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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH A NON-METRIC ϕ-SYMMETRIC CONNECTION

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Abstract. We study lightlike hypersurfaces M of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. We characterize the geometry of lightlike hypersurfaces of such a $\bar{M}$.

1. Introduction

The notion of non-metric ϕ-symmetric connection was introduced by Jin [8]. Semi-symmetric non-metric connection [1] and quarter-symmetric non-metric connection [2] are two impotent examples of this connection. It is defined as follow: An affine connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *non-metric ϕ-symmetric connection* if it and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_X \bar{g})(Y, Z) = -\theta(Y)\phi(X, Z) - \theta(Z)\phi(X, Y), \tag{1.1}$$

$$\bar{T}(X, Y) = \theta(Y)JX - \theta(X)JY, \tag{1.2}$$

for any vector fields X, Y and Z on $\bar{M}$, where ϕ and J are tensor fields of types $(0, 2)$ and $(1, 1)$ respectively, and θ is a 1-form on $\bar{M}$.

The objective of this paper is to study the geometry of lightlike hypersurfaces of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection, in which the tensor field J in (1.2) is identical with the indefinite almost contact structure tensor J of $\bar{M}$, the tensor field ϕ in (1.1) is identical with the fundamental 2-form associated with the tensor field J, *i.e.*,

$$\phi(X, Y) = \bar{g}(JX, Y), \tag{1.3}$$

and the 1-form θ, defined by (1.1) and (1.2), is identical with the structure 1-form θ of the indefinite almost contact structure $(J, \zeta, \theta, \bar{g})$ of $\bar{M}$.

Denote $\widetilde{\nabla}$ by the unique Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to the metric $\bar{g}$. It is known [8] that *a linear connection*

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$\bar{\nabla}$ on $\bar{M}$ is non-metric ϕ-symmetric connection if and only if it satisfies

$$\bar{\nabla}_X Y = \widetilde{\nabla}_X Y + \theta(Y)JX. \tag{1.4}$$

In this paper, by saying that non-metric ϕ-symmetric connection we shall mean the non-metric ϕ-symmetric connection defined by (1.4).

2. Lightlike hypersurfaces

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exist a (1, 1)-type tensor field J, a vector field ζ which is called the *structure vector field*, and a 1-form θ such that

$$J^2X = -X + \theta(X)\zeta,\ \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\ \theta(\zeta) = 1, \tag{2.1}$$

for any vector fields X and Y on $\bar{M}$, where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively. The set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$. From (2.1), we show that

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(X) = \epsilon\bar{g}(X, \zeta), \quad \bar{g}(JX, Y) = -\bar{g}(X, JY).$$

In the entire discussion of this article, we shall assume that the structure vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Definition. An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite trans-Sasakian manifold* if, for any vector fields X and Y on $\bar{M}$, there exist two smooth functions α and β such that

$$(\widetilde{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)JX\}.$$

We say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β).

The notion of indefinite trans-Sasakian manifold was introduced by Oubina [9]. Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of indefinite trans-Sasakian manifold such that

$$\alpha = 1,\ \beta = 0; \quad \alpha = 0,\ \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}$$

By directed calculation from (1.4), we obtain the following relation:

$$(\bar{\nabla}_X J)Y = (\widetilde{\nabla}_X J)Y + \theta(Y)\{X - \theta(X)\zeta\}.$$

Thus, replacing the Levi-Civita connection $\tilde{\nabla}$ by the non-metric ϕ-symmetric connection $\bar{\nabla}$, the equation in the above Definition is reformed to

$$\begin{aligned}(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)JX\} \\ + \theta(Y)\{X - \theta(X)\zeta\}.\end{aligned} \tag{2.2}$$

Replacing Y by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X\zeta) = 0$, we obtain

$$\bar{\nabla}_X\zeta = -(\alpha - 1)JX + \beta(X - \theta(X)\zeta). \tag{2.3}$$

Let (M, g) be a lightlike hypersurface of $(\bar{M}, \bar{g})$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. It is known [5] that the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM, of rank 1, and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. For any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$ respectively. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.7}$$

where ∇ and ∇^* are the induced linear connections on M and $S(TM)$ respectively, B and C are the local second fundamental forms on M and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on M and $S(TM)$ respectively, and τ and σ are 1-forms on M.

