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ISOMETRIC WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE OF $\mathbb{C}^N$

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Abstract. This paper gives complete characterization for isometric weighted composition operators on the Fock space of $\mathbb{C}^N$. The main result shows that an isometric weighted composition operator on the Fock space of $\mathbb{C}^N$ is a unitary operator.

1. Introduction

It is generally considered that weighted composition operators derived from the characterization of isometric operators on Hardy space and Bergman space [4, 5]. So it is natural to consider the condition for a weighted composition operator on a function space to be isometric. In [6], isometric weighted composition operators on the Hardy space are characterized. In [2], Bonet et al. discussed isometric weighted composition operators on weighted Banach spaces of type H^∞. Compared with the characterization of isometric composition operators (see [1, 8, 9, 10] etc.), the characterization of isometric weighted composition operators seems more difficult. For example, a complete characterization of such operators on the Bergman space is still unsolved [6].

In this paper, we completely characterize isometric weighted composition operators on the Fock space of $\mathbb{C}^N$, which extends the corresponding result on the Fock space of complex plane in [7].

Recall that the Fock space $\mathcal{F}^2$ is the space of analytic functions f on $\mathbb{C}^N$ (the N dimensional complex Euclidean space) for which

$$\|f\|^2 = \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |f(z)|^2 \exp(-\frac{|z|^2}{2}) dm_{2N}(z),$$

where $|z|$ denotes the norm for $z \in \mathbb{C}^N$ and dm_{2N} is usual Lebesgue measure on $\mathbb{C}^N$. It is well-known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with

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inner product

$$\langle f,g\rangle=\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} f(z)\overline{g(z)}\exp(-\frac{|z|^2}{2})dm_{2N}(z),\quad f,g\in\mathcal{F}^2$$

and reproducing kernel function

$$K_w(z)=\exp(\frac{\langle z,w\rangle}{2}),\quad w,z\in\mathbb{C}^N,$$

here $\langle z,w\rangle$ denotes the inner product for $z,w\in\mathbb{C}^N$ and $|z|^2=\langle z,z\rangle$. Without confuse, we do not distinguish the inner product symbol in $\mathcal{F}^2$ from that in $\mathbb{C}^N$.

A weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ with ψ an analytic function on $\mathbb{C}^N$ and φ an analytic self-map of $\mathbb{C}^N$ is defined as

$$C_{\psi,\varphi}f=\psi(f\circ\varphi),\quad f\in\mathcal{F}^2.$$

Our main result reads as follows.

Theorem 1.1. *Let ψ be an analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$ if and only if $C_{\psi,\varphi}$ is a unitary operator.*

2. Isometric weighted composition operator

In this section, we give the proof of the main result. The following two well-known lemmas are basic properties of weighted composition operators on Hilbert spaces of analytic functions with reproducing kernel functions.

Lemma 2.1. *Let $\psi_1,\dots,\psi_n$ be analytic functions on $\mathbb{C}^N$ and $\varphi_1,\dots,\varphi_n$ be analytic self-map of $\mathbb{C}^N$. If $C_{\psi_1,\varphi_1},\dots,C_{\psi_n,\varphi_n}$ are bounded operators on $\mathcal{F}^2$, then*

$$C_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}\cdots C_{\psi_n,\varphi_n}=C_{\psi_1(\psi_2\circ\varphi_1)\cdots(\psi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1),\ \varphi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1}.$$

Lemma 2.2. *Let ψ be an analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then for $z\in\mathbb{C}^N$,*

$$C^*_{\psi,\varphi}K_z=\overline{\psi(z)}K_{\varphi(z)}.$$

The following result reveals that a bounded weighted composition operator on the Fock space has a simple composite symbol, which is a key factor for complete characterization of various of weighted composition operators on the Fock space.

Lemma 2.3 ([13, Proposition 7])**.** *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then there exists an operator A on $\mathbb{C}^N$, $|A|\le 1$, $b\in\mathbb{C}^N$ such that*

$$\begin{gathered}\varphi(z)=Az+b,\\ \sup_{z\in\mathbb{C}^N}|\psi(z)|^2\exp(\tfrac{|\varphi(z)|^2-|z|^2}{2})<\infty.\end{gathered}\tag{2.1}$$

In particular, when A is invertible, the condition (2.1) *is sufficient also. Here $|A|$ denotes the norm of A.*

Obviously, a unitary operator is isometric. In [12], unitary operators and their spectrum on the Fock space of $\mathbb{C}^N$ are characterized completely.

