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MULTIPLE SOLUTIONS FOR A CLASS OF QUASILINEAR SCHRÖDINGER SYSTEM IN $\mathbb{R}^N$

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Abstract. This paper is concerned with the quasilinear Schrödinger system

$$(0.1)\qquad \begin{cases} -\Delta u + a(x)u - \Delta(u^2)u = F_u(u,v) + h(x) & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v - \Delta(v^2)v = F_v(u,v) + g(x) & x \in \mathbb{R}^N, \end{cases}$$

where $N \geq 3$. The potential functions $a(x), b(x) \in L^\infty(\mathbb{R}^N)$ are bounded in $\mathbb{R}^N$. By using mountain pass theorem and the Ekeland variational principle, we prove that there are at least two solutions to system (0.1).

1. Introduction and main result

In this paper, we are interested in the existence of solutions for the quasilinear Schrödinger system

$$(1.1)\qquad \begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h_1(x,u,v), & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v - \Delta(v^2)v = h_2(x,u,v), & x \in \mathbb{R}^N. \end{cases}$$

The system is related to the existence of solitary wave solutions for quasilinear Schrödinger equation

$$(1.2)\qquad iz_t = -\Delta z + W(x)z - h(|z|^2)z - \kappa\Delta(l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N,$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is a real constant and l, h are real functions. Quasilinear equations of the form (1.2) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of nonlinear terms l. For instance, when $l(s) = s$, Eq. (1.2) can be used to model a superfluid film equation in plasma physics (see Kurihura [9]). In the case $l(s) = \sqrt{1+s}$, Eq. (1.2) models the self-channeling of a high-power ultra short laser in matter

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(see [10]). For more mathematical models in physics described by (1.2), see [14] and the references therein.

As we know, that the solitary wave solution of the form $z(t,x) = e^{-i\omega t}u(x)$ satisfies (1.2) with $l(s) = s$ if and only if the function $u(x)$ solves the equation of elliptic type

$$-\Delta u + V(x)u - \kappa\Delta(u^2)u = \theta(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - \omega$, $\omega \in \mathbb{R}$ and $\theta(u) = l(u^2)u$. Without loss of generality we assume $\kappa = 1$.

In the last decades, Eq. (1.3) has received great interest and there are recent mathematical studies in the existence of solutions for (1.3). Among others we refer to [6, 7, 11, 13] and the references therein.

There are also several papers concerned with the quasilinear Schrödinger system (1.1). Guo and Tang in [8] have studied with $h_1(x,u,v) = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u$ and $h_2(x,u,v) = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v$ and the potentials $V_1(x) = \lambda a(x) + 1$ and $V_2(x) = \lambda b(x) + 1$, where λ is a positive parameter. By using Nehari manifold method and concentration compactness principle, they proved that there exists a ground state solution which localize near the potential well $int\{a^{-1}(0)\} = int\{b^{-1}(0)\}$ for λ sufficiently large. Severo and Silva in [15] employed minimax methods in a suitable Orlicz space to establish the existence of standing wave solutions for the quasilinear Schrödinger system (1.1) involving subcritical nonlinearities, their conclusion is under the assumptions on the potentials V_1 and V_2:

($\mathcal{V}_1$) The functions $V_1, V_2 : \mathbb{R}^N \to \mathbb{R}$ are continuous and satisfy

$$\inf_{\mathbb{R}^N} V_1 > 0, \quad \inf_{\mathbb{R}^N} V_2 > 0;$$

($\mathcal{V}_2$) There exists $M_0 > 0$ such that for all $M \geq M_0$

$$meas(\{x \in \mathbb{R}^N | V_i(x) \leq M\}) < \infty, \ \ i = 1, 2.$$

The assumption ($\mathcal{V}_2$) is also essential in [8] and it guarantees the compactness of the embedding $\mathcal{X} \hookrightarrow [L^s(\mathbb{R}^N)]^2$ (see Lemma 2.2 in [15]). But in this paper, we will prove the existence of solutions for quasilinear Schrödinger equations with general bounded potential. As the domain is the whole space $\mathbb{R}^N$, a main difficulty when dealing with this problem is the lack of compactness of Sobolev embedding theorem. So, motivated by Alves and Souto [1], we develop a new technique to verify the Cerami condition and then prove the existence of multiple solutions by mountain pass theorem and the Ekeland variational principle.

In this work, we study the following quasilinear Schrödinger system

$$\begin{cases} -\Delta u + a(x)u - \Delta(u^2)u = F_u(u,v) + h(x) & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v - \Delta(v^2)v = F_v(u,v) + g(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N \geq 3$.

Throughout this paper, we make the following assumptions:

(H_1) The functions $a(x), b(x) \in C(\mathbb{R}^N)$ and satisfy $a_0 \le a(x) \le a_1$, $b_0 \le b(x) \le b_1$ in $\mathbb{R}^N$ for some positive constants a_0, b_0, a_1, b_1.

(H_2) The nonnegative function $F(u,v) \in C^1(\mathbb{R}^2)$ is positively homogeneous of degree $d \in (4, 2\cdot 2^*)$ where $2^* = \frac{2N}{N-2}$, that is, $F(tu,tv) = t^d F(u,v)$ $(t>0)$ for any $(u,v) \in \mathbb{R}^2$. Also, assume $F_u(u,v), F_v(u,v)$ are increasing function about u, v. Furthermore, there exists the constant $c_0 > 0$ such that for any $(u,v) \in \mathbb{R}^2$

$$0 \le F(u,v), F_u(u,v)u, F_v(u,v)v \le c_0(|u|^d + |v|^d). \tag{1.5}$$

(H_3) $h, g \in L^\infty_{loc}(\mathbb{R}^N) \cap L^\mu(\mathbb{R}^N)$ with $\mu = \frac{2\cdot 2^*}{2\cdot 2^* - 1}$.