Due to [6], it is known that, for any lightlike hypersurface M of an indefinite almost contact manifold $\bar{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are vector subbundles of $S(TM)$, of rank 1. In the following, we shall assume that ζ is tangent to M. Călin [4] proved that if ζ is tangent to M, then it belongs to $S(TM)$. In this case, there exists two non-degenerate almost complex distributions D_o and D with respect to J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = TM^\perp \oplus_{orth} J(TM^\perp) \oplus_{orth} D_o,$$
$$TM = D \oplus J(tr(TM)).$$

Consider two null vector fields U and V and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{2.8}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.9}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $FX = JSX$. Applying J to (2.9) and using $(2.1)_1$ and (2.8), we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \tag{2.10}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the non-metric ϕ-symmetric connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss-Codazzi equations for M and $S(TM)$ such that

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_{N}Y - B(Y,Z)A_{N}X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \\ &- \tau(Y)B(X,Z) + B(T(X,Y),Z)\}N, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \bar{R}(X,Y)N = -\nabla_X(A_{N}Y) &+ \nabla_Y(A_{N}X) + A_{N}[X,Y] \\ &+ \tau(X)A_{N}Y - \tau(Y)A_{N}X \\ &+ \{B(Y,A_{N}X) - B(X,A_{N}Y) + 2d\tau(X,Y)\}N, \end{aligned} \tag{2.12}$$

$$\begin{aligned} R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_{\xi}Y - C(Y,PZ)A^*_{\xi}X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \sigma(X)C(Y,PZ) \\ &+ \sigma(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi, \end{aligned} \tag{2.13}$$

$$\begin{aligned} R(X,Y)\xi = -\nabla^*_X(A^*_{\xi}Y) &+ \nabla^*_Y(A^*_{\xi}X) + A^*_{\xi}[X,Y] \\ &- \sigma(X)A^*_{\xi}Y + \sigma(Y)A^*_{\xi}X \\ &+ \{C(Y,A^*_{\xi}X) - C(X,A^*_{\xi}Y) - 2d\sigma(X,Y)\}\xi. \end{aligned} \tag{2.14}$$

3. Non-metric ϕ-symmetric connections

Let $(\bar{M},\bar{g})$ be an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection $\bar{\nabla}$. Using (1.1), (1.2), (1.3), (2.4) and (2.9), we obtain

$$\begin{aligned} (\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) &+ B(X,Z)\eta(Y) \\ &- \theta(Y)\phi(X,Z) - \theta(Z)\phi(X,Y), \end{aligned} \tag{3.1}$$

$$T(X,Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}$$

$$B(X,Y) - B(Y,X) = \theta(Y)u(X) - \theta(X)u(Y), \tag{3.3}$$

$$\begin{aligned} &\phi(X,\xi) = u(X), \qquad \phi(X,N) = v(X), \\ &\phi(X,V) = 0, \qquad\quad \phi(X,U) = -\eta(X), \end{aligned} \tag{3.4}$$

where T is the torsion tensor with respect to ∇ and η is a 1-form such that

$$\eta(X) = \bar{g}(X,N).$$

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y,\xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X,\xi) = B(\xi,X) = 0. \tag{3.5}$$

The local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_{\xi}X,Y) + \theta(Y)u(X), \qquad \bar{g}(A^*_{\xi}X,N) = 0, \tag{3.6}$$

$$C(X, PY) = g(A_N X, PY) + \theta(PY)v(X), \quad \bar{g}(A_N X, N) = 0, \tag{3.7}$$

and $\tau = \sigma$. From (2.3), (2.6), (3.5) and the fact that $\tau = \sigma$, we obtain

