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THE BOUNDARY HARNACK PRINCIPLE IN HÖLDER DOMAINS WITH A STRONG REGULARITY

Hyejin Kim

Abstract. We prove the boundary Harnack principle and the Carleson type estimate for ratios of solutions u/v of non–divergence second order elliptic equations $Lu = a_{ij}D_{ij}u + b_iD_iu = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$. We assume that $b_i \in L^n(\Omega)$ and Ω is a Hölder domain of order $\alpha \in (0, 1)$ satisfying a strong regularity condition.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We consider second order elliptic equations in *non-divergence* form,

$$Lu := a_{ij}D_{ij}u + b_iD_iu = 0 \quad \text{in } \Omega, \tag{1}$$

with measurable coefficients a_{ij} and b_i. Assume that a_{ij} satisfy the *uniform ellipticity condition* with the *ellipticity constant* $\nu \in (0, 1]$:

$$a_{ij} = a_{ji}, \qquad \nu|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2}$$

Throughout this paper, we use notations $D_i := \partial/\partial x_i$, $D_{ij} := D_iD_j$ and the summation convention over repeated indices is imposed. We denote

$$S := S(\Omega) := \int_\Omega |\mathbf{b}|^n dx < \infty, \quad \text{where } \mathbf{b} := (b_1, \dots, b_n). \tag{3}$$

The operator L in (1) is considered as a second order operator acting on the functions $u \in W(\Omega) := W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})$, which implies that u, D_iu, $D_{ij}u$ belong to the Lebesgue space $L^n(\Omega')$ for any open set $\Omega' \subset \overline{\Omega'} \subset \Omega$, and the equality in (1), or inequalities $Lu \leq 0$, $Lu \geq 0$, are understood almost everywhere (a.e.) in Ω.

Definition 1. A domain $\Omega \subset \mathbb{R}^n$ is called a *Hölder domain of order* α for $\alpha \in (0, 1]$ if for every $z \in \partial\Omega$, there exist a neighborhood U of z, an orthonomal

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coordinate system CS, and a function $\Psi : \mathbb{R}^{n-1} \to [-\Lambda, \Lambda]$, which is a Hölder continuous function of order α, i.e.,

$$(4)\quad |\Psi(\tilde{x}) - \Psi(\tilde{y})| \le \Lambda (|\tilde{x} - \tilde{y}|^{\alpha} \wedge 1), \quad \tilde{x}, \tilde{y} \in \mathbb{R}^{n-1} \text{ and a constant } \Lambda > 1,$$

such that $\Omega \cap U := \{x = (x_1, \ldots, x_n) \in U : x_n > \Psi(x_1, \ldots, x_{n-1}) \text{ in CS}\}$. Here $a \wedge b$ denotes the minimum of a and b and a notation $\partial\Omega$ is the boundary of an open set $\Omega \subset \mathbb{R}^n$.

In other words, a Hölder domain Ω of order α is a set whose boundary is locally represented by the graph of a Hölder function Ψ of order α and the constant Λ.

Definition 2. If there exists a constant $\mu \in (0, 1)$ such that the Lebesgue measure

$$(5)\qquad |B_r(z) \setminus \Omega| \ge \mu \cdot |B_r| \quad \text{for all} \quad z \in \Gamma \text{ and } r > 0,$$

where $B_r(z)$ is a ball of radius $r > 0$ centered at $z \in \mathbb{R}^n$, then a subset $\Gamma \subset \partial\Omega$ is *strongly regular*, or satisfies a *strong regularity condition.*

Lemma 1.1. *Let Ω be a Hölder domain of order $\alpha \in (0, 1]$ and $z \in \partial\Omega$. From* (4)*, there exists a constant $K > 1$ depending only on α, Λ and R such that*

$$(6)\qquad d(x) \le \delta(x) \le K d^{\alpha}(x), \qquad \forall x \in \Omega_R(z) := \Omega \cap B_R(z),$$

where $d(x) := dist(x, \partial\Omega)$ is a distance function and $\delta(x) := x_n - \Psi(\tilde{x})$ with $x = (\tilde{x}, x_n) \in \Omega \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

Proof. First, if $d(x) = \delta(x)$, then (6) is trivial with $K = R^{1-\alpha}$ because of $d(x) \le R$ for all $x \in \Omega_R(z)$.