Remark 1.1. By the assumption (H_1), we have the so-called Euler identity

$$F_u(u,v)u + F_v(u,v)v = dF(u,v),\ \forall (u,v) \in \mathbb{R}^2.$$

Obviously, the function $F(u,v) = |u|^\alpha |u|^\beta (\alpha, \beta > 1, \alpha + \beta = d)$ and $F(u,v) = (u^2+v^2)^{d/2}$ satisfy (H_2).

Remark 1.2. For convenience, we assume that $a_0 = b_0 = 1$ in (H_1).

By (H_1) the norm for $X = Y = H^1(\mathbb{R}^N)$ can be defined by

$$\|u\|_X = (\int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)|u|^2)dx)^{1/2} \ \ \forall u \in X, \tag{1.6}$$

and

$$\|u\|_Y = (\int_{\mathbb{R}^N} (|\nabla u|^2 + b(x)|u|^2)dx)^{1/2} \ \ \forall u \in Y. \tag{1.7}$$

For the product space $E = X \times Y$, we introduce the norm

$$\|(u,v)\|_E = \|u\|_X + \|v\|_X, \ \ \forall (u,v) \in E. \tag{1.8}$$

Then E is the reflexive Banach space endowed with the norm $\|(u,v)\|_E$.

It is well known that there is a constant $S > 0$ such that

$$S(\int_{\mathbb{R}^N} |u|^{2^*})^{2/2^*} dx \le \int_{\mathbb{R}^N} |\nabla u|^2 dx,\ \forall u \in C_0^\infty(\mathbb{R}^N). \tag{1.9}$$

From the approximation argument, we see that (1.9) holds on $H^1(\mathbb{R}^N)$.

Definition 1.1. A pair of functions $(u,v) \in E$ is said to be a weak solution of problem (1.4) if for any $(\varphi, \psi) \in E$ there holds

$$\begin{aligned} &\int_{\mathbb{R}^N} [(1+2u^2)\nabla u \nabla \varphi + 2u|\nabla u|^2 \varphi + (1+2v^2)\nabla v \nabla \psi \\ &\qquad + 2v|\nabla v|^2 \psi + a(x)u\varphi + b(x)v\psi]dx \\ &= \int_{\mathbb{R}^N} (F_u(u,v)\varphi + F_v(u,v)\psi)dx + \int_{\mathbb{R}^N} (h(x)\varphi + g(x)\psi)dx. \end{aligned} \tag{1.10}$$

Our main result in this paper is as follows.

Theorem 1.3. *Let* (H_1)-(H_3) *hold. Then there exists* $m_0 > 0$ *such that for* $0 < \|h\|_\mu + \|g\|_\mu \le m_0$, *problem* (1.4) *admits at least two solutions in* E.

Remark 1.4. From the proof of Theorem 1.3 below, we found that it is admissible to suppose that one of the functions $h(x)$ and $g(x)$ is zero in $\mathbb{R}^N$.

This paper is organized as follows: In the forthcoming section, with a convenient change of variable, we establish the variational framework for problem (1.4). In Section 3, we verify that the energy functional associated to problem (1.4) satisfies the Cerami condition. Section 4 is devoted to the proof of Theorem 1.3 by using mountain pass theorem and the Ekeland variational principle.

2. Preliminaries

We observe that the natural energy functional associated to problem (1.4) is given by

$$\begin{aligned} I(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N}[(1+2u^2)|\nabla u|^2 + (1+2v^2)|\nabla v|^2 + a(x)u^2 + b(x)v^2]dx \\ &- \int_{\mathbb{R}^N} F(u,v)dx - \int_{\mathbb{R}^N}(h(x)u + g(x)v)dx. \end{aligned} \tag{2.1}$$

It should be pointed out that the functional I is not well defined in general in E. To overcome this difficulty, we apply an argument developed by [6] (see also [12]). we make the change of variables by $u = f(z), v = f(w)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2|f(t)|^2}} \quad \text{on } [0,+\infty) \quad \text{and} \quad f(-t) = -f(t) \quad \text{on } (-\infty,0). \tag{2.2}$$

Let us collect some properties of the change of variables f, which will be used frequently in the sequel of the paper. Proofs may be found in [6] and [4].

Lemma 2.1. *The function* $f(t)$ *satisfies the following properties*:

(f_1) f *is uniquely defined, odd, increasing and invertible*;
(f_2) $0 < f'(t) \le 1, \forall t \in \mathbb{R}$;
(f_3) $|f(t)| \le |t|, \forall t \in \mathbb{R}$;
(f_4) $f(t)/t \to 1$ *as* $t \to 0$;
(f_5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to \infty$;
(f_6) $f(t)/2 \le tf'(t) \le f(t), \forall t > 0$;
(f_7) $|f(t)| \le 2^{1/4}|t|^{1/2}, \forall t \in \mathbb{R}$;
(f_8) *There exists a positive constant* C *such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{1/2}, & |t| \ge 1; \end{cases}$$

(f_9) *For* $n < \tau \le n+1 (n \in \mathbb{N})$, $t \in \mathbb{R}$, $|f(\tau t)| \le (n+1)|f(t)|$.