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi. \tag{3.8}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N) = 0$ and using (1.1), (2.3), (2.5) and $(3.4)_2$, we have

$$g(A_N X, \zeta) = -\alpha v(X) + \beta\eta(X). \tag{3.9}$$

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then $\beta = 0$.*

Proof. Taking $X = U$ and $X = \xi$ to (3.9) by turns, we obtain

$$\theta(A_N U) = 0, \qquad \theta(A_N \xi) = \beta. \tag{3.10}$$

From the Gauss equation (2.11) and (3.5), we see that

$$-\bar{g}(\bar{R}(X, Y)N, \xi) = \bar{g}(\bar{R}(X, Y)\xi, N) = \bar{g}(R(X, Y)N, \xi).$$

From this equation, (2.12), (2.14) and the fact that $\tau = \sigma$, we obtain

$$B(Y, A_N X) - B(X, A_N Y) = C(X, A^*_\xi Y) - C(Y, A^*_\xi X).$$

Using this equation, (3.6) and (3.7), we obtain

$$\theta(A_N X)u(Y) - \theta(A_N Y)u(X) = \theta(A^*_\xi Y)v(X) - \theta(A^*_\xi X)v(Y).$$

Replacing Y by U to this equation and using $(3.10)_1$, we have

$$\theta(A_N X) = \theta(A^*_\xi U)v(X).$$

Taking $X = \xi$, we get $\theta(A_N \xi) = 0$. From this and $(3.10)_2$, we get $\beta = 0$. □

Corollary 3.2. *There exist no lightlike hypersurfaces of an indefinite Kenmotsu manifold with a non-metric ϕ-symmetric connection.*

Applying $\bar{\nabla}_X$ to (2.8) and (2.9) and using (2.2), (2.4), (2.5), (2.8), (2.9), (3.1), $(3.4)_4$, (3.7), (3.8) and the fact that $\beta = 0$, we have

$$B(X, U) = C(X, V), \tag{3.11}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - \alpha\eta(X)\zeta, \tag{3.12}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V, \tag{3.13}$$

$$\begin{aligned}(\nabla_X F)(Y) = {} & u(Y)A_N X - B(X, Y)U - (\alpha - 1)\theta(Y)X \\ & + \{\alpha g(X, Y) - \theta(X)\theta(Y)\}\zeta,\end{aligned} \tag{3.14}$$

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY), \tag{3.15}$$

$$(\nabla_X v)(Y) = v(Y)\tau(X) - g(A_N X, FY) - (\alpha - 1)\theta(Y)\eta(X). \tag{3.16}$$

From (3.7), we show that (3.9) satisfying $\beta = 0$ is equivalent to

$$C(X, \zeta) = -(\alpha - 1)v(X). \tag{3.17}$$

Substituting (2.9) into (2.3) such that $\beta = 0$ and using (2.4), we have

$$\nabla_X \zeta = -(\alpha - 1)FX, \qquad B(X, \zeta) = -(\alpha - 1)u(X). \tag{3.18}$$

4. Recurrent and Lie recurrent lightlike hypersurfaces

Definition. The structure tensor field F of M is said to be *recurrent* [7] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.1. *There exist no recurrent lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection.*

Proof. If M is recurrent, then, from the above definition and (3.14), we get

$$\begin{aligned}\varpi(X)FY &= u(Y)A_N X - B(X,Y)U - (\alpha-1)\theta(Y)X \\ &\quad + \{\alpha g(X,Y) - \theta(X)\theta(Y)\}\zeta.\end{aligned} \tag{4.1}$$

Replacing Y by ξ to this equation and using (3.5) and the fact that $F\xi = -V$, we get $-\varpi(X)V = 0$. Taking the scalar product with U to this result, we obtain $\varpi = 0$. It follows that F is parallel with respect to the connection ∇.

Taking $Y = \zeta$ to (4.1) and using $(3.18)_2$, we get

$$(\alpha - 1)\{-X + u(X)U + \theta(X)\zeta\} = 0.$$

It follows that $\alpha = 1$. Thus $\bar{M}$ is an indefinite Sasakian manifold.

