θ is zero, we know

$$\lim_{\|u\|_X \to \infty} (J(u) + \lambda \Psi(u)) = +\infty$$

for all $u \in X$ and all $\lambda \in \mathbb{R}$.

To check all assumptions in Lemma 3.3, we verify the assumption (6). Let $s_0 \neq 0$ be from (F2). For $\varrho \in (0,1)$, define

$$(8) \qquad u_\varrho(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B_N(x_0, r_0) \\ |s_0| & \text{if } x \in B_N(x_0, \varrho r_0) \\ \frac{|s_0|}{r_0(1-\varrho)}(r_0 - |x - x_0|) & \text{if } x \in B_N(x_0, r_0) \setminus B_N(x_0, \varrho r_0). \end{cases}$$

It is clear that $0 \leq u_\varrho(x) \leq |s_0|$ for all $x \in \Omega$, and so $u_\varrho \in X$. Moreover, the fact that (see Theorem 2.8 of [19]) $L^{p(x)}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ implies

$$\|u_\varrho\|_X^{\alpha_0} \geq \|\nabla u_\varrho\|_{L^{p(x)}(\Omega)}^{\alpha_0} \geq C_7 \int_\Omega |\nabla u_\varrho|^{p_-} \, dx = \frac{C_7 |s_0|^{p_-} (1 - \varrho^N)}{(1-\varrho)^{p_-}} r_0^{N-p_-} \omega_N > 0$$

for a positive constant C_7, where α_0 is either p_+ or p_- and ω_N is the volume of $B_N(0,1)$. Also, by using assumption (F2), we get

$$-\Psi(u_\varrho) = \int_{B_N(x_0, \varrho r_0)} F(x, |s_0|) \, dx$$

$$+ \int_{B_N(x_0, r_0) \setminus B_N(x_0, \varrho r_0)} F(x, \frac{|s_0|}{r_0(1-\varrho)}(r_0 - |x - x_0|)) \, dx > 0.$$

By condition (F3), there exist positive constants $\eta \in (0,1]$ and C_8 such that

$$(9) \qquad F(x,s) < C_8 |s|^{\kappa(x)} < C_8 |s|^{\kappa_-}$$

for almost all $x \in \Omega$ and for all $s \in [-\eta, \eta]$. Let us consider two positive constants M_1 and M_2 given by

$$M_1 = \sup_{|s|>1} \frac{C(|s| + |s|^{\gamma_+})}{|s|^{\kappa_-}} \quad \text{and} \quad M_2 = \sup_{\eta < |s| < 1} \frac{C(|s| + |s|^{\gamma_-})}{|s|^{\kappa_-}}$$

for some positive constant C. Then it follows from (9) and (F1) that

$$F(x,s) < M |s|^{\kappa_-}$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$, where $M = \max\{C_8, M_1, M_2\}$. Fix a real number μ such that $0 < \mu < 1$. When $\alpha_1 \|u\|_X^{p_+} \leq \mu < 1$, where $\alpha_1 = \min\{c_*, 1\}/p_+$ and c_* is the positive constant from (J4), then by Lemma 2.2, we have

$$(10) \qquad -\Psi(u) = \int_\Omega F(x,u) \, dx < M \int_\Omega |u|^{\kappa_-} \, dx \leq C_9 \|u\|_X^{\kappa_-} \leq C_{10} \mu^{\frac{\kappa_-}{p_+}}$$

for some positive constants C_9 and C_{10}. Since $\kappa_- > p_+$, the relation (10) implies that

$$(11) \qquad \lim_{\mu \to 0+} \frac{\sup_{\alpha_1 \|u\|_X^{p_+} \leq \mu} -\Psi(u)}{\mu} = 0.$$