So, after the change of variables, we can write $I(u,v)$ as

$$
\begin{aligned}
J(z,w) &= I(f(z),f(w)) \\
&= \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla z|^2+|\nabla w|^2+a(x)f^2(z)+b(x)f^2(w)]dx \\
&\quad -\int_{\mathbb{R}^N}F(f(z),f(w))dx-\int_{\mathbb{R}^N}(h(x)f(z)+g(x)f(w))dx,
\end{aligned}
\tag{2.3}
$$

which is well defined on the space E under the assumptions (H_1)-(H_3). Our hypotheses imply that $J\in C^1(E,\mathbb{R})$, and

$$
\begin{aligned}
&J'(z,w)(\varphi,\psi) \\
&= \int_{\mathbb{R}^N}[\nabla z\nabla\varphi+\nabla w\nabla\psi+a(x)f(z)f'(z)\varphi+b(x)f(w)f'(w)\varphi]dx \\
&\quad -\int_{\mathbb{R}^N}[F_uf'(z)\varphi+F_vf'(w)\psi]dx-\int_{\mathbb{R}^N}[h(x)f'(z)\varphi+g(x)f'(w)\psi]dx
\end{aligned}
\tag{2.4}
$$

for any $(\varphi,\psi)\in E$. Moreover, the critical points of the functional J are the weak solutions of the following equations:

$$
\begin{cases}
-\Delta z+a(x)f(z)f'(z)=F_u(f(z),f(w))f'(z)+h(x)f'(z), & x\in\mathbb{R}^N,\\
-\Delta w+b(x)f(w)f'(w)=F_v(f(z),f(w))f'(w)+h(x)f'(w), & x\in\mathbb{R}^N,
\end{cases}
\tag{2.5}
$$

and $(f(z),f(w))$ is a weak solution of (1.4).

Remark 2.2. Using (f_7), we see from Hölder inequality and (1.9) that, for any measurable region $\Omega\subset\mathbb{R}^N$ and $(z,w)\in E$, there are constants $C_h,C_g>0$ such that

$$
\int_\Omega|hf(z)|dx\le C_h\|h\|_{L^\mu(\Omega)}\|z\|_X^{1/2},\quad \int_\Omega|gf(w)|dx\le C_g\|g\|_{L^\mu(\Omega)}\|w\|_Y^{1/2}.
\tag{2.6}
$$

To obtain the existence of solutions to problem (1.4), we need to prove that the functional J defined by (2.3) satisfies the Cerami condition.

We first recall that a sequence $\{(z_n,w_n)\}$ in E is called Cerami sequence of J if $\{J(z_n,w_n)\}$ is bounded and

$$
(1+\|(z_n,w_n)\|_E)\|J'(z_n,w_n)\|_{E^*}\to 0 \quad\text{as}\quad n\to\infty.
\tag{2.7}
$$

The functional J satisfies the Cerami condition if any Cerami sequence possesses a convergent subsequence in E.

Lemma 2.3. *Assume (H_1)-(H_3). If $\{(z_n,w_n)\}\subset E$ is a Cerami sequence, then $\{(z_n,w_n)\}$ is bounded in E.*

Proof. Set $\varphi_n=\frac{f(z_n)}{f'(z_n)}$, $\psi_n=\frac{f(w_n)}{f'(w_n)}$. Then, using (f_2) and (f_6) in Lemma 2.1, we have

$$
|\varphi_n|\le 2|z_n|,\ |\psi_n|\le 2|w_n|,\ |\nabla\varphi_n|\le 2|\nabla z_n|,\ |\nabla\psi_n|\le 2|\nabla w_n|.
$$

So we get

$$
\begin{aligned}
&J'(z_n,w_n)(\varphi_n,\psi_n)\\
&\le 2\int_{\mathbb{R}^N}[|\nabla z_n|^2+|\nabla w_n|^2+a(x)f^2(z_n)+b(x)f^2(w_n)]dx \qquad (2.8)\\
&\quad -\int_{\mathbb{R}^N}[F_u f(z_n)+F_v f(w_n)]dx-\int_{\mathbb{R}^N}[h(x)f(z_n)+g(x)f(w_n)]dx.
\end{aligned}
$$

Since $\{(z_n,w_n)\}$ is a Cerami sequence in E, there is a constant $C_1>0$ such that

$$
\begin{aligned}
C_1 &\ge J(z_n,w_n)-\frac{1}{d}J'(z_n,w_n)(\varphi_n,\psi_n)\\
&\ge (\frac{1}{2}-\frac{2}{d})\int_{\mathbb{R}^N}[|\nabla z_n|^2+|\nabla w_n|^2+a(x)f^2(z_n)+b(x)f^2(w_n)]dx \qquad (2.9)\\
&\quad +(1-\frac{1}{d})\int_{\mathbb{R}^N}[h(x)f(z_n)+g(x)f(w_n)]dx.
\end{aligned}
$$

Set $A_n^2=\int_{\mathbb{R}^N}[|\nabla z_n|^2+|\nabla w_n|^2+a(x)f^2(z_n)+b(x)f^2(w_n)]dx$. Then as in Remark 2.2, we can obtain that, there is a constant $C_2>0$ such that

$$
\int_{\mathbb{R}^N}|h(x)f(z_n)+g(x)f(w_n)|dx\le C_2A_n^{1/2}. \qquad (2.10)
$$

So $\{A_n^2\}$ is bounded. As Chen did in [4], we can obtain that there is a constant $C_0>0$ such that

$$
A_n\ge C_0\|(z_n,w_n)\|_E. \qquad (2.11)
$$

So $\{(z_n,w_n)\}$ is bounded in E. □

Since the sequence $\{(z_n,w_n)\}$ given by Lemma 3.1 is a bounded sequence in E, there exist a constant $M>0$ and $(z,w)\in E$, and a subsequence of $\{(z_n,w_n)\}$, still denoted by $\{(z_n,w_n)\}$, such that $\|(z_n,w_n)\|_E\le M$, $\|(z,w)\|_E \le M$ and

(2.12)

$$
\begin{aligned}
&(z_n,w_n)\rightharpoonup(z,w)\text{ weakly in }E,\ z_n(x)\to z(x),\ w_n(x)\to w(x)\text{ a.e. in }\mathbb{R}^N,\\
&(z_n,w_n)\to(z,w)\text{ in }L^r_{loc}(\mathbb{R}^N)\times L^s_{loc}(\mathbb{R}^N),\ \forall\, r,s\in[1,2^*).
\end{aligned}
$$

Lemma 2.4. *Let* (H_1)-(H_3) *hold. If the sequence* $\{(z_n,w_n)\}$ *satisfies* (2.12), *then*

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}F(f(z_n),f(w_n))=\int_{\mathbb{R}^N}F(f(z),f(w)), \qquad (2.13)
$$

