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RIGIDITY OF IMMERSED SUBMANIFOLDS IN A HYPERBOLIC SPACE

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Abstract. Let $M^n, 2 \leq n \leq 6$ be a complete noncompact hypersurface immersed in $\mathbb{H}^{n+1}$. We show that there exist two certain positive constants $0 < \delta \leq 1$, and β depending only on δ and the first eigenvalue $\lambda_1(M)$ of Laplacian such that if M satisfies a (δ-SC) condition and $\lambda_1(M)$ has a lower bound then $H^1(L^2(M)) = 0$. Excepting these two conditions, there is no more additional condition on the curvature.

1. Introduction

It is well-known that the structures of ends or the number of ends of a noncompact immersed submanifold in a Riemannian manifold is related to the space of bounded harmonic functions with finite energy (see [1, 11, 12]). In fact, Li and Tam, in [11], proved that the number of non-parabolic ends of any complete Riemannian manifold is bounded by the dimension of $H^1(L^2(M))$, here we denote by $H^1(L^2(M))$ the space of bounded harmonic functions with finite energy. Due to their result, if the space $H^1(L^2(M))$ is trivial, then the submanifold has at most one non-parabolic end. Therefore, it is very interesting to study vanishing property of $H^1(L^2(M))$. There are several work have been done in this direction. For example, in [13], Lei Ni proved that if $M^n, n \geq 3$ is a complete minimal immersed hypersurface in $\mathbb{R}^{n+1}$, then M does not admit any non-trivial L^2 harmonic one-form, consequently, M has only one end. When the ambient space N is a hyperbolic space, Seo [14] proved that there are non L^2 harmonic one form on a complete super stable minimal hypersurface in a hyperbolic space if the first eigenvalue $\lambda_1(M)$ of Laplacian is bounded from below by a certain positive number depending only on the dimension of M. Later, Fu and Yang [7] improved the result of Seo by giving a better lower bound of $\lambda_1(M)$. Recently, in [9], Kim and Yun studied complete oriented noncompact hypersurface M^n in a complete Riemannian manifold of nonnegative sectional curvature. They defined a (SC) condition on M and proved that if M satisfies

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the (SC) condition and $2 \le n \le 4$, then there is no non-trivial L^2 harmonic one forms on M. It is important to note that in [9], the authors did not assume the minimality of such a hypersurface nor the constant mean curvature condition. Finally, in [5], Seo and the author investigate complete hypersurfaces immersed in $\mathbb{R}^{n+1}$ and improve the results in [9].

In this paper, motivated by [5, 9], we consider a complete noncompact immersed hypersurface in a hyperbolic space. We will not require the minimality of such a hypersurface nor the constant mean curvature condition in our research. Now, in order to establish our result, first we give a definition. Let M^n be an immersed hypersuface in $\mathbb{H}^{n+1}$. For a constant $0 < \delta \le 1$, we say that M satisfies the (δ-SC) condition if for any function $\phi \in \mathcal{C}_0^1(M)$

$$\delta \int_M (-n + |A|^2)\phi^2 \le \int_M |\nabla \phi|^2, \tag{1.1}$$

where A is the second fundamental form of M. Note that if $\delta = 1$, then the condition (1.1) means the index of the operator $\Delta + (-n + |A|^2)$ is zero (see [7]). In this case, we also say that M satisfies a (SC) condition or M is stable. Now, we state our main theorem.

Theorem 1.1. *Let $2 \le n \le 6$. Let M^n be a complete hypersurface immersed in a hyperbolic space $\mathbb{H}^{n+1}$. Suppose that M satisfies (SC) condition and*

$$\lambda_1(M) > \frac{2n - (n-1)^{3/2}}{(n + 2\sqrt{n-1})(n-1)^{3/2}},$$

then $\mathbb{H}^1(L^2(M)) = 0$ and M has at most one nonparabolic end.