For $p \in \mathbb{C}^N$, let k_p be the normalization of K_p, $\varphi_p(z) = z - p,\ z \in \mathbb{C}^N$ and $U_p = C_{k_p,\varphi_p}$.

Lemma 2.4 ([12, Proposition 2.3])**.** *U_p is a unitary operator on $\mathcal{F}^2$ and $U_p^{-1} = U_{-p}$.*

Lemma 2.5 ([12, Theorem 1.1])**.** *Let ψ be an analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a unitary operator on $\mathcal{F}^2$ if and only if there exists a unitary operator A on $\mathbb{C}^N$, $b \in \mathbb{C}^N$ and a constant s with $|s| = 1$ such that $\varphi(z) = Az + b$ and $\psi(z) = sk_{-A^*b}(z), z \in \mathbb{C}^N$.*

In fact, by Lemma 2.5 and Lemma 6 in [13], we have the following result.

Lemma 2.6. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and $\varphi(z) = Az + b, z \in \mathbb{C}^N$ with A an operator on $\mathbb{C}^N$, $b \in \mathbb{C}^N$. Then $C_{\psi,\varphi}$ is nonzero multiples of a unitary operator on $\mathcal{F}^2$ if and only if A is unitary.*

Since isometric operator is a nonzero bounded operator, by Lemma 2.3, we only consider weighted composition operator $C_{\psi,\varphi}$ with $\varphi(z) = Az + b,\ z \in \mathbb{C}^N$ for some operator A on $\mathbb{C}^N$, $|A| \le 1$ and $b \in \mathbb{C}^N$.

The following result is adjustment of [7, Lemma 4.1].

Lemma 2.7 ([7, Lemma 4.1])**.** *Let f be a measurable function on $\mathbb{C}$. If there exist positive constants ϵ and M such that*

$$|f(z)| \le M e^{\frac{|z|^2 - \epsilon|z|^2}{2}},\ z \in \mathbb{C},$$

and for any nonnegative integers m, k,

$$\int_{\mathbb{C}} f(z) z^m \bar{z}^k e^{-\frac{|z|^2}{2}} dm_2(z) = 0,$$

then $f = 0$ almost everywhere on $\mathbb{C}$.

Now we extend Lemma 2.7 to the high dimensional case.

A N multi-index refers to an ordered N-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ of nonnegative integers $\alpha_i,\ i = 1, 2, \ldots, N$. For $z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$.

Lemma 2.8. *Let f be a measurable function on $\mathbb{C}^N$. If there exist positive constants ϵ and M such that*

$$|f(z)| \le M \exp(\frac{|z|^2 - \epsilon|z|^2}{2}),\ z \in \mathbb{C}^N,$$

and for any N multi-indexes α, β,

$$\int_{\mathbb{C}^N} f(z) z^\alpha \bar{z}^\beta \exp(-\frac{|z|^2}{2}) dm_{2N}(z) = 0,$$

then $f = 0$ *a.e. on* $\mathbb{C}^N$.

Proof. Since

$$\int_{\mathbb{C}^N} |f(z) z^\alpha \bar{z}^\beta| \exp(-\frac{|z|^2}{2}) dm_{2N}(z) \le M \int_{\mathbb{C}^N} |z^\alpha \bar{z}^\beta| \exp(-\frac{\epsilon}{2}|z|^2) dm_{2N}(z) < \infty.$$

Fubini Theorem applies for $\int_{\mathbb{C}^N} f(z) z^\alpha \bar{z}^\beta \exp(-\frac{|z|^2}{2}) dm_{2N}(z)$.

We prove the conclusion by induction on N.

It is Lemma 2.7 when $N = 1$. Assume the conclusion holds for $N = j$. We consider the case $N = j + 1$.