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}F_u(f(z_n),f(w_n))f'(z_n)z_n=\int_{\mathbb{R}^N}F_u(f(z),f(w))f'(z)z, \qquad (2.14)
$$

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}F_v(f(z_n),f(w_n))f'(w_n)w_n=\int_{\mathbb{R}^N}F_v(f(z),f(w))f'(w)w. \qquad (2.15)
$$

Proof. From (2.12), one sees that $z_n \to z$, $w_n \to w$ in $L^{d/2}(B_r)$ for any $r > 0$. Then from (1.5) we know

$$\lim_{n\to\infty} \int_{B_r} F(f(z_n), f(w_n)) = \int_{B_r} F(f(z), f(w)). \tag{2.16}$$

In the following we prove that, for any small $\epsilon > 0$, there exists $r_0 > 0$ such that for any $n \in \mathbb{N}$ and $r \geq r_0$,

$$\int_{B_r^c} F(f(z_n), f(w_n)) \leq \epsilon, \quad \int_{B_r^c} F(f(z), f(w)) \leq \epsilon. \tag{2.17}$$

By (H_2) and (f_7), for any small $\varepsilon > 0$, there exist $S_0 > s_0 > 0$ such that

$$|F(f(z), f(w))| \leq \begin{cases} \varepsilon|(z,w)|^2 & \text{if } |(z,w)| \leq s_0, \\ \varepsilon|(z,w)|^{2^*} & \text{if } |(z,w)| \geq S_0, \end{cases} \tag{2.18}$$

where $(z, w) \in \mathbb{R}^2$ and $|(z, w)| = \sqrt{z^2 + w^2}$. This shows that

$$\begin{aligned} &|F(f(z), f(w))| \\ &\leq \varepsilon(|(z,w)|^2 + |(z,w)|^{2^*}) + \chi_{[s_0,S_0]}(\sqrt{z^2+w^2})|F(f(z), f(w))|, \ \forall (u,v) \in \mathbb{R}^2, \end{aligned} \tag{2.19}$$

where χ_A denotes the characteristic function associated to the mensurable subset $A \subset R$. We then obtain

$$\begin{aligned} &\int_{B_r^c} |F(f(z_n), f(w_n))|dx \\ &\leq \varepsilon Q(z_n, w_n) + \beta_0 \int_{A_n \bigcap B_r^c} \chi_{[s_0,S_0]}(\sqrt{z^2+w^2})dx, \ \forall n \in \mathbb{N}, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} A_n &= \{x \in \mathbb{R}^N | s_0 \leq |(z_n, w_n)| \leq S_0\}, \\ \beta_0 &= \max_{s_0 \leq |(z_n,w_n)| \leq S_0} |F(f(z_n), f(w_n))| \end{aligned}$$

and

$$Q(z_n, w_n) = \int_{\mathbb{R}^N} (|(z_n, w_n)|^2 + |(z_n, w_n)|^{2^*})dx \quad \forall n \in \mathbb{N}. \tag{2.21}$$

From (H_2), we know $0 < \beta_0 \leq c_0 S_0^d$. It follows from (2.12) that there exists a constant $M_1 > 0$ such that

$$Q(z_n, w_n) \leq \int_{\mathbb{R}^N} (z_n^2 + w_n^2 + 2^{2^*-1}(|z_n|^{2^*} + |w_n|^{2^*}))dx \leq M_1. \tag{2.22}$$

Then we have

$$s_0^2|A_n| \leq \int_{A_n} |(z_n, w_n)|^2 dx \leq Q(z_n, w_n) \leq M_1, \tag{2.23}$$

where $|A_n| = meas(A_n)$. Relation (2.23) shows that $\sup_{n\in\mathbb{N}} |A_n| \le s_0^{-2} M_1 \equiv \rho < \infty$. We now assert that $\lim_{r\to\infty} |A_n \cap B_r^c| = 0$ uniformly in $n \in \mathbb{N}$. To begin with, we show that

$$\lim_{r\to\infty} |A_n \cap B_r^c| = 0 \quad \text{for all } n \in \mathbb{N}. \tag{2.24}$$

In fact, if the assertion is not true, then there exist $n_0 \ge 1$, $\delta > 0$, and $r_j \to \infty$ such that

$$|A_{n_0} \cap B_{r_j}^c| \ge \delta, \qquad \forall j \in \mathbb{N}. \tag{2.25}$$

Clearly, $|A_{n_0} \cap B_{r_j}^c| \le |A_{n_0}| \le \rho$, $\forall j \in \mathbb{N}$.

Denote $\Omega_j = B_{r_j}^c \setminus \overline{B_{r_{j+1}}^c}$, $\forall j \in \mathbb{N}$. It is easy to see that

$$B_{r_j}^c = \bigcup_{k=j}^{\infty} \Omega_k,\ \forall j \in \mathbb{N}, \quad \Omega_k \cap \Omega_m = \emptyset,\ \text{if } k \ne m. \tag{2.26}$$

Thus we have

$$|A_{n_0} \cap B_{r_j}^c| = \sum_{k=j}^{\infty} |A_{n_0} \cap \Omega_k| \ge \delta, \quad \forall j \in \mathbb{N}$$

and the series $\sum_{k=1}^{\infty} |A_{n_0} \cap \Omega_k| = \infty$. This is a contradiction. Thus, the limit (2.24) is proved. In the following, we show that this limit is true uniformly in $n \in \mathbb{N}$.