Next, assume that $d(x) < \delta(x)$. There exists a point $y = (\tilde{y}, \Psi(\tilde{y})) \in \partial\Omega$ such that $d(x) = |x - y|$ for $x = (\tilde{x}, x_n) \in \Omega_R(z)$. So we have

$$\begin{aligned} \delta(x) = |x_n - \Psi(\tilde{x})| \;&\le\; |x_n - \Psi(\tilde{y})| + |\Psi(\tilde{x}) - \Psi(\tilde{y})| \\ &\le d(x) + \Lambda (|\tilde{x} - \tilde{y}|^{\alpha} \wedge 1) \;\le\; R^{1-\alpha} d^{\alpha}(x) + \Lambda\, d^{\alpha}(x). \end{aligned}$$

Thus, in this case $K = R^{1-\alpha} + \Lambda$, depending only on α, Λ, and R. □

It was shown in [2] that the boundary Harnack principle holds for non–divergence elliptic equations with a measurable bounded drift $b_i \in L^{\infty}(\Omega)$ in Hölder domains of an order $\alpha \in (0, 1]$ provided a boundary of the domain satisfies a strong regularity condition. In this paper, we extended the result in [2] to a measurable unbounded coefficient $b_i \in L^n(\Omega)$ to prove the Carleson type estimate and the boundary Harnack principle.

In [3, 4], under the assumption of a measurable unbounded drift, the boundary Harnack principle with weak regularity condition was proved in the twisted Hölder domains of $\alpha \in (1/2, 1]$ by using the interior Harnack inequality and the growth lemma [7]. And, in [5], more direct proof of the boundary Harnack principle for the ratios u/v of positive solutions to (1) was given in John domains, which are special cases of twisted Hölder domains of order $\alpha = 1$.

We will use the similar approach to prove the Carleson type estimate and the boundary Harnack principle in the Hölder domains of $\alpha \in (0,1]$. Note that the assumption of $b_i \in L^n(\Omega)$ is the most possible generalization of the drift coefficient, as the interior Harnack inequality fails when the assumption $b_i \in L^n(\Omega)$ is weakened by $b_i \in L^{n-\varepsilon}(\Omega)$ with an arbitrary small $\varepsilon > 0$ [7].

In fact, the classes of Lipschitz domains and Hölder domains of order $\alpha = 1$ are identical and Safonov proved that the boundary Harnack principle holds for non–divergence elliptic equations with a unbounded drift in Lipschitz domains in [S10]. Thus we will prove that the boundary Harnack principle holds for non–divergence elliptic equations with a unbounded drift in Hölder domains of order $\alpha \in (0,1)$. In addition, a strong regularity on Ω is assumed in this paper.

The main purpose of the paper is to prove:

Theorem 1.2 (Boundary Harnack principle)**.** *Let $\Omega \subset B_{R_0}(z)$, for some $z \in \Omega$, be a bounded Hölder domain of order $\alpha \in (0,1)$ and, for $y_0 \in \partial\Omega$ and $0 < 2R \le R_0$, let $\Gamma := \partial\Omega \cap B_{2R}(y_0)$ be strongly regular with a constant μ in Definition 2. Let $x_0 \in \Omega$ with $d(x_0) > 0$ where $d(x) := dist(x, \partial\Omega)$ is a distance function, and $u, v \in W(\Omega)$ such that*

$$u \ge 0, \quad v > 0 \quad \text{in } \Omega; \qquad Lu = Lv = 0 \quad \text{a.e. in } \Omega,$$

and $u = 0$ on Γ. Then we have

$$\sup_{\Omega_R(y_0)} \frac{u}{v} \le N \cdot \frac{u(x_0)}{v(x_0)}, \tag{7}$$

where the constant N depends only on $n, \nu, S, \mu, \alpha, \Lambda, R, R_0$, and $d(x_0)$.

The rest of the paper is organized as follows. In Section 2, we describe the growth lemma and the interior Harnack inequality. From the interior Harnack inequality, we derive the upper and lower estimates for the positive solution of second order equations (1). Then, by using these estimates and the interior Harnack inequality, we prove the Carleson type estimates in Hölder domains of $\alpha \in (0,1)$. The following Section 3 contains the proof of Theorem 1.2, the boundary Harnack principle, which is our main theorem in the paper.