Taking the scalar product with ζ to (4.1) and using (3.9), we get

$$g(X,Y) - \theta(X)\theta(Y) - v(X)u(Y) = 0.$$

Taking the skew-symmetric part of this equation, we obtain

$$u(X)v(Y) - u(Y)v(X) = 0.$$

Taking $X = U$ and $Y = V$ to this result, we have $1 = 0$. It is a contradiction. Thus we have our theorem. □

Corollary 4.2. *There exist no lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a non-metric ϕ-symmetric connection such that the structure tensor field F is parallel with respect to the connection ∇ of M.*

Definition. The structure tensor field F of M is said to be *Lie recurrent* [7] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X,Y].$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A lightlike hypersurface M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.3. *Let M be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. Then*

(1) *F is Lie parallel,*
(2) *$\alpha = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold,*
(3) *the 1-form τ satisfies $\tau = 0$.*

Proof. (1) Using the above definition, (2.10), (3.2) and (3.14), we get

$$\begin{aligned}\vartheta(X)FY = & -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X \\ & - \{B(X,Y) - \theta(Y)u(X)\}U + \alpha\{g(X,Y)\zeta - \theta(Y)X\}.\end{aligned} \tag{4.2}$$

Taking $Y = \xi$ to (4.2) and using (2.9), we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X. \tag{4.3}$$

Taking the scalar product with V and ζ to (4.3) by turns, we have

$$u(\nabla_V X) = 0, \qquad \theta(\nabla_V X) = 0. \tag{4.4}$$

Replacing Y by V to (4.2) and using the fact that $\theta(V) = 0$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U + \alpha u(X)\zeta.$$

Applying F to this equation and using (2.10) and (4.4), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X.$$

Comparing this equation with (4.3), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with ζ to (4.2) and using (3.9), we have

$$-\, g(\nabla_{FY}X, \zeta) + \alpha\{g(X,Y) - v(X)u(Y) - \theta(X)\theta(Y)\} = 0.$$

Taking $X = U$ to this equation and using (3.12) and the fact that $\eta(FY) = v(Y)$, we get $\alpha v(Y) = 0$. It follows that $\alpha = 0$.

(3) Taking the scalar product with N to (4.2) and using $(3.7)_2$, we have

$$-\,\bar{g}(\nabla_{FY}X, N) + \bar{g}(F\nabla_Y X, N) = 0. \tag{4.5}$$

Replacing X by ξ to (4.5) and using (2.7), (2.8), (2.9) and (3.6), we have

$$B(X,U) = \tau(FX). \tag{4.6}$$

Replacing X by U to (4.6) and using (3.11) and the fact that $FU = 0$, we get

$$C(U,V) = B(U,U) = 0. \tag{4.7}$$

Replacing X by V to (4.5) and using (2.10), (3.6) and (3.13), we have

$$B(FY,U) = -\tau(Y).$$

Taking $Y = U$ and $Y = \zeta$ and using the fact that $FU = F\zeta = 0$, we obtain

$$\tau(U) = 0, \qquad \tau(\zeta) = 0. \tag{4.8}$$

Replacing X by U to (4.2) and using (2.10), (3.3) and (3.9), we get

$$u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U = 0.$$

Taking the scalar product with V and using (3.7), (3.11) and (4.7), we get

$$B(X,U) = -\tau(FX).$$

Comparing this with (4.6), we obtain $\tau(FX) = 0$. Replacing X by FY to this result and using (3.9) and (4.8), we have $\tau = 0$. □

Theorem 4.4. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a non-metric ϕ-symmetric connection. If V or U is parallel with respect to the induced connection ∇ on M, then $\tau = 0$ and $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold.*

Proof. (1) If V is parallel with respect to ∇, then, from (3.13), we have

$$F(A^*_\xi X) - \tau(X)V = 0.$$

Taking the scalar product with U to this equation, we have $\tau = 0$. Applying F to the last equation and using (2.10), (3.6) and $(3.18)_2$, we obtain

$$A^*_\xi X = -\alpha u(X)\zeta + u(A^*_\xi X)U.$$

Taking the scalar product with U and using (3.6), we have $B(X, U) = 0$. Thus $B(\zeta, U) = 0$. Taking $X = U$ and $Y = \zeta$ to (3.3), we get $B(U, \zeta) = 1$. On the other hand, replacing X by U to $(3.18)_2$, we have $B(U, \zeta) = -\alpha + 1$. From the above two results, we get $\alpha = 0$ and