For any arbitrarily fixed nonnegative m, n and any $z' \in \mathbb{C}^j$, let

$$F_{n,m}(z') = \int_{\mathbb{C}} f(z', z_{j+1}) z_{j+1}^n \bar{z}_{j+1}^m e^{-\frac{|z_{j+1}|^2}{2}} dm_2(z_{j+1}).$$

Then

$$\begin{aligned} |F_{n,m}(z')| &\le \int_{\mathbb{C}} |f(z', z_{j+1})| |z_{j+1}|^{m+n} e^{-\frac{|z_{j+1}|^2}{2}} dm_2(z_{j+1}) \\ &\le M \int_{\mathbb{C}} e^{\frac{1-\epsilon}{2}(|z'|^2 + |z_{j+1}|^2)} |z_{j+1}|^{n+m} e^{-\frac{|z_{j+1}|^2}{2}} dm_2(z_{j+1}) \\ &= M e^{\frac{1-\epsilon}{2}|z'|^2} \int_{\mathbb{C}} |z_{j+1}|^{n+m} e^{-\frac{\epsilon |z_{j+1}|^2}{2}} dm_2(z_{j+1}) \\ &\le M' e^{\frac{1-\epsilon}{2}|z'|^2}, \end{aligned}$$

here $M' = M \int_{\mathbb{C}} |z_{j+1}|^{n+m} e^{-\frac{\epsilon |z_{j+1}|^2}{2}} dm_2(z_{j+1}) < \infty$.

For any j multi-indexes α', β',

$$\begin{aligned} &\int_{\mathbb{C}^j} F_{n,m}(z')\ (z')^{\alpha'}\ (\bar{z'})^{\beta'} \exp(-\frac{|z'|^2}{2}) dm_{2j}(z') \\ = &\int_{\mathbb{C}^j} \left(\int_{\mathbb{C}} f(z', z_{j+1}) z_{j+1}^n \bar{z}_{j+1}^m e^{-\frac{|z_{j+1}|^2}{2}} dm_2(z_{j+1}) \right) (z')^{\alpha'} (\bar{z'})^{\beta'} e^{-\frac{|z'|^2}{2}} dm_{2j}(z') \\ = &\int_{\mathbb{C}^{j+1}} f(z', z_{j+1}) (z')^{\alpha'} z_{j+1}^n (\bar{z'})^{\beta'} \bar{z}_{j+1}^m \exp(-\frac{|z'|^2 + |z_{j+1}|^2}{2}) dm_{2(j+1)}(z', z_{j+1}) \\ = &\int_{\mathbb{C}^{j+1}} f(z) z^\alpha \bar{z}^\beta \exp(-\frac{|z|^2}{2}) dm_{2(j+1)}(z), \end{aligned}$$

where $z = (z', z_{j+1}) \in \mathbb{C}^{j+1}$, $\alpha = (\alpha', n)$, $\beta = (\beta', m)$ are $j + 1$ multi-indexes. By the given condition,

$$\int_{\mathbb{C}^j} F_{n,m}(z')\ (z')^{\alpha'} (\bar{z'})^{\beta'} \exp(-\frac{|z'|^2}{2}) dm_{2j}(z') = 0,$$

which implies that $F_{n,m}(z')$ satisfies the condition as a function on $\mathbb{C}^j$. By assumption on $N = j$, $F_{n,m} = 0$ a.e. on $\mathbb{C}^j$.

For any $z' \in \mathbb{C}^j$ such that $F_{n,m}(z') = 0$, $n, m = 0, 1, 2, \ldots$, let

$$g_{z'}(z_{j+1}) = f(z', z_{j+1}),\ z_{j+1} \in \mathbb{C},$$

then

$$|g_{z'}(z_{j+1})| \le M_{z'} e^{\frac{1-\epsilon}{2}|z_{j+1}|^2},$$

here $M_{z'} = Me^{\frac{1-\epsilon}{2}|z'|^2}$, and for any nonnegative integer n, m

$$\int_{\mathbb{C}} g_{z'}(z_{j+1}) z_{j+1}^n \bar{z}_{j+1}^m e^{-\frac{|z_{j+1}|^2}{2}} dm_2(z_{j+1}) = F_{n,m}(z') = 0.$$

which implies $g_{z'}(z_{j+1})$ satisfies the condition in Lemma 2.7 as a function on $\mathbb{C}$. So $g_{z'} = 0$ a.e. on $\mathbb{C}$.