Through the paper, N, c (with indices or without) denote different constants depending only on the prescribed quantities such as n, ν, S, etc. The dependence is indicated in the parentheses: $N = N(n, \nu, S, \ldots)$, $c = c(n, \nu, S, \ldots)$. In addition, we will use a notation $|\Omega|$ which is its Lebesgue measure. We also denote by $|\gamma|$ the length of a rectifiable curve γ in $\mathbb{R}^n$.

2. Auxiliary statements

The following two statements, a growth lemma and the interior Harnack inequality, are main ingredients to prove our results and Safonov proved them in [7] (Lemma 2.5 and Theorem 3.1).

Lemma 2.1 (Growth lemma)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $u \in W(\Omega)$, $x_0 \in \Omega$, and $r > 0$ be such that*

$$u \geq 0, \quad Lu \geq 0 \text{ a.e. in } \Omega; \text{ and } u = 0 \text{ on } (\partial\Omega) \cap B_{2r}(x_0). \tag{8}$$

We claim that for an arbitrary constant $\mu_1 \in (0,1)$, there is a constant $\beta_1 = \beta_1(n, \nu, S, \mu_1) \in (0,1)$, such that from the estimate for the Lebesgue measure

$$|B_r(x_0) \setminus \Omega| \geq \mu_1 \cdot |B_r| \tag{9}$$

it follows

$$\sup_{\Omega_r(x_0)} u \leq \beta_1 \cdot \sup_{\Omega_{2r}(x_0)} u. \tag{10}$$

Theorem 2.2 (Interior Harnack inequality)**.** *Let u be a function in $W(B_{8r})$, where $B_{8r} := B_{8r}(x_0)$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$, and let*

$$u > 0 \text{ in } B_{8r}; \qquad Lu := a_{ij} D_{ij} u + b_i D_i u = 0 \text{ a.e. in } B_{8r}.$$

Then

$$\sup_{B_r} u \leq N_0 \cdot \inf_{B_r} u, \text{ where } N_0 = N_0(n, \nu, S) \geq 1, \quad S := \int_{B_{8r}} |\mathbf{b}|^n dx. \tag{11}$$

From this inequality, if $u \in W(\Omega)$, $u > 0$ in Ω, and $Lu = 0$ a.e. in Ω, then

$$N_0^{-1} u(y) \leq u(x) \leq N_0 u(y) \quad \text{for} \quad x, y \in \Omega \quad \text{with} \quad |x - y| \leq d(x)/8. \tag{12}$$

By iterating these inequalities, the following upper and lower estimates for $u(x)$ can be derived.

Theorem 2.3. *Let Ω be a Hölder domain of order $\alpha \in (0,1)$, and let $u \in W(\Omega)$ be a function satisfying $u > 0$ in Ω, $Lu = 0$ a.e. in Ω. Then*

$$u(x) \leq N_1 \exp\big(c\, d^{\alpha-1}(x)\big) u(x_0) \quad \text{for all } x \in \Omega, \tag{13}$$

$$u(x) \geq N_1^{-1} \exp\big(-c\, d^{\alpha-1}(x)\big) u(x_0) \quad \text{for all } x \in \Omega, \tag{14}$$

with constants $N_1 = N_1(n, \nu, S, \alpha, \Lambda, R, d(x_0))$ and $c = c(n, \nu, S, \alpha, \Lambda, R, d(x_0))$ and $x_0 \in \Omega$ with $0 < d(x_0) < R$.