$$A^*_\xi X = u(A^*_\xi X)U. \tag{4.9}$$

(2) If U is parallel with respect to ∇, then, from (3.12), we have

$$F(A_N X) + \tau(X)U - \alpha\eta(X)\zeta = 0.$$

Taking the scalar product with ζ and V to this equation by turns, we get $\alpha = 0$ and $\tau = 0$ respectively. Applying F to the last equation and using (2.10), (3.9) and the fact that $\alpha = \beta = 0$, we obtain

$$A_N X = u(A_N X)U. \tag{4.10}$$

As $\alpha = \beta = 0$ in (1) and (2), $\bar{M}$ is an indefinite cosymplectic manifold. □

5. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form*, denote it by $\bar{M}(f_1, f_2, f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(X, Y)Z = {} & f_1\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} \\ & + f_2\{\bar{g}(X, JZ)JY - \bar{g}(Y, JZ)JX + 2\bar{g}(X, JY)JZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ & + \bar{g}(X, Z)\theta(Y)\zeta - \bar{g}(Y, Z)\theta(X)\zeta\} \end{aligned} \tag{5.1}$$

for any vector fields X, Y and Z on $\bar{M}$.

The generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ was introduced by Alegre *et al*. [3]. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of indefinite generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms.

Comparing the transversal components of (2.11) and (5.1), we have

$$\begin{aligned}(5.2)\qquad &(\nabla_X B)(Y,Z)-(\nabla_Y B)(X,Z)\\ &+\ \tau(X)B(Y,Z)-\tau(Y)B(X,Z)\\ &-\ \theta(X)B(FY,Z)+\theta(Y)B(FX,Z)\\ &=f_2\{u(Y)\bar{g}(X,JZ)-u(X)\bar{g}(Y,JZ)+2u(Z)\bar{g}(X,JY)\}.\end{aligned}$$

Taking the scalar product with N to (2.11) and using $(3.7)_2$, we have

$$\bar{g}(\bar{R}(X,Y)PZ,N)=\bar{g}(R(X,Y)PZ,N).$$

Substituting (2.13) and (5.1) into the last equation and using (3.2), we get

$$\begin{aligned}(5.3)\qquad &(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)\\ &-\ \tau(X)C(Y,PZ)+\tau(Y)C(X,PZ)\\ &-\ \theta(X)C(FY,PZ)+\theta(Y)C(FX,PZ)\\ &=f_1\{g(Y,PZ)\eta(X)-g(X,PZ)\eta(Y)\}\\ &+\ f_2\{v(Y)\bar{g}(X,JPZ)-v(X)\bar{g}(Y,JPZ)+2v(PZ)\bar{g}(X,JY)\}\\ &+\ f_3\{\theta(X)\eta(Y)-\theta(Y)\eta(X)\}\theta(PZ).\end{aligned}$$

Theorem 5.1. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a non-metric ϕ-symmetric connection. Then α is a constant, $\beta=0$, and the functions f_1, f_2 and f_3 are satisfied*

$$f_2=f_3=f_1-\alpha(\alpha-1).$$

Proof. Applying ∇_Y to (3.11) and using (3.12), (3.13) and $(3.18)_2$, we get

$$\begin{aligned}(\nabla_X B)(Y,U)=&(\nabla_X C)(Y,V)\\ &-2\tau(X)C(Y,V)-\alpha(\alpha-1)u(Y)\eta(X)\\ &-g(A^*_\xi X,F(A_N Y))-g(A^*_\xi Y,F(A_N X)).\end{aligned}$$

Substituting this equation and (3.11) into (5.2) with $Z=U$, we get

$$\begin{aligned}&(\nabla_X C)(Y,V)-(\nabla_Y C)(X,V)\\ &-\ \tau(X)C(Y,V)+\tau(Y)C(X,V)\\ &-\ \theta(X)C(FY,V)+\theta(Y)C(FX,V)\\ &-\ \alpha(\alpha-1)\{u(Y)\eta(X)-u(X)\eta(Y)\}\\ &=f_2\{u(Y)\eta(X)-u(X)\eta(Y)+2\bar{g}(X,JY)\}.\end{aligned}$$