The paper is organized as follows. In Section 2, we introduce an auxiliary lemma. Then, we prove the main Theorem 1.1. Finally, in Section 3, we give a sufficient condition to ensure a δ-SC property on immersed hypersurfaces.

2. Immersed submanifolds with positive spectrum

In this section, we will consider a complete hypersurface of lower dimension immersed in a hyperbolic space. To begin with, we first prove the following lemma.

Lemma 2.1. *Let M^n be a complete immersed submanifold in $\mathbb{H}^{n+p}$. Then*

$$Ric_M \ge -(n-1) - \frac{\sqrt{n-1}}{2}|A|^2. \tag{2.1}$$

Proof. By [10], it is well-known that

$$\begin{aligned}\mathrm{Ric_M} \ge &- (n-1) - \frac{n-1}{n}|A|^2 \\ &+ \frac{1}{n^2}\left\{2(n-1)|H|^2 - (n-2)\sqrt{n-1}|H|\sqrt{n|A|^2 - |H|^2}\right\}.\end{aligned} \tag{2.2}$$

Claim: If $b := \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}$. Then we have

$$2(n-1)|H|^2 - (n-2)\sqrt{n-1}|H|\sqrt{n|A|^2 - |H|^2} \geq -bn^2|A|^2. \tag{2.3}$$

Suppose that the claim is proved, then by (2.2), we have

$$\begin{aligned} Ric_M &\geq -(n-1) - \left\{ \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2} + \frac{n-1}{n} \right\} |A|^2 \\ &= -(n-1) - \frac{\sqrt{n-1}}{2}|A|^2. \end{aligned}$$

Hence, we have proven the conclusion of Lemma 2.1. The rest of this part is to verify the above Claim. Indeed, If $|A| = 0$, then $H = 0$, here we used $|H|^2 \leq n|A|^2$. Thus the inequality (2.3) is trivial. Now we assume that $|A| > 0$. The inequality (2.3) is equivalent to

$$\frac{(n-2)\sqrt{n-1}}{n^2}\frac{|H|}{|A|}\sqrt{n - \frac{|H|^2}{|A|^2}} - \frac{2(n-1)}{n^2}\frac{|H|^2}{|A|^2} \leq b.$$

We define $f_n(t)$ on $[0, \sqrt{n}]$ by

$$f_n(t) = \frac{(n-2)\sqrt{n-1}}{n^2}t\sqrt{n-t^2} - \frac{2(n-1)}{n^2}t^2.$$

Suppose that there is a constant $B > 0$ such that $B \geq \max_{[0,\sqrt{n}]} f_n(t)$. Then

$$(n-2)\sqrt{n-1}t\sqrt{n-t^2} \leq 2(n-1)t^2 + Bn^2, \forall t \in [0, \sqrt{n}]$$

or equivalently,

$$(n-2)^2(n-1)x(n-x) \leq 4(n-1)^2x^2 + 4B(n-1)n^2x + B^2n^4, \tag{2.4}$$

where $x := t^2$ for all $x \in [0, n]$. A simple computation shows that the inequality (2.4) holds true if

$$B \geq \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}.$$

Now, choose $b = \frac{(n-2)^2\sqrt{n-1}}{2n(\sqrt{n-1}+1)^2}$. The claim is proved. Thus, the proof is complete. □

We have the following vanishing theorem.

Theorem 2.2. *Let $2 \leq n \leq 6$. Let M^n be a complete hypersurface immersed in a hyperbolic space $\mathbb{H}^{n+1}$. Suppose that M satisfies (δ-SC) condition for some $\frac{n-2}{2\sqrt{n-1}} < \delta \leq 1$, if the first eigenvalue of M has lower bound*

$$\lambda_1 = \lambda_1(M) \geq (\sqrt{n-1}+1)^2 \left(\frac{2\sqrt{n-1}}{n-2} - \frac{1}{\delta} \right)^{-1},$$

then any harmonic one-form ω on M is trivial, provided that

$$\int_{B(R)}|\omega|^{2\beta}<o(R^2),$$

where β is a constant satisfying

$$\frac{1-\sqrt{1-D\frac{n-2}{n-1}}}{D}<\beta<\frac{1+\sqrt{1-D\frac{n-2}{n-1}}}{D}$$

and

$$D=\frac{\sqrt{n-1}}{2\delta}+\frac{1}{\lambda_1}\left(\frac{n\sqrt{n-1}}{2}+(n-1)\right).$$