Proof. Let us denote

$$M := \sup_{\Omega} \exp\big(-c\, d^{\alpha-1}(x)\big) u(x), \quad d(x) := \operatorname{dist}(x, \partial\Omega). \tag{15}$$

Since $\exp\big(-c\, d^{\alpha-1}(x)\big) \to 0$ as $d \to 0^+$, there exists $z_0 \in \Omega$ such that

$$M := \sup_{\Omega} \exp\big(-c\, d^{\alpha-1}(x)\big) u(x) = \exp\big(-c\, d^{\alpha-1}(z_0)\big) u(z_0).$$

The constant $c > 0$ will be specified later. Note that the twisted Hölder domains of order α contain Hölder domain of order α ([3, 4]). Thus there exists a rectifiable curve $\gamma(z_0, x_0)$ such that $|\gamma(z_0, y)| \leq d(y) + \delta(y)$ for $y \in \gamma(z_0, x_0)$ (see [3], Lemma 1.3 or Theorem 2.1), and the rectifiable curve $\gamma(z_0, x_0)$ is chosen as a union of subcurves $\gamma(w_j, w_{j+1})$ such that $w_0 = z_0$ and $|\gamma(w_j, w_{j+1})| \leq r_j$ for each j with $r_j := d(w_j)/8$ and $r_j \leq r_{j+1}$.

Next, with this rectifiable curve $\gamma(z_0, x_0)$, let us assume that $d(x_0) \geq 2\, d(z_0)$. Then we will show that, if $z_1 \in \gamma(z_0, x_0)$ with $d(z_1) = 2\, d(z_0)$ and $d(z) \leq d(z_1)$ for all $z \in \gamma(z_0, z_1)$, then

$$u(z_0) \leq \exp\left(c_0\, d^{\alpha-1}(z_0)\right) \cdot u(z_1). \tag{16}$$

The constant c_0 will be determined later. Note that, from (6), there exists a constant K depending only on Λ, α, and R such that $d(x_0) < K d^\alpha(x_0) =: R_0$. Since $2d(z_0) \leq d(x_0) \leq K d^\alpha(x_0)$, there is a constant $c_2 = c_2(\Lambda, \alpha, R, d(x_0))$ such that $1 \leq c_2\, d^{\alpha-1}(z_0)$. In addition, since the ball of radius $d(z_1)$ centered at z_1 touches $\partial\Omega$ at some point $z^* \in \partial\Omega$ and $d(z_1) \leq K d^\alpha(x_0) =: R_0$, $z_1 \in \Omega_{2R_0}(z^*)$ and $|\gamma(z_0, z_1)| \leq d(z_1) + \delta(z_1)$. So, there is a constant $c_1 = c_1(\Lambda, \alpha, R, d(x_0))$ such that $|\gamma(z_0, z_1)| \leq c_1 d^\alpha(z_1) = 2^\alpha c_1\, d^\alpha(z_0)$. The curve $\gamma(z_0, z_1)$ is divided into p subcurves $\gamma(w_{j-1}, w_j)$ for $1 \leq j \leq p$ with $w_0 = z_0$ and $w_p = z_1$. Then,

$$\begin{aligned} p &\leq 1 + \frac{|\gamma(z_0, z_1)|}{r_0} \leq 1 + \frac{2^\alpha c_1 d^\alpha(z_0)}{d(z_0)/8} \\ &\leq \left(c_2 + 2^{\alpha+3} c_1\right)\, d^{\alpha-1}(z_0) = c_3\, d^{\alpha-1}(z_0), \end{aligned} \tag{17}$$

where $c_3 = c_3(\Lambda, \alpha, R, d(x_0))$. Since $|w_j - w_{j+1}| \leq r_j = d(w_j)/8$ for each j, the interior Harnack inequality (11) implies

$$u(z_0) = u(w_0) \leq N_0 u(w_1) \leq \cdots \leq N_0^p u(w_p) = N_0^p u(z_1). \tag{18}$$

This is equivalent to (16) with $c_0 = c_3 \ln N_0 > 0$.

Meanwhile, by the definition of M in (15) and the result (16),

$$\begin{aligned} M &= \exp\left(-c\, d^{\alpha-1}(z_0)\right) u(z_0) \\ &\leq \exp\left(-c\, d^{\alpha-1}(z_0) + c_0\, d^{\alpha-1}(z_0)\right) u(z_1) \\ &\leq \exp\left(c\left(2^{\alpha-1} - 1\right) + c_0\right) d^{\alpha-1}(z_0)\big)\; M. \end{aligned} \tag{19}$$

Since $2^{\alpha-1} < 1$, if we choose a constant c in such a way that $c(1 - 2^{\alpha-1}) > c_0$, the right hand side of (19) is strictly less than M, which is a contradiction. This implies that $d(z_1) = 2\, d(z_0)$ is impossible. Therefore, for this choice of c, we have $d(x_0) < 2\, d(z_0)$. In this case, a point x_0 can be still reached from a point z_0 in a finite number of steps such that $u(z_0) \leq \exp\left(c_0\, d^{\alpha-1}(z_0)\right) \cdot u(x_0)$.