Comparing this equation with (5.3) such that $PZ=V$, we obtain

$$\{f_1-f_2-\alpha(\alpha-1)\}\{u(X)\eta(Y)-u(Y)\eta(X)\}=0.$$

Taking $X=U$ and $Y=\xi$ to this equation, we have

$$f_1-f_2=\alpha(\alpha-1).$$

Applying ∇_Y to (3.17) and using (3.7), (3.16) and $(3.18)_1$, we have

$$(\nabla_X C)(Y,\zeta) = -(X\alpha)v(Y) + (\alpha-1)\{g(A_{_N}X,FY) + g(A_{_N}Y,FX) - v(Y)\tau(X) + (\alpha-1)\theta(Y)\eta(X)\}.$$

Substituting this and (3.17) into (5.3) with $PZ=\zeta$, we get

$$\{f_1 - f_3 - \alpha(\alpha-1)\}\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\} = (X\alpha)v(Y) - (Y\alpha)v(X).$$

Taking $X=\zeta$, $Y=\xi$ and $X=U$, $Y=V$ to this by turns, we have

$$f_1 - f_3 = \alpha(\alpha-1), \qquad U\alpha = 0.$$

Applying ∇_Y to $(3.18)_2$ and using (3.15) and $(3.18)_1$, we have

$$(\nabla_X B)(Y,\zeta) = -(X\alpha)u(Y) + \alpha\{u(Y)\tau(X) + B(X,FY) + B(Y,FX)\}.$$

Substituting this into (5.2) such that $Z=\zeta$ and using $(3.18)_2$, we have

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Taking $Y=U$ to this result and using the fact that $U\alpha=0$, we have $X\alpha=0$. Therefore α is a constant. This completes the proof of the theorem. □

Definition. (1) A lightlike hypersurface M is called *totally umbilical* [5] if there exists a smooth function ρ on a coordinate neighborhood $\mathcal{U}$ such that

$$B(X,Y) = \rho g(X,Y).$$

In case $\rho = 0$, we say that M is *totally geodesic*.

(2) A screen distribution $S(TM)$ is called *totally umbilical* [5] in M if there exists a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X,PY) = \gamma g(X,Y).$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(3) A lightlike hypersurface M is called *screen conformal* [6] if there exists a non-vanishing smooth function φ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X,PY) = \varphi B(X,Y).$$

Theorem 5.2. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with a non-metric ϕ-symmetric connection. If one of the following three conditions is satisfied,*

(1) *M is totally umbilical,*
(2) *$S(TM)$ is totally umbilical, and*
(3) *M is screen conformal,*

then $\bar{M}(f_1,f_2,f_3)$ is a flat indefinite Sasakian manifold, that is,

$$\alpha = 1,\ \beta = 0; \qquad f_1 = f_2 = f_3 = 0.$$

Proof. (1) If M is totally umbilical, then $(3.18)_2$ is reduced to

$$\rho\theta(X) = -(\alpha - 1)u(X).$$

Taking $X = \zeta$ and $X = U$ by turns, we have $\rho = 0$ and $\alpha = 1$ respectively. As $\rho = 0$, M is totally geodesic. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and f_1, f_2 and f_3 are satisfied $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking $Z = U$ to (5.2) and using the fact that $B = 0$, we have

$$f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\} = 0.$$

Taking $X = \xi$ and $Y = U$ to this equation, we get $f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(2) If $S(TM)$ is totally umbilical, then (3.17) is reduced to

$$\gamma\theta(X) = -(\alpha - 1)v(X).$$

Taking $X = \zeta$ and $X = V$ by turns, we have $\gamma = 0$ and $\alpha = 1$ respectively. As $\gamma = 0$, $S(TM)$ is totally geodesic in M. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking $PZ = V$ to (5.3) and using the fact that $C = 0$, we have

$$f_1\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2f_2\,\bar{g}(X, JY) = 0.$$

Taking $X = \xi$ and $Y = U$ to this equation, we obtain $f_1 + 2f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(3) If M is screen conformal, then, from (3.17) and $(3.18)_2$, we have

$$(\alpha - 1)\{v(X) - \varphi u(X)\} = 0.$$

Taking $X = V$ to this equation, we have $\alpha = 1$. As $\alpha = 1$ and $\beta = 0$, $\bar{M}$ is an indefinite Sasakian manifold and $f_1 = f_2 = f_3$ by Theorem 5.1.