Let us check the assumption (7) in Lemma 3.5. Fix a real number μ_0 such that

$$0 < \mu < \mu_0 < \alpha_1 \min\left\{\|u_\varrho\|_X^{p_+}, \|u_\varrho\|_X^{p_-}, 1\right\} \leq \alpha_1,$$

where u_ϱ was defined in (8). By Lemma 2.2 and (J5), we have

$$J(u_\varrho) = \int_\Omega J_0(x, \nabla u_\varrho)\, dx + \int_\Omega \frac{1}{p(x)} |u_\varrho|^{p(x)}\, dx$$
$$\geq \int_\Omega \frac{c_*}{p_+} |\nabla u_\varrho|^{p(x)}\, dx + \int_\Omega \frac{1}{p_+} |u_\varrho|^{p(x)}\, dx \geq \alpha_1 \|u_\varrho\|_X^{p_+} \geq \mu_0 > \mu$$

for $\|u_\varrho\|_X < 1$ and

$$J(u_\varrho) = \int_\Omega J_0(x, \nabla u_\varrho)\, dx + \int_\Omega \frac{1}{p(x)} |u_\varrho|^{p(x)}\, dx$$
$$\geq \int_\Omega \frac{c_*}{p_+} |\nabla u_\varrho|^{p(x)}\, dx + \int_\Omega \frac{1}{p_+} |u_\varrho|^{p(x)}\, dx \geq \alpha_1 \|u_\varrho\|_X^{p_-} \geq \mu_0 > \mu$$

for $\|u_\varrho\|_X > 1$. From (11), we obtain

$$(12) \qquad \sup_{\alpha_1 \|u\|_X^{p_+} \leq \mu} -\Psi(u) \leq \frac{\mu}{2} \frac{-\Psi(u_\varrho)}{J(u_\varrho)} < \mu \frac{-\Psi(u_\varrho)}{J(u_\sigma)}.$$

For any $u \in J^{-1}((-\infty, \mu])$, we obtain that $J(u) \leq \mu$ and so

$$\frac{c_*}{p_+} \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega \frac{1}{p_+} |u|^{p(x)}\, dx$$
$$\leq \int_\Omega J_0(x, \nabla u)\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx = J(u) \leq \mu.$$

Hence we deduce

$$\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \leq \frac{1}{\alpha_1} \mu < \frac{1}{\alpha_1} \mu_0 < 1.$$

This inequality implies that $\|u\|_X < 1$. It follows that

$$\alpha_1 \|u\|_X^{p_+} < \int_\Omega J_0(x, \nabla u)\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx = J(u) \leq \mu.$$

So we can get

$$J^{-1}((-\infty, \mu]) \subset \left\{ u \in X : \alpha_1 \|u\|_X^{p_+} \leq \mu \right\}.$$

Then

$$\sup_{u \in J^{-1}((-\infty, \mu])} -\Psi(u) \leq \sup_{\alpha_1 \|u\|_X^{p_+} \leq \mu} -\Psi(u) < \mu \frac{-\Psi(u_\varrho)}{J(u_\varrho)},$$

that is,

$$\sup_{u \in J^{-1}((-\infty, \mu])} -\Psi(u) < \mu \frac{-\Psi(u_\varrho)}{J(u_\varrho)}.$$

Thus we can choose $\mu > 0, u_0 = 0$, and $u_1 = u_\varrho$ such that relations $J(u_\varrho) \leq \mu$ and (7) are satisfied. Also there exists ρ such that

$$\sup_{u \in J^{-1}((-\infty, \mu])} -\Psi(u) < \rho < \mu \frac{-\Psi(u_\varrho)}{J(u_\varrho)}.$$

Set $I = [0, +\infty)$. According to Lemma 3.5, we obtain that

$$\sup_{\lambda \geq 0} \inf_{u \in X} (J(u) + \lambda(\Psi(u) + \rho)) < \inf_{u \in X} \sup_{\lambda \geq 0} (J(u) + \lambda(\Psi(u) + \rho)).$$

Therefore, J and Ψ satisfy all the assumptions of Lemma 3.3. This completes the proof. $\qquad\square$

Theorem 3.6 gives no further information on the size and location of the open set Λ. Hence we will investigate the localization of the interval for the existence of at least three solutions for problem (N) by applying the three critical points theorems given in [9]. To do this, we consider the following eigenvalue problem:

$$\text{(E)} \quad \begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda m(x)\, |u|^{p(x)-2}\, u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Under some conditions, we show the positivity of the infimum of all eigenvalues for problem (E). From the analogous argument as in the proof of Theorem 4.1 in [9], we get the following result.

Proposition 3.7. *Assume that* (H1) *holds. Moreover, suppose that*

(H2) $m \in L^\infty(\Omega)$ *and* $m(x) > 0$ *for almost all* $x \in \Omega$.