Finally, from the definition of M and (16) with $z_1 = x_0$, we get

$$M \leq u(z_0) \leq \exp\left(c_0\, 2^{1-\alpha}\, d^{\alpha-1}(x_0)\right) u(x_0) := N_1 u(x_0), \tag{20}$$

$$u(x) \leq \exp\left(c\, d^{\alpha-1}(x)\right)\, M \leq N_1 \exp\left(c\, d^{\alpha-1}(x)\right) u(x_0). \tag{21}$$

Since we can approximate u by functions $u + \varepsilon$, $\varepsilon > 0$, we can assume that $u \geq$ constant > 0 on $\overline{\Omega}$. The proof of (13) was mainly based on the inequalities (12). Note the inequalities (12) are still valid with $v = 1/u$. Therefore, the estimate (13) holds for $v(x)$ which is equivalent to (14). $\square$

Theorem 2.4 (Carleson Type Estimate)**.** *Let $\Omega \subset B_{R_0}(z)$, for some $z \in \Omega$, be a bounded Hölder domain of order $\alpha \in (0,1)$ and, for $y_0 \in \partial\Omega$ and $0 < 2R \leq R_0$, let $\Gamma := \partial\Omega \cap B_{2R}(y_0)$ be strongly regular with a constant μ in Definition 2. Assume that u is a function in $W(\Omega)$, $u > 0$ and $Lu = 0$ in Ω, and $u = 0$ on Γ. Then,*

$$\sup_{\Omega\cap B_R(y_0)} u \leq N u(x_0), \tag{22}$$

where the constant N depends only on $n, \nu, S, \mu, R, R_0, \Lambda, \alpha,$ and $d(x_0) > 0$ for $x_0 \in \Omega$.

Proof. Note that $\Gamma \neq \partial\Omega$ because, if $u = 0$ on $\partial\Omega$ and $Lu = 0$ *a.e.* in Ω, by the maximum principle, it implies $u = 0$ in Ω, which contradicts the assumption $u > 0$ in Ω. Since a point x_0 can be replaced by any other interior point in Ω by the interior Harnack inequality with an appropriate replacement of the constant N, we assume that $x_0 \in \Omega_R(y_0) := \Omega \cap B_R(y_0)$.

Let us denote

$$M_0 := \sup_{\Omega} \exp\left(-d_0^{-A}(x)\right) \cdot u(x), \qquad \text{where } d_0(x) := \operatorname{dist}(x, (\partial\Omega)\setminus\Gamma), \tag{23}$$

and a fixed constant A satisfies the equality $(A+1)(\alpha-1) = 1 - A$. Since $d_0(x) = 0$ on $(\partial\Omega)\setminus\Gamma$ and $u = 0$ on Γ, for any constant $A = 2/\alpha - 1 > 1$, there exists $z_0 \in \Omega$ such that

$$M_0 = \exp\left(-d_0^{-A}(z_0)\right) \cdot u(z_0). \tag{24}$$

For a small constant $0 < h \leq d_0(z_0)/8$, which will be specified later, consider two cases: (i) $d(z_0) < h$, and (ii) $d(z_0) \geq h$.

Consider the first case. Since $d(z_0) < h \leq d_0(z_0)/8$, there is a point $z^* \in \Gamma$ such that $d(z_0) = |z_0 - z^*| < h$. Since Γ satisfies the strong regularity condition (5) and $B_h(z^*) \subset B_{2h}(z_0)$,

$$|B_{2h}(z_0)\setminus\Omega| \geq |B_h(z^*)\setminus\Omega| \geq \mu|B_h| = \mu_0|B_{2h}|, \text{ where } \mu_0 = 2^{-n}\mu \in (0,1).$$