Proof. We use the method in [7]. Let ω be a harmonic 1-form as in Theorem 2.2. The Bochner formula and the refine Kato's identity imply

$$|\omega|\Delta|\omega|\geq\frac{1}{n-1}|\nabla|\omega||^2+Ric_M(\omega,\omega).$$

By Lemma 2.1, this shows that

$$|\omega|\Delta|\omega|\geq\frac{1}{n-1}|\nabla|\omega||^2-(n-1)|\omega|^2-\frac{\sqrt{n-1}}{2}|A|^2|\omega|^2.$$

Now, for any $\alpha>0$, we have

$$\begin{aligned}|\omega|^\alpha\Delta|\omega|^\alpha &= |\omega|^\alpha\Big(\alpha(\alpha-1)|\omega|^{\alpha-2}|\nabla|\omega||^2+\alpha|\omega|^{\alpha-1}\Delta|\omega|\Big)\\
&= \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2+\alpha|\omega|^{2\alpha-2}|\omega|\Delta|\omega|\\
&\geq \frac{\alpha-1}{\alpha}|\nabla|\omega|^\alpha|^2+\alpha|\omega|^{2\alpha-2}\left(\frac{1}{n-1}|\nabla|\omega||^2-(n-1)|\omega|^2\right.\\
&\quad\left.-\frac{\sqrt{n-1}}{2}|A|^2|\omega|^2\right)\\
&\geq \left(1-\frac{(n-2)}{(n-1)\alpha}\right)|\nabla|\omega|^\alpha|^2-\alpha(n-1)|\omega|^{2\alpha}-\alpha\frac{\sqrt{n-1}}{2}|A|^2|\omega|^{2\alpha}. \qquad (2.5)\end{aligned}$$

Let $q\geq 0$ and $\phi\in\mathcal{C}_0^\infty(M)$. Multiplying both sides of (2.5) by $|\omega|^{2q\alpha}\phi^2$ then integrating over M, we obtain

$$\begin{aligned}&\left(1-\frac{n-2}{(n-1)\alpha}\right)\int_M|\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2\\
\leq&\int_M|\omega|^{(2q+1)\alpha}\phi^2\Delta|\omega|^\alpha+\alpha(n-1)\int_M|\omega|^{2(1+q)\alpha}\phi^2\\
&+\alpha\frac{\sqrt{n-1}}{2}\int_M|A|^2\phi^2|\omega|^{2(q+1)\alpha}\\
=&\ \alpha(n-1)\int_M|\omega|^{2(1+q)\alpha}\phi^2+\alpha\frac{\sqrt{n-1}}{2}\int_M|A|^2\phi^2|\omega|^{2(q+1)\alpha}\end{aligned}$$

$$- (2q+1)\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 - 2\int_M \phi|\omega|^{(2q+1)\alpha}\left\langle \nabla\phi, \nabla|\omega|^\alpha\right\rangle.$$

Hence,

$$\begin{aligned}&\left(2(q+1) - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2 \\ &\leq \alpha(n-1)\int_M |\omega|^{2(1+q)\alpha}\phi^2 + \alpha\frac{\sqrt{n-1}}{2}\int_M |A|^2\phi^2|\omega|^{2(q+1)\alpha} \\ &\quad - 2\int_M \phi|\omega|^{(2q+1)\alpha}\left\langle \nabla\phi, \nabla|\omega|^\alpha\right\rangle. \end{aligned} \tag{2.6}$$