Denote the quantity

$$(13) \qquad \lambda_* = \inf_{u \in X \backslash \{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx}{\int_\Omega m(x)\, |u|^{p(x)}\, dx}.$$

Then λ_* is a positive eigenvalue of problem (E), *that is there is $u_1 \in X$ with $\int_\Omega m(x)|u_1|^{p(x)}\, dx = 1$ such that realizes the infimum in* (13) *and represents an eigenfunction for λ_*. In particular,*

$$(14) \qquad \lambda_* \int_\Omega m(x)\, |u|^{p(x)}\, dx \leq \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx$$

for every $u \in X$.

Proof. In view of Lemma 2.5 with the assumption (H2), we get $\lambda_* > 0$. Denote the functional $\tilde{J}_0, \tilde{\Psi}_0 : X \to \mathbb{R}$ by

$$\tilde{J}_0(u) = \int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx \quad \text{and} \quad \tilde{\Psi}_0(u) = \int_\Omega m(x)|u|^{p(x)}\, dx$$

for any $u \in X$. Then these functionals are continuously Gáteaux differentiable, convex in X, and obviously $\tilde{J}_0'(0) = \tilde{\Psi}_0'(0) = 0$. Moreover, $\tilde{J}_0'(u) = 0$ implies $u = 0$. By Lemma 3.1, $\tilde{J}_0$ is weakly lower semicontinuous on X. From the convexity of $\tilde{\Psi}_0$, we deduce that $\tilde{\Psi}_0$ is weakly lower semicontinuous on X. Note that any C^1 functional on X with compact derivative is sequentially weakly continuous on X (see Corollary 41.9 in [29]). As seen in the proof of Theorem 3.6, $\tilde{\Psi}_0$ is sequentially weakly continuous on X. Since $\tilde{J}_0$ is coercive in X, it follows from an easy contradiction argument that $\tilde{J}_0$ is coercive in $\{u \in X : \tilde{\Psi}_0(u) \leq 1\}$. Consequently all the assumptions of Theorem 6.3.2 in

[5] are verified and thus λ_* is attained in $\{u \in X : \tilde{\Psi}_0(u) = 1\}$. In other words, there is an element u in X with $\int_\Omega m(x)|u|^{p(x)}\, dx = 1$ such that realizes the infimum in (13) and represents an eigenfunction for (E) corresponding to λ_*. Therefore (14) holds. $\square$

In addition, we assume that

(F4) $\limsup_{s\to 0} \frac{|f(x,s)|}{m(x)|s|^{\xi_1(x)-1}} < +\infty$ uniformly for almost all $x \in \Omega$, where
$\xi_1 \in C_+(\overline{\Omega})$ with $p(x) < \xi_1(x) < p^*(x)$ for all $x \in \Omega$.

(G2) $\limsup_{s\to 0} \frac{|g(x,s)|}{|s|^{\xi_2(x)-1}} < +\infty$ uniformly for almost all $x \in \partial\Omega$, where
$\xi_2 \in C_+(\partial\Omega)$ with $p(x) < \xi_2(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

Let us introduce two functions

$$\chi_1(r) = \inf_{u \in \Psi^{-1}((-\infty,r))} \frac{\inf_{v\in\Psi^{-1}(r)} J(v) - J(u)}{\Psi(u) - r},$$

$$\chi_2(r) = \sup_{u \in \Psi^{-1}((r,+\infty))} \frac{\inf_{v\in\Psi^{-1}(r)} J(v) - J(u)}{\Psi(u) - r}$$

for every $r \in (\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u))$. Denote the crucial values

$$\mathcal{C}_f = \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|f(x,s)|}{m(x)\,|s|^{p(x)-1}} \quad \text{and} \quad \mathcal{C}_g = \operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|g(x,s)|}{|s|^{p(x)-1}}.$$

Then the same arguments in [9] imply that $\mathcal{C}_f$ and $\mathcal{C}_g$ are well defined, positive constants, and furthermore the following relations hold:

$$(15) \qquad \operatorname*{ess\,sup}_{s\neq 0, x\in\Omega} \frac{|F(x,s)|}{m(x)\,|s|^{p(x)}} = \frac{\mathcal{C}_f}{p_-} \quad \text{and} \quad \operatorname*{ess\,sup}_{s\neq 0, x\in\partial\Omega} \frac{|G(x,s)|}{|s|^{p(x)}} = \frac{\mathcal{C}_g}{p_-}.$$

The next result represents the differentiable version of the Arcoya and Carmona; see Theorem 3.10 in [3].