Hence for any $z' \in \mathbb{C}^j$ such that $F_{n,m}(z') = 0$, $n, m = 1, 2, \ldots$, $f(z', z_{j+1}) = 0$ a.e. on $\mathbb{C}$. So $f = 0$ a.e. on $\mathbb{C}^{j+1}$. □

Now, we characterize isometric weighted composition operator $C_{\psi,\varphi}$ on $\mathcal{F}^2$ with $\varphi(z) = Az,\ z \in \mathbb{C}^N$ for some operator A on $\mathbb{C}^N$.

Lemma 2.9. *Let ψ be an analytic function on $\mathbb{C}^N$ and $\varphi(z) = Az,\ z \in \mathbb{C}^N$ with A an operator on $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$, then A is invertible.*

Proof. As an operator on $\mathbb{C}^N$, it is enough to prove that A is surjective.

Assume $b \perp \operatorname{ran}(A)$, then for any $z \in \mathbb{C}^N$, $\langle Az, b\rangle = 0$. So $(K_b \circ \varphi)(z) = 1$. Since $C_{\psi,\varphi}$ is isometric,

$$\|\psi\|^2 = \|\psi\ K_b \circ \varphi\|^2 = \|C_{\psi,\varphi} K_b\|^2 = \|K_b\|^2. \tag{2.2}$$

On the other hand, we have $\|\psi\|^2 = \|C_{\psi,\varphi} K_0\|^2 = \|K_0\|^2 = 1$. Combining with equation (2.2), we have $b = 0$. Therefore A is surjective. □

Proposition 2.10. *Let ψ be an analytic function on $\mathbb{C}^N$ and $\varphi(z) = Az,\ z \in \mathbb{C}^N$ with A an invertible operator on $\mathbb{C}^N$, $|A| \le 1$. Then $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$ if and only A is a unitary operator and ψ is a constant of modulus 1.*

Proof. The sufficiency follows from Lemma 2.5 and Lemma 2.6.

Necessity.

For any $f, g \in \mathcal{F}^2$, put $w = Az$, we have

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} f(w)\overline{g(w)} \exp(-\frac{|w|^2}{2}) dm_{2N}(w) \\ &= \frac{1}{(2\pi)^N} |\det A|^2 \int_{\mathbb{C}^N} f(Az)\overline{g(Az)} \exp(\frac{|z|^2 - |Az|^2}{2}) \exp(-\frac{|z|^2}{2}) dm_{2N}(z).\end{aligned}$$

On the other hand,

$$\langle C_{\psi,\varphi} f, C_{\psi,\varphi} g\rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |\psi(z)|^2 f(Az)\overline{g(Az)} \exp(-\frac{|z|^2}{2}) dm_{2N}(z).$$

For any N multi-indexes α, β, let $f(z) = (A^{-1}z)^{\alpha}, g(z) = (A^{-1}z)^{\beta},\ z \in \mathbb{C}^N$, it follows from

$$\langle f, g\rangle = \langle C_{\psi,\varphi} f, C_{\psi,\varphi} g\rangle$$

that

$$\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N}(|\psi(z)|^2 - |\det A|^2 e^{\frac{|z|^2-|Az|^2}{2}}) z^{\alpha}\overline{z}^{\beta}\exp(-\frac{|z|^2}{2}) dm_{2N}(z) = 0. \tag{2.3}$$

Since $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$, it is bounded. By Lemma 2.3, there exists a positive constant M such that $|\psi(z)|^2 \le Me^{\frac{|z|^2-|Az|^2}{2}},\ z \in \mathbb{C}^N$. So

$$\begin{aligned}&\big||\psi(z)|^2 - |\det A|^2 \exp(\frac{|z|^2-|Az|^2}{2})\big| \\ &\le (M + |\det A|^2)\exp(\frac{|z|^2-|Az|^2}{2}),\ z \in \mathbb{C}^N.\end{aligned}$$

Since A is invertible, there exists a positive constant δ such that for any $z \in \mathbb{C}^N$,

$$|Az| \ge \delta|z|.$$

It follows from $|A| \le 1$ that $\delta \le 1$.