In fact, it follows from (2.12) that $(z, w) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and $(z_n(x), w_n(x)) \to (z(x), w(x))$ a.e. in $\mathbb{R}^N$. Therefore, for any small $\varepsilon > 0$, there exists $r_0 > 1$ such that $r > r_0$,

$$\int_{B_r^c} (|z|^2 + |w|^2) dx \le \varepsilon. \tag{2.27}$$

For this $\varepsilon > 0$, we choose $t_1 = r_0, t_j \to \infty$ such that $D_j = B_{t_j}^c \setminus \overline{B_{t_{j+1}}^c}$, $B_{r_0}^c = \cup_{j=1}^{\infty} D_j$ and

$$\int_{D_j} (|z|^2 + |w|^2) dx \le \frac{\varepsilon}{2^j}, \quad \forall j \in \mathbb{N}. \tag{2.28}$$

Obviously, for every fixed $j \in \mathbb{N}$, D_j is a bounded domain and $D_j \cap D_i = \emptyset$ $(j \ne i)$. Furthermore, $s_0 \le |(z_n, w_n)| \le S_0$ in $D_j \cap A_n$. By Fatou's lemma, we have for every $j \in \mathbb{N}$,

$$\begin{aligned} \limsup_{n\to\infty} \int_{D_j \cap A_n} (|z_n|^2 + |w_n|^2) dx &\le \int_{D_j} \limsup_{n\to\infty} (|z_n|^2 + |w_n|^2) dx \\ &\le \int_{D_j} (|z|^2 + |w|^2) dx \le \frac{\varepsilon}{2^j}. \end{aligned} \tag{2.29}$$

Then we obtain

$$
\begin{aligned}
s_0^2 \limsup_{n\to\infty} |A_n \cap B_{r_0}^c| &\le \limsup_{n\to\infty} \int_{B_{r_0}^c \cap A_n} (|z_n|^2 + |w_n|^2)dx \\
&= \limsup_{n\to\infty} \sum_{j=1}^{\infty} \int_{D_j \cap A_n} (|z_n|^2 + |w_n|^2)dx \\
&\le \sum_{j=1}^{\infty} \limsup_{n\to\infty} \int_{D_j \cap A_n} (|z_n|^2 + |w_n|^2)dx \\
&\le \sum_{j=1}^{\infty} \int_{D_j} (|z|^2 + |w|^2)dx \\
&\le \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.
\end{aligned}
\tag{2.30}
$$

Noticing that for any $r > r_0$ and $n \in \mathbb{N}$, we have $(A_n \cap B_r^c) \subset (A_n \cap B_{r_0}^c)$. Therefore, the application of (2.24) and (2.30) yields $\lim_{r\to\infty} |A_n \cap B_r^c| = 0$ uniformly in $n \in \mathbb{N}$.

Then, for each $\varepsilon > 0$, there exist the constants $r_0 > 1$ and $0 < \delta_0 \le \frac{\varepsilon}{\beta_0}$, such that $|A_n \cap B_r^c| < \delta_0$ for any $r > r_0$ and $n \in \mathbb{N}$, and

$$
\int_{A_n \bigcap B_r^c} \chi_{[s_0, S_0]}(\sqrt{z^2 + w^2})dx \le |A_n \cap B_r^c| < \delta_0 \le \frac{\varepsilon}{\beta_0}, \quad \forall n \in \mathbb{N}. \tag{2.31}
$$

Then from (2.20) and the fact $Q(z_n, w_n) \le M_1$, it yields that

$$
\int_{B_r^c} F(f(z_n), f(w_n))dx \le \varepsilon(M_1 + 1), \quad \forall n \in \mathbb{N},\ r > r_0, \tag{2.32}
$$

By Fatou's lemma, we have for every $r > r_0$,

$$
\int_{B_r^c} F(f(z), f(w))dx \le \liminf_{n\to\infty} \int_{B_r^c} F(f(z_n), f(w_n))dx \le \varepsilon(M_1 + 1). \tag{2.33}
$$

Therefore, we get (2.17) from (2.32) and (2.33). Then the application of (2.16) and (2.17) yields the limit (2.13). Note that (f_6) and

$$
0 \le F_u(f(z_n), f(w_n))f'(z_n)z_n, F_v(f(z_n), f(w_n))f'(w_n)w_n \le dF(f(z_n), f(w_n)),
$$

then, arguing as the above, we can conclude the limits (2.14) and (2.15). $\square$

Lemma 2.5. *Let (H_1)-(H_3) hold. If the sequence $\{(z_n, w_n)\}$ satisfies* (2.12), *then the following statements hold*

(i) *For any $\varepsilon > 0$, there exists $r_0 \ge 1$ such that $r \ge r_0$,*

$$
\limsup_{n\to\infty} \int_{B_{2r}^c} [|\nabla z_n|^2 + |\nabla w_n|^2 + a(x)f^2(z_n) + b(x)f^2(w_n)]dx < \varepsilon. \tag{2.34}
$$

(ii)

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)f(z_n)f'(z_n)z_n dx = \int_{\mathbb{R}^N} a(x)f(z)f'(z)zdx, \tag{2.35}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)f(w_n)f'(w_n)w_n dx = \int_{\mathbb{R}^N} b(x)f(w)f'(w)wdx, \tag{2.36}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)f^2(z_n-z)dx = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)f^2(w_n-w)dx = 0. \tag{2.37}$$

(iii) *The weak limit $(z,w)\in E$ is a critical point for functional J.*

Proof. (i) For $r>1$, we choose the function $\eta_r=\eta_r(|x|)\in C^1(\mathbb{R}^N)$ such that

(2.38)

$$\eta_r(|x|)\equiv 1,\ x\in B_{2r}^c,\ \eta_r(|x|)\equiv 0,\ x\in B_r \text{ and } 0\le\eta_r\le 1,\ |\nabla\eta_r|\le\frac{2}{r} \text{ in } \mathbb{R}^N.$$