By the growth lemma,

$$u(z_0) \leq \sup_{\Omega\cap B_{2h}(z_0)} u \leq \beta_1 \cdot \sup_{\Omega\cap B_{4h}(z_0)} u = \beta_1 u(z_1), \tag{25}$$

where $\beta_1 = \beta_1(n,\nu,S,\mu) \in (0,1)$ and $z_1 \in \Omega \cap \partial B_{4h}(z_0)$. Thus we have

$$\begin{aligned} M_0 &= \exp\left(-d_0^{-A}(z_0)\right) u(z_0) \\ &\leq \exp\left(-d_0^{-A}(z_0)\right) \cdot \beta_1 u(z_1) \\ &\leq \exp\left[d_0^{-A}(z_1) - d_0^{-A}(z_0)\right] \cdot \beta_1 M_0. \end{aligned} \tag{26}$$

By the triangle inequality, for $z_1 \in \Omega \cap \partial B_{4h}(z_0)$, $d_0(z_0) \leq 4h + d_0(z_1)$ and the condition $h \leq d_0(z_0)/8$,

$$\begin{aligned} d_0^{-A}(z_1) - d_0^{-A}(z_0) &\leq (d_0(z_0) - 4h)^{-A} - d_0^{-A}(z_0) \\ &\leq 4hA\,(d_0(z_0) - 4h)^{-A-1} \end{aligned} \tag{27}$$

$$\leq 2^{A+3} hA \cdot d_0^{-A-1}(z_0).$$

Now, fix a constant $\varepsilon_1 = \varepsilon_1(n, \nu, S, \alpha, \mu, R_0) > 0$ such that $\beta_1 \mathrm{e}^{\varepsilon_1} < 1$, and choose

$$h := h_0\, d_0^{A+1}(z_0), \text{ where } h_0 = \min\left\{\frac{1}{8R_0^A}, \frac{\varepsilon_1}{2^{A+3}A}\right\}. \tag{28}$$

From this choice of h and (27), it guarantees $h \leq d_0(z_0)/8$ and it gives us a contradiction $M_0 \leq \beta_1 \mathrm{e}^{\varepsilon_1} M_0 < M_0$. Therefore, $d(z_0) < h$ is impossible.

Assume that $d(z_0) \geq h$. By Theorem 2.3 and $d^{\alpha-1}(z_0) \leq h_0^{\alpha-1}\, d_0^{1-A}(z_0)$,

$$u(z_0) \leq N_1 \exp\left(c\, d^{\alpha-1}(z_0)\right) u(x_0) \leq N_1 \exp\left(c\, h_0^{\alpha-1}\, d_0^{1-A}(z_0)\right) u(x_0). \tag{29}$$

Let $c_1 := c\, h_0^{\alpha-1}$, which depends only on the prescribed quantities n, ν, S, α, Λ, R, R_0, $d(x_0)$, and μ. Therefore,

$$\begin{aligned} M_0 = \exp\left(-d_0^{-A}(z_0)\right) u(z_0) &\leq N_1 \exp\left[c_1\, d_0^{1-A}(z_0) - d_0^{-A}(z_0)\right] u(x_0) \\ &= N_1 \exp\left[d_0^{-A}(z_0)\left(c_1\, d_0(z_0) - 1\right)\right] u(x_0) \\ &\leq N_1 \exp\left(c_1^A\right) u(x_0). \end{aligned} \tag{30}$$

The last inequality follows from the elementary inequality $c_1 d_0(z_0) \leq 1 + (c_1 d_0(z_0))^A$. Let $N_2 := N_1 \exp\left(c_1^A\right) < \infty$. Since $d_0(x) > R$ in $\Omega \cap B_R(y_0)$, we have

$$u(x) \leq \exp\left(d_0^{-A}(x)\right) \cdot M_0 \leq \exp\left(R^{-A}\right) M_0 \leq N_2 \exp\left(R^{-A}\right) u(x_0). \tag{31}$$

Finally, $u(x) \leq N u(x_0)$ for all $x \in \Omega \cap B_R(y_0)$, where $N = N_2 \exp\left(R^{-A}\right)$ and the constant N depends only on $n, \nu, S, \mu, R, R_0, \Lambda, \alpha$, and $d(x_0)$. The proof is complete. □

3. Proof of boundary Harnack principle

Finally, we will prove Theorem 1.2 in this section.