Without loss of generality, assume $\delta < 1$. Then

$$\begin{aligned}&\big||\psi(z)|^2 - |\det A|^2 \exp(\frac{|z|^2-|Az|^2}{2})\big| \\ &\le (M + |\det A|^2)\exp(\frac{(1-\delta^2)|z|^2}{2}), \qquad z \in \mathbb{C}^N.\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), it follows from Lemma 2.8 that

$$|\psi(z)|^2 = |\det A|^2 \exp(\frac{|z|^2-|Az|^2}{2}),\ z \in \mathbb{C}^N$$

which implies that the analytic function $\psi(z)\overline{\psi(\bar{w})} - |\det A|^2 \exp(\frac{\langle z,\bar{w}\rangle - \langle Az, A\bar{w}\rangle}{2})$ with variable $(z, w), z, w \in \mathbb{C}^N$ is zero on $\{(z, \bar{z})\ :\ z \in \mathbb{C}^N\}$. Hence

$$\psi(z)\overline{\psi(w)} = |\det A|^2 \exp(\frac{\langle z, w\rangle - \langle Az, Aw\rangle}{2}),\ z, w \in \mathbb{C}^N.$$

Let $w = 0$, then ψ is a constant function, hence

$$\exp(\frac{|z|^2-|Az|^2}{2}),\ z \in \mathbb{C}^N$$

is a constant too. So there exists an integer-valued continuous function $n(z)$ such that

$$|z|^2 - |Az|^2 = 4n(z)\pi i,\ z \in \mathbb{C}^N,$$

here i is the imaginary unit. Let $z = 0$, we have $n = 0$, therefore

$$|z|^2 - |Az|^2 = 0,\ z \in \mathbb{C}^N.$$

So A is an isometric operator, hence a unitary operator on $\mathbb{C}^N$. By Lemma 2.5 and Lemma 2.6, ψ is a constant function. Since $C_{\psi,\varphi}$ is isometric, $|\psi| = 1$. $\square$

In the following, we characterize isometric weighted composition operators on $\mathcal{F}^2$ generally.

Lemma 2.11. *Let ψ be an analytic function on $\mathbb{C}^N$ and $\varphi(z) = Az+b$, $z \in \mathbb{C}^N$ with A an operator on $\mathbb{C}^N$, $|A| \leq 1$ and $b \in \mathbb{C}^N$. If $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$, then A is invertible.*

Proof. Let $b = b_1 + b_2$ with $b_1 \in \operatorname{ran}(A)$, $b_2 \in \operatorname{ran}(A)^\perp = \ker(A^*)$. Then there exists $p \in \mathbb{C}^N$ such that $b_1 = Ap$.

Since $C_{\psi,\varphi}$ is isometric, it follows from Lemma 2.4 that $U_p C_{\psi,\varphi} = C_{\psi_1,\varphi_1}$ is isometric also. Here

$$\varphi_1(z) = (\varphi \circ \varphi_p)(z) = Az + b_2,\ z \in \mathbb{C}^N.$$

So we have

$$\|\psi_1\|^2 = \|C_{\psi_1,\varphi_1} K_0\|^2 = \|K_0\|^2 = 1$$

and

$$\begin{aligned}\|\psi_1\|^2 \exp(|b_2|^2) &= \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |\psi_1(z)|^2 | \exp(\frac{\langle z, A^* b_2\rangle + \langle b_2, b_2\rangle}{2})|^2 e^{-\frac{|z|^2}{2}} dm_{2N}(z) \\ &= \|C_{\psi_1,\varphi_1} K_{b_2}\|^2 = \|K_{b_2}\|^2 \\ &= \exp(\frac{|b_2|^2}{2}),\end{aligned}$$

which implies that $b_2 = 0$. It follows from Lemma 2.9 that A is invertible. □

Proof of Theorem 1.1. The sufficiency is obvious.