Since the sequence $\{(z_n,w_n)\}$ is bounded in E, the sequence $\{(\eta_r\varphi_n,\eta_r\psi_n)\}$ is also bounded in E, where φ_n,ψ_n are defined as in Lemma 3.1. Hence we have $J'(z_n,w_n)(\eta_r\varphi_n,\eta_r\psi_n)=o_n(1)$ as $n\to\infty$, where

$$\begin{aligned} &J'(z_n,w_n)(\eta_r\varphi_n,\eta_r\psi_n)\\ &= \int_{\mathbb{R}^N}[\nabla z_n\nabla\varphi_n+\nabla w_n\nabla\psi_n+a(x)f^2(z_n)+b(x)f^2(w_n)]\eta_r dx\\ &\quad + A_n(r)+B_n(r)+C_n(r) \end{aligned} \tag{2.39}$$

with

$$\begin{aligned} A_n(r) &= -\frac{1}{d}\int_{\mathbb{R}^N}F(f(z_n),f(w_n))\eta_r dx,\\ B_n(r) &= \int_{\mathbb{R}^N}(\nabla z_n\nabla\eta_r\varphi_n+\nabla w_n\nabla\eta_r\psi_n)dx,\\ C_n(r) &= -\int_{\mathbb{R}^N}(h(x)f(z_n)+g(x)f(w_n))\eta_r dx. \end{aligned} \tag{2.40}$$

Then it follows from (2.17) that

$$\lim_{r\to\infty}A_n(r)=0 \text{ uniformly in } n\in\mathbb{N}. \tag{2.41}$$

Similarly, for any $n\in\mathbb{N}$, we have

$$\begin{aligned} |B_n(r)| &\le \int_{\Omega_r}(|\nabla z_n||\nabla\eta_r||\varphi_n|+|\nabla w_n||\nabla\eta_r||\psi_n|)dx\\ &\le \frac{4}{r}\int_{\Omega_r}(|\nabla z_n||z_n|+|\nabla w_n||w_n|)dx\\ &\le \frac{4}{r}(\|\nabla z_n\|_2\|z_n\|_2+\|\nabla w_n\|_2\|w_n\|_2)\le\frac{8}{r}M^2\to 0 \quad \text{as } r\to\infty, \end{aligned} \tag{2.42}$$

where $\Omega_r = B_r^c \backslash \overline{B_{2r}^c}$. From Remark 2.2 we know

$$
\begin{aligned}
|C_n(r)| &\le \int_{B_{2r}^c} |h(x)f(z_n) + g(x)f(w_n)|dx \\
&\le C_h \|h\|_{L^\mu(B_{2r}^c)} \|z_n\|_X^{1/2} + C_g \|h\|_{L^\mu(B_{2r}^c)} \|w_n\|_Y^{1/2} \\
&\le (C_h + C_g)\sqrt{M}\|h\|_{L^\mu(B_{2r}^c)} \to 0 \quad \text{as } r \to \infty.
\end{aligned} \tag{2.43}
$$

Then we have

$$
\begin{aligned}
&\int_{B_{2r}^c} [|\nabla z_n|^2 + |\nabla w_n|^2 + a(x)f^2(z_n) + b(x)f^2(w_n)]dx \\
&\le \int_{\mathbb{R}^N} [\nabla z_n \nabla \varphi_n + \nabla w_n \nabla \psi_n + a(x)f^2(z_n) + b(x)f^2(w_n)]\eta_r dx \\
&= A_n(r) + B_n(r) + C_n(r) + o_n(1).
\end{aligned} \tag{2.44}
$$

This estimate concludes (2.34).

(ii) The limit (2.34) gives

$$
\int_{B_{2r}^c} a(x)f^2(z_n)dx < \varepsilon, \tag{2.45}
$$

and consequently,

$$
\int_{B_{2r}^c} a(x)f^2(z)dx < \varepsilon. \tag{2.46}
$$

Since $z_n \to z$ in $L^2(B_{2r})$, we get

$$
\lim_{n\to\infty} \int_{B_{2r}} a(x)f^2(z_n)dx = \int_{B_{2r}} a(x)f^2(z)dx. \tag{2.47}
$$

Thus we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x)f^2(z_n)dx = \int_{\mathbb{R}^N} a(x)f^2(z)dx. \tag{2.48}
$$

Similarly, we get

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} b(x)f^2(w_n)dx = \int_{\mathbb{R}^N} b(x)f^2(w)dx. \tag{2.49}
$$

Noting (f_6) and arguing as the above, we can conclude the limits (2.35) and (2.36).

Set $\phi(t) = f^2(t)$, then $\phi''(t) = 2(1+2f^2(t))^{-2} > 0$, and $\phi(t)$ is convex and even in $\mathbb{R}$. Hence, by (f_9) we get

$$
\begin{aligned}
\int_{B_{2r}^c} a(x)f^2(z_n - z)dx &\le \frac{1}{2}\int_{B_{2r}^c} a(x)(f^2(2z_n) + f^2(2z))dx \\
&\le 2\int_{B_{2r}^c} a(x)(f^2(z_n) + f^2(z))dx \le 2\varepsilon
\end{aligned} \tag{2.50}
$$

for large n. Since $f^2(z_n - z) \leq |z_n - z|^2$ and $z_n \to z$ in $L^2(B_{2r})$, we have

$$\lim_{n\to\infty} \int_{B_{2r}} a(x) f^2(z_n - z)dx = 0. \tag{2.51}$$

Then

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x) f^2(z_n - z)dx = 0. \tag{2.52}$$

Similarly, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} b(x) f^2(w_n - w)dx = 0. \tag{2.53}$$

(iii) From (2.12), one sees that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla z_n \nabla \varphi dx = \int_{\mathbb{R}^N} \nabla z \nabla \varphi dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{2.54}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla w_n \nabla \psi dx = \int_{\mathbb{R}^N} \nabla w \nabla \psi dx \quad \forall \psi \in C_0^\infty(\mathbb{R}^N). \tag{2.55}$$

Then by Lebegsue dominated convergence theorem, it follows

$$J'(z,w)(\varphi,\psi) = \lim_{n\to\infty} J'(z_n,w_n)(\varphi,\psi) = 0 \quad \forall \varphi,\psi \in C_0^\infty(\mathbb{R}^N). \tag{2.56}$$