Proof of Theorem 1.2. Without loss of generality, we assume that $0 < R \leq 1$, $y_0 = 0$, and $u(x_0) = v(x_0) = 1$. In addition, we also assume $x_0 \in \Omega_R(y_0)$. Under these assumptions, we denote

$$\rho_k := 2^{-k-3}R, \quad R_k := R + 4\rho_k, \quad h_k := \varepsilon_0 \rho_k^{1/\alpha} \quad \text{for } k = 0, 1, \ldots, \tag{32}$$

where ε_0 is a small positive constant, which will be specified later. We also denote,

$$T_k := \Omega_{R_k} \cap \{d(x) < h_k\}, \quad T_k^+ := \Omega_{R_k} \cap \{h_{k+1} \leq d(x) < h_k\}. \tag{33}$$

First, we will show the following:

$$u(x) \leq N_1 u(x_0) \quad \text{for } x \in T_0 := \Omega_{3R/2} \cap \{d(x) < h_0\}, \tag{34}$$

where N_1 depends only on ε_0, n, ν, S, μ, R, R_0, Λ, α, and $d(x_0)$. Assume that $\varepsilon_0 \leq 2^{-1/\alpha}$. This assumption implies

$$2^{1/\alpha}\varepsilon_0 \leq 1 \leq \rho_k^{1-1/\alpha} \quad \text{for } k = 0, 1, 2, \ldots. \tag{35}$$

Note that $\varepsilon_0 < 2^{-1}$ and $h_0 := \varepsilon_0 \rho_0^{1/\alpha} \le \varepsilon_0 \rho_0 < R/16$. For each $x \in T_0$, there is a point $z^* \in \partial\Omega$ such that $|x - z^*| = d(x) < h_0$. By the triangle inequality,

$$|z^*| \le |z^* - x| + |x| < \frac{R}{16} + \frac{3R}{2} \text{ and } x \in \Omega_{h_0}(z^*) \subset \Omega_{2h_0}(z^*) \subset \Omega_{2R}. \tag{36}$$

According to Carleson type estimate in Theorem 2.4 with $y_0 = z^*$ and $R = h_0$, $u(x) \le N_1 u(x_0)$ for an arbitrary point x in T_0, where N_1 depends on ε_0 and the prescribed constants n, ν, S, μ, R, R_0, Λ, α, and $d(x_0)$. Therefore, the proof of (34) is complete.

Second, we will show

$$w(x) := Nv(x) - u(x) \ge 0 \quad \text{for } x \in \Omega_R := \Omega \cap B_R, \tag{37}$$

which is an equivalent form of (7) with an assumption $u(x_0) = v(x_0) = 1$.

From Theorem 2.3, we have

$$u(x) \le N_2\, u(x_0) \text{ and } v(x) \ge N_2^{-1}\, v(x_0) \quad \text{in } \Omega \cap \{d(x) \ge h\}, \tag{38}$$

where h is a positive constant and the constant N_2 depends only on n, ν, S, α, Λ, $d(x_0)$, and h. These inequalities imply

$$\frac{u(x)}{v(x)} \le N_2^2 \frac{u(x_0)}{v(x_0)} \quad \text{for } x \in \Omega \cap \{d(x) \ge h_0\}. \tag{39}$$

In addition, by (38), for $x \in \Omega_R \cap \{d(x) \ge h_0\}$,

$$w(x) \ge N \cdot N_2^{-1} - N_2 \ge 0, \quad \text{if } N \ge N_2^2. \tag{40}$$

Here the constant N_2 depends on n, ν, S, α, Λ, $d(x_0)$, ε_0, and R.