Lemma 3.8. *Let J, Ψ be two functionals on X which are weakly lower semi-continuous and continuously Gâteaux differentiable in X. Let Ψ be nonconstant and H be continuously Gâteaux differentiable with compact derivative H'. Let also $J' : X \to X^*$ be a mapping of type (S_+) and Ψ' be a compact operator. Assume that there exist an interval $I \subset \mathbb{R}$ and a number $\tau > 0$ such that for every $\lambda \in I$ and every $\theta \in [-\tau, \tau]$ the functional $I_{\lambda,\theta} = J + \lambda(\Psi + \theta H)$ is coercive in X. If there exists*

$$(16) \qquad r \in \left(\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u) \right) \quad \text{such that} \quad \chi_1(r) < \chi_2(r)$$

and $(\chi_1(r), \chi_2(r)) \cap I \neq \emptyset$, then for every compact interval $[a,b]$ with $[a,b] \subset (\chi_1(r), \chi_2(r)) \cap I$, there exists $\gamma \in (0, \tau)$ with $|\theta| < \gamma$ such that the functional $I_{\lambda,\theta}$ admits at least three critical points for every $\lambda \in [a,b]$.

By applying Lemma 3.8, we can obtain the following assertion.

Theorem 3.9. *Assume* (J1)–(J4), (H1)–(H3), (F1)–(F2), *and* (G1)–(G2) *hold. Then we have*

(i) *For every $\theta \in \mathbb{R}$, there exists $\ell_* = \min\{1, c_*\}\lambda_*/(\mathcal{C}_f + C_{11}\lambda_* |\theta| \mathcal{C}_g)$ such that problem* (N) *has only the trivial solution for all $\lambda \in [0, \ell_*)$, where c_* is a positive constant from* (J4), *C_{11} is a positive constant, and λ_* is a positive real number in* (13).

(ii) *If furthermore f satisfies* (F4), *then for some positive constant ℓ^* with $\ell^* \geq \ell_*$ and for any compact interval $[a, b]$ with $[a, b] \subset (\ell^*, +\infty)$, there exists $\tau > 0$ such that problem* (N) *has at least two nontrivial solutions for every $\lambda \in [a, b]$ and $\theta \in (-\tau, \tau)$.*

Proof. To apply Lemma 3.8, let us denote the operators J, Ψ, and H by (2) and (4), respectively. Under the assumptions (J1)–(J4), (H1), (F1)–(F2), and (G1), all of the assumptions in Lemma 3.8 except the condition (16) are satisfied.

Now we prove the assertion (i). Let $u \in X$ be a nontrivial weak solution of problem (N). Then it is clear that

$$\int_\Omega a(x, \nabla u) \cdot \nabla v \, dx + \int_\Omega |u|^{p(x)-2} uv \, dx = \lambda \int_\Omega f(x, u)v \, dx + \lambda\theta \int_{\partial\Omega} g(x, u)v \, dS$$

for all $v \in X$. If we put $v = u$, then it follows from (J4), (14), and the definitions of $\mathcal{C}_f$ and $\mathcal{C}_g$ that

$$\min\{1, c_*\}\lambda_* \left(\int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{p(x)} \, dx \right)$$

$$\leq \lambda_* \left(\int_\Omega a(x, \nabla u) \cdot \nabla u \, dx + \int_\Omega |u|^{p(x)} \, dx \right)$$

$$= \lambda_* \lambda \left(\int_\Omega f(x, u)u \, dx + \theta \int_{\partial\Omega} g(x, u)u \, dS \right)$$

$$\leq \lambda_* \lambda \left(\int_\Omega \frac{f(x, u)}{m(x) |u|^{p(x)-1}} m(x) |u|^{p(x)} \, dx + \theta \int_{\partial\Omega} \frac{g(x, u)}{|u|^{p(x)-1}} |u|^{p(x)} \, dS \right)$$

$$\leq \lambda \mathcal{C}_f \left(\int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{p(x)} \, dx \right) + \lambda_* \lambda |\theta| \mathcal{C}_g \int_{\partial\Omega} |u|^{p(x)} \, dS$$

$$\leq \lambda(\mathcal{C}_f + C_{11}\lambda_* |\theta| \mathcal{C}_g) \left(\int_\Omega |\nabla u|^{p(x)} \, dx + \int_\Omega |u|^{p(x)} \, dx \right).$$