Since the set $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ is dense in E, we have $J'(z,w)(\varphi,\psi) = 0$, $\forall \varphi,\psi \in E$. In particular, $J'(z,w)(z,w) = 0$. Hence, (z,w) is a critical point of J in E. This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let* (H_1)-(H_3) *hold. If* $\{(z_n,w_n)\} \subset E$ *is a Cerami sequence satisfies* (2.12)*, then* $(z_n,w_n) \to (z,w)$ *in* E*, that is, the functional* J *satisfies the Cerami condition in* E*.*

Proof. By (2.12), we know

$$\lim_{n\to\infty} \int_{B_r} h(x) f'(z_n) z_n dx = \int_{B_r} h(x) f'(z) z dx. \tag{2.57}$$

On the other hand, we see from (2.12) and Remark 2.2 that

$$\int_{B_r^c} |h(x) f'(z_n) z_n| dx \leq \int_{B_r^c} |h(x) f(z_n)| dx \leq C_h \|h\|_{L^\mu(B_r^c)} M^{1/2} \to 0 \text{ as } r \to \infty, \tag{2.58}$$

and

$$\int_{B_r^c} |h(x) f'(z) z| dx \leq \int_{B_r^c} |h(x) f(z)| dx \leq C_h \|h\|_{L^\mu(B_r^c)} M^{1/2} \to 0 \quad \text{as} \quad r \to \infty. \tag{2.59}$$

Thus we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x) f'(z_n) z_n dx = \int_{\mathbb{R}^N} h(x) f'(z) z dx. \tag{2.60}$$

Since $J'(z_n, w_n) \to 0$ in E^* as $n \to \infty$ and (z, w) is a critical point of J, we derive

(2.61)
$$\begin{aligned} o_n(1) &= J'(z_n, w_n)(z_n, 0) \\ &= \int_{\mathbb{R}^N} (|\nabla z_n|^2 + a(x) f(z_n) f'(z_n) z_n) dx - \int_{\mathbb{R}^N} (F_u + h(x)) f'(z_n) z_n dx, \end{aligned}$$

and

$$\begin{aligned} 0 &= J'(z, w)(z, 0) \\ &= \int_{\mathbb{R}^N} (|\nabla z|^2 + a(x) f(z) f'(z) z) dx - \int_{\mathbb{R}^N} (F_u + h(x)) f'(z) z dx. \end{aligned} \tag{2.62}$$

Using the limits (2.14), (2.35) and (2.60) we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla z_n|^2 dx = \int_{\mathbb{R}^N} |\nabla z|^2 dx. \tag{2.63}$$

Similarly we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \int_{\mathbb{R}^N} |\nabla w|^2 dx. \tag{2.64}$$

The application of Brezis-Lieb lemma in [3] yields

$$\lim_{n\to\infty} \|\nabla(z_n - z)\|_2^2 = \lim_{n\to\infty} \|\nabla(w_n - w)\|_2^2 = 0. \tag{2.65}$$

As in the proof of (2.11), we see that

(2.66)
$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla(z_n - z)|^2 + |\nabla(w_n - w)|^2 + a(x) f^2(z_n - z) + b(x) f^2(w_n - w)) dx \\ &\geq C_0 \|(z_n - z, w_n - w)\|_E. \end{aligned}$$

Then we get from (2.37) and (2.65) that $(z_n, w_n) \to (z, w)$ in E. This completes the proof of Lemma 2.6.

□

3. Proof of main result

To prove our result, We will make use of the Mountain Pass Theorem in [2] (see also [16]).

Lemma 3.1 (Mountain Pass Theorem)**.** *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ with $J(0) = 0$. Suppose $J(u)$ satisfies Cerami condition and*

(A_1) there are $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ when $\|u\|_E = \rho$,

(A_2) there is $e \in E$, $\|e\|_E > \rho$ such that $J(e) < 0$.

Define $\Gamma = \{\gamma \in C^1([0,1], E) | \gamma(0) = 0, \gamma(1) = e\}$. Then

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \geq \alpha$$

is a critical value of $J(u)$.

Lemma 3.2. *Assume* (H_1)-(H_3). *Then there exists* $m_0 > 0$ *such that for* $\|h\|_\mu, \|g\|_\mu \le m_0$, $J(z,w)$ *defined by* (2.3) *satisfies the assumptions* (A_1), (A_2) *in Lemma 3.1.*

Proof. It follows from (H_2) and (f_7) that, there is a constant $C_3 > 0$ such that

$$\int_{\mathbb{R}^N} F(f(z), f(w))dx \le C_3 \|(z,w)\|_E^{d/2}. \tag{3.1}$$

Using Remark 2.2 we get

$$\begin{aligned}\int_{\mathbb{R}^N} |h(x)f(z) + g(x)f(w)|dx &\le (c_h\|h\|_\mu + c_g\|g\|_\mu)\|(z,w)\|_E^{1/2} \\ &\le \varepsilon\|(z,w)\|_E^2 + C_\varepsilon(\|h\|_\mu^{4/3} + \|g\|_\mu^{4/3})\end{aligned} \tag{3.2}$$

with any small $\varepsilon > 0$, $C_\varepsilon = C(\varepsilon) > 0$. Thus it follows by (2.11) that

$$J(z,w) \ge \frac{C_0}{2}\|(z,w)\|_E^2 - C_3\|(z,w)\|_E^{d/2} - \varepsilon\|(z,w)\|_E^2 - C_\varepsilon(\|h\|_\mu^{4/3} + \|g\|_\mu^{4/3}). \tag{3.3}$$

Fix $\varepsilon \le \frac{C_0}{4}$, then $2 < d/2$ implies that, there exist $m_0, \rho, \alpha > 0$ such that $J(u) \ge \alpha$ with $\|z,w\|_E = \rho$ and $\|h\|_\mu, \|g\|_\mu \le m_0$ for each $h, g \in L^\mu(\mathbb{R}^N)$. (A_1) in Lemma 3.1 is true.