On the other hand, since M satisfies the (δ-SC) condition and $\mathbb{H}^{n+1}$ has non-negative constant sectional curvature, we have for any $\phi \in \mathcal{C}_0^\infty(M)$

$$\int_M |\nabla\phi|^2 \geq \delta\int_M (-n + |A|^2)\phi^2.$$

Replacing ϕ by $|\omega|^{(q+1)\alpha}\phi$ in the above inequality, we obtain

$$\delta\int_M |\omega|^{2(q+1)\alpha}|A|^2\phi^2 \leq \int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 + n\delta\int_M |\omega|^{2(q+1)\alpha}\phi^2. \tag{2.7}$$

Combining (2.6) and (2.7), we infer

$$\begin{aligned}&\left(2(q+1) - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2 \\ &\leq \frac{\alpha\sqrt{n-1}}{2\delta}\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 - 2\int_M \phi|\omega|^{(2q+1)\alpha}\left\langle \nabla\phi, \nabla|\omega|^\alpha\right\rangle. \\ &\quad + \alpha\left\{\frac{n\sqrt{n-1}}{2} + n - 1\right\}\int_M |\omega|^{2(q+1)\alpha}\phi^2. \end{aligned} \tag{2.8}$$

Moreover, by variational characterization of λ_1, we have

$$\int_M |\omega|^{2(q+1)\alpha}\phi^2 \leq \frac{1}{\lambda_1}\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2. \tag{2.9}$$

Hence, (2.8) implies

$$\begin{aligned}&\left(2(q+1) - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2 \\ &\leq \left\{\frac{\alpha\sqrt{n-1}}{2\delta} + \frac{\alpha}{\lambda_1}\left(\frac{n\sqrt{n-1}}{2} + n - 1\right)\right\}\int_M |\nabla(|\omega|^{(q+1)\alpha}\phi)|^2 \\ &\quad - 2\int_M \phi|\omega|^{(2q+1)\alpha}\left\langle \nabla\phi, \nabla|\omega|^\alpha\right\rangle \end{aligned}$$

or equivalently,

$$\left(2(q+1) - \frac{n-2}{(n-1)\alpha}\right)\int_M |\omega|^{2q\alpha}\phi^2|\nabla|\omega|^\alpha|^2$$

$$\leq D\alpha(q+1)^2\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 + D\alpha\int_M |\omega|^{2(q+1)\alpha}|\nabla\phi|^2$$

$$+\Big(D\alpha(q+1)-1\Big)\int_M 2|\omega|^{(2q+1)\alpha}\phi\,\langle\nabla\phi,\nabla|\omega|^\alpha\rangle\,. \tag{2.10}$$

For any $\varepsilon > 0$, the Schwarz inequality implies

$$\Big(D\alpha(q+1)-1\Big)\int_M 2|\omega|^{(2q+1)\alpha}\phi\,\langle\nabla\phi,\nabla|\omega|^\alpha\rangle$$

$$\leq |1-D\alpha(q+1)|\int_M 2|\omega|^{(2q+1)\alpha}|\phi|.|\nabla\phi|.|\nabla|\omega|^\alpha|$$

$$\leq |1-D\alpha(q+1)|\left(\varepsilon\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 + \frac{1}{\varepsilon}\int_M |\omega|^{2(q+1)\alpha}|\nabla\phi|^2\right). \tag{2.11}$$

From (2.10) and (2.11), we conclude that

(2.12)

$$\left\{2(q+1)-\frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 - |1-D\alpha(q+1)|\varepsilon\right\}\int_M \phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2$$

$$\leq \left\{D\alpha + \frac{|1-D\alpha(q+1)|}{\varepsilon}\right\}\int_M |\omega|^{2(q+1)\alpha}|\nabla\phi|^2.$$