Thus if u is a nontrivial weak solution of problem (N), then necessarily $\lambda \geq \ell_* = \min\{1, c_*\}\lambda_*/(\mathcal{C}_f + C_{11}\lambda_* |\theta| \mathcal{C}_g)$ for C_{11} is a positive constant, as claimed.

Next, we show the assertion (ii). As shown in the proof of Theorem 3.6, there exists u_ϱ in X, which is defined in (8), such that $\Psi(u_\varrho) < 0$. Then the crucial number

$$\ell^* = \chi_1(0) = \inf_{u \in \Psi^{-1}((-\infty, 0))} \left(-\frac{J(u)}{\Psi(u)} \right)$$

is well defined. Let u be in X with $u \not\equiv 0$. From the assumption (J4) and relation (15), we obtain

$$
\begin{aligned}
\frac{J(u)}{|\Psi(u)|} &= \frac{\int_\Omega J_0(x, \nabla u)\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}{\int_\Omega F(x, u)\, dx} \\
&\geq \frac{\frac{c_*}{p_+} \int_\Omega |\nabla u|^{p(x)}\, dx + \frac{1}{p_+} \int_\Omega |u|^{p(x)}\, dx}{\int_\Omega \frac{|F(x,u)|}{m(x)|u|^{p(x)}} m(x) |u|^{p(x)}\, dx} \\
&\geq \frac{\frac{\min\{1,c_*\}}{p_+} \left(\int_\Omega |\nabla u|^{p(x)}\, dx + \int_\Omega |u|^{p(x)}\, dx \right)}{\frac{\mathcal{C}_f}{p_-} \int_\Omega m(x) |u|^{p(x)}\, dx} \\
&\geq \frac{\min\{1, c_*\}p_-}{\mathcal{C}_f p_+} \lambda_* \geq \frac{\min\{1, c_*\}\lambda_*}{\mathcal{C}_f} \\
&\geq \frac{\min\{1, c_*\}\lambda_*}{(\mathcal{C}_f + C_{11}\lambda_* |\theta| \mathcal{C}_g)} = \ell_*.
\end{aligned}
$$

Hence we have $\ell^* \geq \ell_*$. Now we claim that there exists a real number r satisfying the condition (16). For any $u \in \Psi^{-1}((-\infty, 0))$, we deduce that

$$
\begin{aligned}
\chi_1(r) &= \inf_{u \in \Psi^{-1}((-\infty, r))} \frac{\inf_{v \in \Psi^{-1}(r)} J(v) - J(u)}{\Psi(u) - r} \\
&\leq \frac{\inf_{v \in \Psi^{-1}(r)} J(v) - J(u)}{\Psi(u) - r} \leq \frac{J(u)}{r - \Psi(u)}
\end{aligned}
$$

for all $r \in (\Psi(u), 0)$. It implies that

$$
\limsup_{r \to 0-} \chi_1(r) \leq -\frac{J(u)}{\Psi(u)}
$$

for all $u \in \Psi^{-1}((-\infty, 0))$. Hence we have

$$
\limsup_{r \to 0-} \chi_1(r) \leq \chi_1(0) = \ell^*.
$$

By the assumption (F4), there exists a positive real number M_* such that

$$
(17) \qquad\qquad |F(x, s)| \leq M_* m(x) |s|^{\xi_1(x)}
$$

for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$. In fact, denote

$$
M_3 = \limsup_{s \to 0} \frac{|F(x, s)|}{m(x) |s|^{\xi_1(x)}}.
$$

Then there exists $\delta > 0$ such that $|F(x, s)| \leq (M_3 + 1)m(x)|s|^{\xi_1(x)}$ for almost all $x \in \Omega$ and for all $s \in \mathbb{R}$ with $|s| < \delta$. Let s be fixed with $|s| \geq \delta$. It follows from (15) that