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (5.3) and using (5.2), we have

$$\begin{aligned}
&\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)\\
&= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}\\
&\quad + f_2\{[v(Y) - \varphi u(Y)]\bar{g}(X, JPZ) - [v(X) - \varphi u(X)]\bar{g}(Y, JPZ)\\
&\qquad + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X, JY)\}\\
&\quad + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\end{aligned}$$

Replacing Y by ξ to the last equation and using (3.5), we obtain

$$\begin{aligned}
&\{\xi\varphi - 2\varphi\tau(\xi)\}B(X, PZ)\\
&= f_1 g(X, PZ) + f_2\{v(X) - \varphi u(X)\}u(PZ)\\
&\quad + 2f_2\{v(PZ) - \varphi u(PZ)\}u(X) - f_3\theta(X)\theta(PZ).
\end{aligned}$$

Taking $X = V,\ PZ = U$ and then, $X = U,\ PZ = V$ by turns, we have

$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(V, U) = f_1 + f_2,$$

$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(U,V) = f_1 + 2f_2,$$

respectively. As $B(U,V) = B(V,U)$ by (3.3), from the last two equations we show that $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. □

Theorem 5.3. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a non-metric ϕ-symmetric connection. If one of the following three conditions is satisfied,*

(1) *V is parallel with respect to the induced connection ∇,*
(2) *U is parallel with respect to the induced connection ∇, and*
(3) *M is Lie recurrent,*

then $\bar{M}(f_1, f_2, f_3)$ is a flat indefinite cosymplectic manifold, i.e.,

$$\alpha = \beta = 0, \qquad f_1 = f_2 = f_3 = 0.$$

Proof. (1) If V is parallel with respect to ∇, then, by (1) of Theorem 4.4, we have (4.9) and $\tau = \alpha = \beta = 0$. As $\alpha = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking the scalar product with U to (4.9) and using (3.6) and (3.11), we get

$$C(X, V) = 0.$$

Applying ∇_X to $C(Y, V) = 0$ and using the fact that V is parallel, we obtain

$$(\nabla_X C)(Y, V) = 0.$$

Substituting the last two equations into (5.3) such that $PZ = V$, we have

$$f_1\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2f_2\bar{g}(X, JY) = 0.$$

Taking $X = \xi$ and $Y = U$, we obtain $f_1 + 2f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$.

(2) If U is parallel with respect to ∇, then, by (2) of Theorem 4.4, we have (4.10) and $\tau = \alpha = \beta = 0$. As $\alpha = 0$, $f_1 = f_2 = f_3$ by Theorem 5.1.

Taking the scalar product with U to (4.10) and using (3.7), we get

$$C(X, U) = 0.$$

Applying ∇_X to $C(Y, U) = 0$ and using the fact that U is parallel, we obtain

$$(\nabla_X C)(Y, U) = 0.$$

Substituting the last two equations into (5.3) such that $PZ = U$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$, we obtain $f_1 + f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$.

(3) As $\alpha = 0$, we get $f_1 = f_2 = f_3$. As $\tau(FX) = 0$, from (4.6), we have

$$B(X, U) = 0, \qquad B(U, X) = \theta(X).$$

Applying ∇_Y to the first equation and using (3.12), we have

$$(\nabla_X B)(Y, U) = -B(Y, F(A_N X)).$$

Substituting the last two equations into (5.2) such that $Z = U$, we have

$$B(X, F(A_N Y)) - B(Y, F(A_N X))$$

$$= f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Taking $X = \xi$ and $Y = U$ to this, we obtain $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$. $\square$

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