Now, choose α, q such that

$$2(q+1)-\frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 > 0.$$

Then, from (2.12), we see that if $\varepsilon > 0$ is small enough, then there exists a positive constant C depending on $\varepsilon, q, \alpha, \delta, \lambda_1$ such that

$$\int_M |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 \leq C\int_M |\omega|^{2(q+1)\alpha}|\nabla\phi|^2, \tag{2.13}$$

provided that

$$2(q+1)-\frac{n-2}{(n-1)\alpha} - D\alpha(q+1)^2 > 0. \tag{2.14}$$

Let $\beta = (q+1)\alpha$, it is easy to see that (2.14) is equivalent to

$$2\beta - \frac{n-2}{n-1} - D\beta^2 > 0.$$

This inequality is always satisfied by the assumptions

$$\frac{1-\sqrt{1-D\frac{n-2}{n-1}}}{D} < \beta < \frac{1+\sqrt{1-D\frac{n-2}{n-1}}}{D}.$$

Now, let ϕ be a smooth function on $[0,\infty)$ such that $\phi \geq 0,\ \phi = 1$ on $[0,R]$ and $\phi = 0$ in $[2R,\infty)$ with $|\phi'| \leq \frac{2}{R}$, then considering $\phi\circ r$, where r is the function in the definition of $B(R)$, we obtain from (2.13)

$$\int_{B(R)} |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 \leq \frac{4C}{R^2}\int_M |\omega|^{2\beta}.$$

Let $R \to \infty$, by the assumption $\int_{B(R)} |\omega|^{2\beta} = 0(R^2)$ we have that $|\omega|$ is constant. By (2.9), we obtain

$$|\omega|^{2\beta} \int_M \phi^2 \leq \frac{4}{\lambda_1 R^2} \int_M |\omega|^{2\beta}.$$

Let $R \to \infty$ again, we conclude that $|\omega| = 0$. Hence, ω is trivial. The proof is finished. □

Now, we will give a proof of Theorem 1.1.

Proof of Theorem 1.1. Since M satisfies the (SC) condition, $\delta = 1$. Hence, we can repeat the proof of Theorem 2.2, to obtain $H^1(L^{2\beta}(M)) = 0$, provided that

$$\frac{1 - \sqrt{1 - D\frac{n-2}{n-1}}}{D} < \beta < \frac{1 + \sqrt{1 - D\frac{n-2}{n-1}}}{D},$$

where

$$D = \frac{\sqrt{n-1}}{2} + \frac{1}{\lambda_1}\left(\frac{n\sqrt{n-1}}{2} + (n-1)\right).$$

Note that the vanishing property of $H^1(L^2(M))$ can be verified if we can choose $\beta = 1$. In fact, by above inequalities, it is sufficient to show that

$$|1 - D| \ < \sqrt{1 - D\frac{n-2}{n-1}},$$

namely, $D < \frac{n}{n-1}$. This is satisfied by the assumption

$$\lambda_1(M) > \frac{2n - (n-1)^{3/2}}{(n + 2\sqrt{n-1})(n-1)^{3/2}}.$$

The proof is complete. □

3. δ-stable condition

In this section, we give a sufficient condition for immersed hypersurfaces to be satisfying the (δ-SC) condition. First, recall that we have the following Sobolev type inequality proved by Hoffman and Spruck [8].

Lemma 3.1. *Let M^n be a submanifold immersed in $\mathbb{H}^{n+p}$. Then there exists a positive constant $C_1 > 0$ such that for any function $\phi \in \mathcal{C}^1_0(M)$, we have*

$$\left(\int_M |\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq C_1 \left(\int_M |\nabla\phi| + \int_M |H\phi|\right) \tag{3.1}$$

Proof. See [8], Theorem 2.1. □

From Lemma 3.1, we have the following Sobolev inequality proved by Carron [3] (also see [6]) and rigidity property of complete manifolds with finite total mean curvature.