$$
|F(x, s)| \leq \frac{\mathcal{C}_f}{p_-} |s|^{p(x) - \xi_1(x)} m(x) |s|^{\xi_1(x)}
$$

$$\leq \frac{\mathcal{C}_f(\delta^{p_- -(\xi_1)_+} + \delta^{p_+ -(\xi_1)_-})}{p_-} m(x)\, |s|^{\xi_1(x)}$$

for almost all $x \in \Omega$. Hence the relation (17) holds, where

$$M_* = \max\left\{ M_3 + 1, \mathcal{C}_f(\delta^{p_- -(\xi_1)_+} + \delta^{p_+ -(\xi_1)_-})/p_- \right\}.$$

Then the relation (17) implies that

$$|\Psi(u)| \leq \int_\Omega M_* m(x)\, |u|^{\xi_1(x)}\, dx \leq 2C_{12} M_* \|m\|_{L^\infty(\Omega)} \|u\|_X^\alpha$$

for a positive constant C_{12} and for all $u \in X$, where α is either $(\xi_1)_+$ or $(\xi_1)_-$. If $r < 0$ and $v \in \Psi^{-1}(r)$, then it follows from the assumption (J5) that

$$\begin{aligned}
r = \Psi(v) &\geq -2C_{11} M_* \|m\|_{L^\infty(\Omega)} \|v\|_X^\alpha \\
&\geq -2C_{11} M_* \|m\|_{L^\infty(\Omega)} \left(\max\{1, p_+/c_*\} J(v)\right)^{\frac{\alpha}{\beta}},
\end{aligned}$$

where β is either p_+ or p_-. Since $u = 0 \in \Psi^{-1}((r, +\infty))$, we get

$$\chi_2(r) \geq \frac{1}{|r|} \inf_{v \in \Psi^{-1}(r)} J(v) \geq \frac{|r|^{\frac{\beta}{\alpha}-1}}{(2C_{11}M_*)^{\frac{\beta}{\alpha}} \|m\|_{L^\infty(\Omega)}^{\frac{\beta}{\alpha}}} \frac{1}{\max\{1, p_+/c_*\}},$$

and then $\lim_{r \to 0-} \chi_2(r) = +\infty$ because $\alpha > \beta$. Then we deduce that

$$\limsup_{r \to 0-} \chi_1(r) \leq \chi_1(0) = \ell^* < \lim_{r \to 0-} \chi_2(r) = +\infty.$$

This confirms that for all integers $n \geq n^* = 2 + [\ell^*]$, there exists a negative sequence $\{r_n\}$ such that $r_n \to 0$ as $n \to \infty$ with $\chi_1(r_n) < \ell^* + 1/n < n < \chi_2(r_n)$. By Lemma 3.4, we set $I = \mathbb{R}$. In conclusion, since $u \equiv 0$ is a critical point of $I_{\lambda,\theta}$, according to Lemma 3.8, for every compact interval $[a, b]$ with

$$[a, b] \subset (\ell^*, +\infty) = \bigcup_{n=n^*}^{\infty} \left[\ell^* + \frac{1}{n},\ n\right] \subset \bigcup_{n=n^*}^{\infty} (\chi_1(r_n),\ \chi_2(r_n)),$$

there exists $\tau > 0$ such that problem (N) admits at least two nontrivial solutions for all $\lambda \in [a, b]$ and $\theta \in (-\tau, \tau)$. This completes the proof. $\square$

References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), no. 4, 349–381.

[2] S. Aouaoui, *On some degenerate quasilinear equations involving variable exponents*, Nonlinear Anal. **75** (2012), no. 4, 1843–1858.

[3] D. Arcoya and J. Carmona, *A nondifferentiable extension of a theorem of Pucci and Serrin and applications*, J. Differential Equations **235** (2007), no. 2, 683–700.

[4] J.-H. Bae, Y.-H. Kim, and C. Zhang, *Multiple solutions for the equations involving nonhomogeneous operators of $p(x)$-Laplace type in $\mathbb{R}^N$*, submitted.