Necessity. Since $C_{\psi,\varphi}$ is an isometric operator on $\mathcal{F}^2$, $C_{\psi,\varphi}$ is bounded. By Lemma 2.3, there exists an operator A on $\mathbb{C}^N$, $|A| \leq 1$ and $b \in \mathbb{C}^N$ such that

$$\varphi(z) = Az + b,\ z \in \mathbb{C}^N.$$

By Lemma 2.11, A is invertible and A^{-1} exists. Let $p = A^{-1}b$, $\psi_1 = k_p(\psi \circ \varphi_p)$, $\varphi_1 = \varphi \circ \varphi_p$, then $U_p C_{\psi,\varphi} = C_{\psi_1,\varphi_1}$ and

$$\varphi_1(z) = \varphi(\varphi_p(z)) = A(z-p) + b = Az,\ z \in \mathbb{C}^N.$$

Since $C_{\psi,\varphi}$ is isometric and U_p is unitary, C_{ψ_1,φ_1} is isometric also. By Proposition 2.10 and Lemma 2.6, C_{ψ_1,φ_1} is a unitary operator on $\mathcal{F}^2$, so is $C_{\psi,\varphi}$. □

The main result shows that an isometric weighted composition operator on the Fock space is a unitary. In the end, we summarize some known classes of weighted composition operators on the Fock space which are essentially unitary operators.

Theorem 2.12. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, then the following statements are equivalent.*

(1) *$C_{\psi,\varphi}$ is nonzero multiples of a unitary operator.*

(2) *$C_{\psi,\varphi}$ is nonzero multiples of an isometric operator.*

(3) $C_{\psi,\varphi}$ *is nonzero multiples of a co-isometric operator.*
(4) $C_{\psi,\varphi}$ *is invertible.*
(5) $C_{\psi,\varphi}$ *is Fredholm.*

Proof. The equivalence of (1) and (2) is the conclusion of Theorem 1.1.

The equivalence of (1) and (3) is the conclusion of [14, Corollary 2.6].

The equivalence of (1) and (4) is the conclusion of [14, Theorem 1.1].

We only need to prove that (5) implies (4).

For $N = 1$, it is the conclusion of [3, Theorem 1.1]. In the following, we assume that $N \geq 2$ and $C_{\psi,\varphi}$ is a Fredholm operator on $\mathcal{F}^2$.

If $\psi(z) = 0$ for $z \in \mathbb{C}^N$, then

$$C^*_{\psi,\varphi} K_z = \overline{\psi(z)} K_{\varphi(z)} = 0,$$

which implies that $K_z \in \ker(C^*_{\psi,\varphi})$. Since $C_{\psi,\varphi}$ is a Fredholm operator, $\ker(C^*_{\psi,\varphi})$ is finite-dimensional. It follows that ψ has at most finite number of zeros. Since ψ is an analytic function of N variables and $N \geq 2$, it follows from the well-known result "No analytic function of more than one complex variable has any isolated zeros" [11, Remarks 14.1.4] that ψ has no zeros.

Since $\ker(C^*_{\psi,\varphi})$ is finite dimensional, φ is not a constant vector map. Let $\varphi(z) = \varphi(w)$ for $z, w \in \mathbb{C}^N, z \neq w$, then there are infinitely disjoint subsets $\{p_n\}$ and $\{q_n\}$ in $\mathbb{C}^N$ such that $\varphi(p_n) = \varphi(q_n)$, $n = 1, 2, \ldots$. So we have

$$C^*_{\psi,\varphi}\Big(\frac{K_{p_n}}{\overline{\psi(p_n)}} - \frac{K_{q_n}}{\overline{\psi(q_n)}}\Big) = 0,$$

which contradicts to the condition that $\ker(C^*_{\psi,\varphi})$ is finite-dimensional. It follows that φ is injective.

By the boundedness of $C_{\psi,\varphi}$ and Lemma 2.5, there exists an operator A on $\mathbb{C}^N$ with $|A| \leq 1$, $b \in \mathbb{C}^N$ and a positive constant M such that

$$\varphi(z) = Az + b,$$

$$|\psi(z)|^2 \exp\Big(\frac{|\varphi(z)|^2 - |z|^2}{2}\Big) \leq M, \quad z \in \mathbb{C}^N.$$

The injectiveness of φ implies that A is injective. As an injective operator on $\mathbb{C}^N$, A is invertible.