Lemma 3.2. *Let $M^n, n \geq 3$ be an oriented complete sub-manifold immersed in $\mathbb{H}^{n+p}$. Suppose that $||H||_n = \int_M |H|^n < \infty$, then for any $\phi \in \mathcal{C}_0^1(M)$, we have*

$$\left(\int_M |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla\phi|^2, \tag{3.2}$$

where

$$C_s = \left(\frac{4C_1(n-1)}{n-2}\right)^2$$

and C_1 is the constant in Lemma 3.1. Moreover, each end of M must be non parabolic.

Proof. The proof of the Lemma is given in [3] (see also [6]). For the completeness, we include the detail here. By the assumption that $\int_M |H|^n < \infty$, there exists a compact subset $D \subset M$ such that

$$\left(\int_{M\setminus D} |H|^n\right)^{1/n} \leq \frac{1}{2C_1}.$$

Let $h \in \mathcal{C}_0^1(M)$, the Hölder inequality implies,

$$\begin{aligned} C_1 \int_{M\setminus D} |Hh| &\leq C_1 \left(\int_{M\setminus D} |H|^n\right)^{1/n} \left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \\ &\leq \frac{1}{2}\left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}. \end{aligned}$$

Hence, by (3.1), we have

$$\left(\int_{M\setminus D} |h|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq 2C_1 \int_{M\setminus D} |\nabla h|.$$

Now, replacing h by $\phi^{\frac{2(n-1)}{n-2}}$, we infer

$$\begin{aligned} \left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-1}{n}} &\leq \frac{4C_1(n-1)}{n-2} \int_{M\setminus D} |\phi^{\frac{n}{n-2}} \nabla\phi| \\ &\leq \frac{4C_1(n-1)}{n-2} \left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{1/2} \left(\int_{M\setminus D} |\nabla\phi|^2\right)^{1/2}. \end{aligned}$$

Therefore,

$$\left(\int_{M\setminus D} |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s \int_{M\setminus D} |\nabla\phi|^2$$

for all $\phi \in \mathcal{C}_0^1(M \setminus D)$. By [2] (also see [3]), we obtain the Sobolev inequality

$$\left(\int_M |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla\phi|^2$$

for all $\phi \in \mathcal{C}_0^1(M)$. By Theorem 2.4 and Proposition 2.5 in [6], each end of M is non-parabolic. The proof is complete. □

Theorem 3.3. *Let M^n be an immersed hypersurface in $\mathbb{H}^n, n \geq 3$. If $||A||_n \leq \frac{1}{\sqrt{\delta C_s}}$ where C_s is the constant in Lemma 3.2, then M satisfies the (δ-SC) condition.*

Proof. We only need to show that, for any $\phi \in \mathcal{C}_0^1(M)$,

$$\int_M \left(|\nabla\phi|^2 - \delta(-n + |A|^2)\phi^2\right) \geq 0.$$

By the assumption on the total scalar curvature, we have $||H||_n \leq \sqrt{n}||A||_n < \infty$, hence we can use the Sobolev inequality in Lemma 3.2 to get

$$\int_M \left(|\nabla\phi|^2 - \delta(-n + |A|^2)\phi^2\right) \geq \frac{1}{C_s}\left(\int_M |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} - \delta\int_M |A|^2\phi^2.$$

Moreover, Hölder inequality implies

$$\int_M |A|^2\phi^2 \leq \left(\int_M |A|^n\right)^{\frac{2}{n}} \left(\int_M \phi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Combining above two inequalities, we obtain

$$\int_M \left(|\nabla\phi|^2 - \delta(-n+|A|^2)\phi^2\right) \geq \left\{\frac{1}{C_s} - \delta\left(\int_M |A|^n\right)^{\frac{2}{n}}\right\} \left(\int_M |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \geq 0$$

here we used $||A||_n \leq \frac{1}{\sqrt{\delta C_s}}$. The proof is complete. □

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