[5] M. S. Berger, *Nonlinearity and Functional Analysis*: *Lectures on Nonlinear Problems in Mathematical Analysis*, Academic Press, 1977.

[6] G. Bonanno and A. Chinni, *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, J. Math. Anal. Appl. **418** (2014), no. 2, 812–827.

[7] M.-M. Boureanu and F. Preda, *Infinitely many solutions for elliptic problems with variable exponent and nonlinear boundary conditions*, Nonlinear Differential Equations Appl. **19** (2012), no. 2, 235–251.

[8] M.-M. Boureanu and D.-N. Udrea, *Existence and multiplicity result for elliptic problems with $p(\cdot)$-growth conditions*, Nonlinear Anal. Real World Appl. **14** (2013), no. 4, 1829–1844.

[9] F. Colasuonno, P. Pucci, and C. Varga, *Multiple solutions for an eigenvalue problem involving p-Laplacian type operators*, Nonlinear Anal. **75** (2012), no. 12, 4496–4512.

[10] D. E. Edmunds and J. Rákosník, *Sobolev embedding with variable exponent*, Studia Math. **143** (2000), no. 3, 267–293.

[11] X. Fan, *Boundary trace embedding theorems for variable exponent Sobolev spaces*, J. Math. Anal. Appl. **339** (2008), no. 2, 1395–1412.

[12] X. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl. **262** (2001), no. 2, 749–760.

[13] X. Fan and Q. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), no. 8, 1843–1852.

[14] X. Fan, Q. Zhang, and D. Zhao, *Eigenvalue of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302** (2005), no. 2, 306–317.

[15] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446.

[16] V. K. Le, *On a sub-supersolution method for variational inequalities with Leray-Lions operators in variable exponent spaces*, Nonlinear Anal. **71** (2009), no. 7–8, 3305–3321.

[17] S. D. Lee, K. Park, and Y.-H. Kim, *Existence and multiplicity of solutions for equations involving nonhomogeneous operators of $p(x)$-Laplace type in $\mathbb{R}^N$*, Bound. Value Probl. **261** (2014), 1–17.

[18] J. Liu and X. Shi, *Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x), q(x))$-Laplacian*, Nonlinear Anal. **71** (2009), no. 1–2, 550–557.

[19] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), no. 4, 592–618.

[20] K. R. Rajagopal and M. Růžička, *Mathematical modeling of electrorheological materials*, Contin. Mech. Thermodyn. **13** (2001), no. 1, 59–78.

[21] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems*, Math. Comput. Modelling **32** (2000), no. 11–13, 1485–1494.

[22] ———, *On a three critical points theorem*, Arch. Math. (Basel) **75** (2000), no. 3, 220–226.

[23] ———, *A three critical points theorem revisited*, Nonlinear Anal. **70** (2009), no. 9, 3084–3089.

[24] ———, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), no. 9, 4151–4157.

[25] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics, **1748**, Springer, Berlin, 2000.

[26] I. Sim and Y.-H. Kim, *Existence of solutions and positivity of the infimum eigenvalue for degenerate elliptic equations with variable exponents*, Discrete Contin. Dyn. Syst. Supplement **2013** (2013), 695–707.

[27] P. Winkert, *Multiplicity results for a class of elliptic problems with nonlinear boundary condition*, Commun. Pure Appl. Anal. **12** (2013), no. 2, 785–802.

[28] J. Yao, *Solutions for Neumann boundary value problems involving $p(x)$-Laplace operators*, Nonlinear Anal. **68** (2008), no. 5, 1271–1283.

[29] E. Zeidler, *Nonlinear Functional Analysis and its Applications III*, Springer, New York, 1985.
[30] ———, *Nonlinear Functional Analysis and its Applications II/B*, Springer, New York, 1990.
[31] J.-H. Zhao and P.-H. Zhao, *Existence of infinitely many weak solutions for the p-Laplacian with nonlinear boundary conditions*, Nonlinear Anal. **69** (2008), no. 4, 1343–1355.

Yun-Ho Kim
Department of Mathematics Education
Sangmyung University
Seoul 110-743, Korea
E-mail address: kyh1213@smu.ac.kr

Kisoeb Park
Department of Mathematics
Incheon National University
Incheon 406-772, Korea
E-mail address: kisoeb@gmail.com