Since $C_{\psi,\varphi}$ is a Fredholm operator on $\mathcal{F}^2$, there exist bounded operator T and compact operator S on $\mathcal{F}^2$ such that

$$T(C_{\psi,\varphi})^* = I + S,$$

where I is the identity on $\mathcal{F}^2$. Hence

$$\|TC^*_{\psi,\varphi} k_z\| \geq \|k_z\| - \|Sk_z\| = 1 - \|Sk_z\|, \ z \in \mathbb{C}^N. \tag{2.5}$$

Since S is compact on $\mathcal{F}^2$ and k_z weakly converges to 0 as $|z| \to \infty$, we have

$$\lim_{|z|\to\infty} \|Sk_z\| = 0.$$

So there exists a positive constant r such that

$$\|Sk_z\| < \frac{1}{2}, \ |z| > r. \tag{2.6}$$

By $(C_{\psi,\varphi})^* K_z = \overline{\psi(z)} K_{\varphi(z)}$, we obtain

$$\begin{aligned} \|T\|^2 \ |\psi(z)|^2 \ \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}) &= \|T\|^2 \ |\psi(z)|^2 \ \frac{\|K_{\varphi(z)}\|^2}{\|K_z\|^2} \\ &= \|T\|^2 \frac{\|C^*_{\psi,\varphi} K_z\|^2}{\|K_z\|^2} \\ &\geq \|T(C_{\psi,\varphi})^* k_z\|^2. \end{aligned} \tag{2.7}$$

Let $L' = \frac{1}{4\|T\|^2}$, it follows from (2.5)-(2.7) that

$$L' \leq |\psi(z)|^2 \ \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}), \ |z| \geq r.$$

Since ψ has no zeros, $|\psi(z)|^2 \ \exp(\frac{|\varphi(z)|^2 - |z|^2}{2})$ is a positive continuous function on $\{z \in \mathbb{C}^N \ : \ |z| \leq r\}$. So there exists a positive constant l such that

$$l \leq |\psi(z)|^2 \ \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}), \ |z| \leq r.$$

Let $L = \min\{L', l\}$, then

$$L \leq |\psi(z)|^2 \ \exp(\frac{|\varphi(z)|^2 - |z|^2}{2}) \leq M, \ z \in \mathbb{C}^N.$$

By [13, Theorem 1], $C_{\psi,\varphi}$ is invertible. □

References

[1] R. F. Allen and F. Colonna, *On the isometric composition operators on the Bloch space in C^n*, J. Math. Anal. Appl. **355** (2009), no. 2, 675–688.

[2] J. Bonet, M. Lindström, and E. Wolf, *Isometric weighted composition operators on weighted Banach spaces of type H^∞*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4267–4273.

[3] L. Feng and L. Zhao, *A note on weighted composition operators on the Fock space*, Commun. Math. Res. **31** (2015), no. 3, 281–284.

[4] F. Forelli, *The isometries of H^p*, Canad. J. Math. **16** (1964), 721–728.

[5] C. J. Kolaski, *Isometries of weighted Bergman spaces*, Canad. J. Math. **34** (1982), no. 4, 910–915.

[6] R. Kumar and J. R. Partington, *Weighted composition operators on Hardy and Bergman spaces*, Recent advances in operator theory, operator algebras, and their applications, 157–167, Operator Theory: Advances and Applications 153, Birkhäuser, Basel, 2005.

[7] T. Le, *Normal and isometric weighted composition operators on the Fock space*, Bull. Lond. Math. Soc. **46** (2014), no. 4, 847–856.

[8] M. J. Martín and D. Vukotić, *Isometries of the Dirichlet space among the composition operators*, Proc. Amer. Math. Soc. **134** (2006), no. 6, 1701–1705.

[9] ———, *Isometries of the Bloch space among the composition operators*, Bull. Lond. Math. Soc. **39** (2007), no. 1, 151–155.

[10] ———, *Isometries of some classical function spaces among the composition operators*, Recent Advances in Operator-related Function Theory, 133–138, Contemporary Mathematics, 393, Amer. Math. Soc., Providence, RI, 2006.
[11] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, New-York, Springer-Verlag, 1980.
[12] L. Zhao, *Unitary weighted composition operators on the Fock space of* $\mathbb{C}^n$, Complex Anal. Oper. Theory **8** (2014), no. 2, 581–590.
[13] ———, *Invertible weighted composition operators on the Fock space of* $\mathbb{C}^N$, J. Funct. Spaces **2015** (2015), Art. ID 250358, 5 pp.
[14] ———, *A note on invertible weighted composition operators on the Fock pace of* $\mathbb{C}^N$, J. Math. Res. Appl. **36** (2016), no. 3, 359–362.

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