We now verify (A_2). Fix $(z,w) \in E$ with $z \not\equiv 0$, $w \not\equiv 0$ and $z, w \ge 0$. Set

$$\Omega_t = \{x \in \mathbb{R}^N | tz(x) \ge 1, tw(x) \ge 1\}. \tag{3.4}$$

Choose large $t_0 > 0$ such that $\int_{\Omega_t} F(\sqrt{z}, \sqrt{w})dx > 0$ for $t > t_0$. Using (H_2) and (f_8) we have

$$\int_{\Omega_t} F(f(tz), f(tw))dx \ge \int_{\Omega_t} F(C\sqrt{tz}, C\sqrt{tw})dx = C^d t^{d/2} \int_{\Omega_t} F(\sqrt{z}, \sqrt{w})dx. \tag{3.5}$$

Then, by (f_9) we see that for $t > t_0$

$$\begin{aligned}J(tz,tw) &\le \frac{t^2}{2}\|(z,w)\|_E^2 - C^d t^{d/2} \int_{\Omega_t} F(\sqrt{z}, \sqrt{w})dx \\ &\quad + (t+1)\int_{\mathbb{R}^N} (|h(x)f(z)| + |g(x)f(w)|)dx\end{aligned} \tag{3.6}$$

and $J(tz,tw) \to -\infty$ as $t \to \infty$ since $2 < d/2$. Therefore, there exists t_1 large enough, such that $J(t_1 z, t_1 w) < 0$ and (A_2) in Lemma 3.1 is true. This completes the proof of Lemma 3.2. □

Proof of Theorem 1.3. By Lemma 2.6 and Lemma 3.2, $J(z,w)$ satisfies all assumptions in Lemma 3.1. Then there exists $(z_1, w_1) \in E$ such that (z_1, w_1) is a solution of (1.4). Furthermore, $J(z_1, w_1) \ge \alpha > 0$.

We now seek another solution. Choose $\varphi, \psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (h(x)\varphi + g(x)\psi)dx > 0. \tag{3.7}$$

Then, by (f_8) it follows that, for small $t > 0$

$$J(t\varphi, t\psi) \leq \frac{t^2}{2}\|(\varphi, \psi)\|_E^2 - Ct\int_{\mathbb{R}^N}(h(x)\varphi + g(x)\psi)dx < 0. \tag{3.8}$$

So, for any open ball $B_\tau \subset E$ we have

$$-\infty < c_\tau = \inf_{\overline{B}_\tau} J(z, w) < 0. \tag{3.9}$$

Thus,

$$c_\rho = \inf_{\overline{B}_\rho} J(z, w) < 0 \quad \text{and} \quad \inf_{\partial B_\rho} J(z, w) > 0 \tag{3.10}$$

where $\rho > 0$ is given in Lemma 3.2. Let $\varepsilon_n \to 0$ such that

$$0 < \varepsilon_n < \inf_{\partial B_\rho} J(z, w) - \inf_{B_\rho} J(z, w). \tag{3.11}$$

Then, by Ekelands variational principle in [17], there exists $\{z_n, w_n\} \subset \overline{B}_\rho$ such that

$$c_\rho \leq J(z_n, w_n) < c_\rho + \varepsilon_n \tag{3.12}$$

and

$$J(z_n, w_n) < J(z, w) + \varepsilon_n\|(z_n - z, w_n - w)\|_E, \quad \forall (z, w) \in \overline{B}_\rho,\ (z, w) \neq (z_n, w_n). \tag{3.13}$$

Then it follows that

$$J(z_n, w_n) < c_\rho + \varepsilon_n \leq \inf_{B_\rho} J(z, w) + \varepsilon_n < \inf_{\partial B_\rho} J(z, w), \tag{3.14}$$

so that $(z_n, w_n) \in B_\rho$. We now consider the functional $F : \overline{B}_\rho \to \mathbb{R}$ given by

$$F(z, w) = J(z, w) + \varepsilon_n\|(z_n - z, w_n - w)\|_E, \quad (z, w) \in \overline{B}_\rho. \tag{3.15}$$

Then (3.13) shows that $F(z_n, w_n) < F(z, w)$, $(z, w) \in \overline{B}_\rho$, $(z, w) \neq (z_n, w_n)$ and thus (z_n, w_n) is a strict local minimum of F. Moreover,

$$\frac{1}{t}(F(z_n + tz', w_n + tw') - F(z_n, w_n)) \geq 0 \quad \text{for small } t > 0 \text{ and } \forall (z', w') \in B_1. \tag{3.16}$$

Hence,

$$\frac{1}{t}(J(z_n + tz', w_n + tw') - J(z_n, w_n)) + \varepsilon_n\|(z', w')\|_E \geq 0. \tag{3.17}$$

Passing to the limit as $t \to 0^+$, it follows that

$$J'(z_n, w_n)(z', w') + \varepsilon_n\|(z', w')\|_E \geq 0,\ \ \forall (z', w') \in B_1. \tag{3.18}$$

Replacing (z', w') by $(-z', -w')$, we get

$$-J'(z_n, w_n)(z', w') + \varepsilon_n\|(z', w')\|_E \geq 0,\ \ \forall (z', w') \in B_1. \tag{3.19}$$

So that $\|J'(z_n, w_n)\| \leq \varepsilon_n$.

Therefore, there is a sequence $\{z_n, w_n\} \subset B_\rho$ such that $J(z_n, w_n) \to c_\rho < 0$, and $J'(z_n, w_n) \to 0$ in E^*. By Lemma 2.6, $\{z_n, w_n\}$ has a convergent subsequence in E, still denoted by $\{z_n, w_n\}$, such that $(z_n, w_n) \to (z_2, w_2)$ in E. Thus (z_2, w_2) is a solution of (1.4) with $J(z_2, w_2) < 0$. Then the proof of Theorem 1.3 is completed. $\square$

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