The remaining part of Ω_R, i.e., $\Omega_R \cap \{d(x) < h_0\}$, is covered by the union of sets T_k^+ since $R < R_k < 3R/2$ for $k = 0, 1, 2, \ldots$. Thus we will show that $w(x) \ge 0$ in T_k^+ for each k. To prove this, we will show the stronger inequality:

$$M_k := \inf_{T_k^+} w \ge \sup_{T_k} (-w)_+ =: m_k \quad \text{for } k = 0, 1, 2, \ldots, \tag{41}$$

where $(-w)_+ = \max\{-w, 0\}$. To prove (41), we use the principle of mathematical induction. In the basis case $k = 0$, by (34) and (38) with $h = h_1$, we have

$$M_0 \ge N \cdot N_2^{-1} - N_1 \ge N_1 \ge m_0 \quad \text{if } N \ge 2N_2N_1. \tag{42}$$

Suppose the estimate (41) is true for some $k > 0$. Then,

$$\begin{aligned} w \ge M_k \ge 0 \quad &\text{on} \quad \overline{T_k^+}, \\ w = Nv \ge 0 \quad &\text{on} \quad \Omega_{R_k} \cap \{d = 0\}. \end{aligned}$$

Since $\Omega_{R_k} \cap \{d(x) = h_{k+1}\} \subset \overline{T_k^+}$, the open set $O_k := T_k \cap \{w < 0\}$ is contained in $\Omega_{R_k} \cap \{d(x) < h_{k+1}\}$ and $w = 0$ on $\partial O_k \cap \Omega_{R_k}$. For fixed k, a function $m(R)$ is defined by

$$m(R) := \sup_{T_k \cap B_R} (-w)_+ \quad \text{for } 0 < R \le R_k. \tag{43}$$

By the definition of T_k, $m_k = m(R_k)$, and $m_{k+1} = m(R_{k+1})$ since $w \geq 0$ on T_k^+.

Next, we will show

$$m(\rho - 2h_k) \leq \beta_1 \cdot m(\rho) \quad \text{for } \rho \in (2h_k, R_k], \tag{44}$$

where a constant $\beta_1 \in (0,1)$ depends only on n, ν, S, and μ. Note that $2h_k = 2\varepsilon_0 \rho_k^{1/\alpha} < \rho_k$ since $\varepsilon_0 < 2^{-1}$ and $1 \leq \rho_k^{1-1/\alpha}$. If $m(\rho - 2h_k) = 0$, this is trivial. So let us assume $m(\rho - 2h_k) > 0$. Since the function w satisfies $Lw = 0$ in O_k and $w = 0$ on $\partial O_k \cap B_{\rho - 2h_k}$, by the maximal principle, there exists a point $z_0 \in O_k \cap (\partial B_{\rho-2h_k})$ such that $0 < m(\rho - 2h_k) = -w(z_0)$. Since $O_k \subset \Omega_{R_k} \cap \{d(x) < h_{k+1}\}$, $d(z_0) = |z_0 - z_1| \leq h_{k+1} = 2^{-1/\alpha} h_k \leq h_k/2$ for some point $z_1 \in \partial\Omega$. Using the strong regularity property (5) with the domain Ω, we have

$$|B_{h_k}(z_0) \setminus \Omega| \geq |B_{h_k/2}(z_0) \setminus \Omega| \geq \mu |B_{h_k/2}| = \mu \cdot 2^{-n} |B_{h_k}|. \tag{45}$$

Note that $w = 0$ on $(\partial O_k) \cap B_{R_k}$. By applying the growth Lemma 2.1 to the function $-w$ in $O_k = T_k \cap \{w < 0\}$ with $r = h_k$ and $\mu_1 = \mu \cdot 2^{-n}$,

$$-w(z_0) \leq \sup_{O_k \cap B_{h_k}(z_0)} (-w) \leq \beta_1 \sup_{O_k \cap B_{2h_k}(z_0)} (-w) \leq \beta_1 \sup_{O_k \cap B_\rho} (-w)\,. \tag{46}$$

Here the last inequality follows because $(\partial O_k) \cap B_{2h_k}(z_0) \subset (\partial O_k) \cap B_\rho$, where a point $z_0 \in O_k \cap (\partial B_{\rho-2h_k})$. Since $m(R) = \sup_{T_k \cap B_R} (-w)_+ = \sup_{O_k \cap B_R} (-w)_+$,

$$m(\rho - 2h_k) = -w(z_0) \leq \beta_1 \sup_{O_k \cap B_\rho} (-w) = \beta_1 m(\rho).$$

The proof of (44) is complete.

Let $\widetilde{R}_k := R + 3\rho_k$. Then $R_{k+1} < \widetilde{R}_k < R_k$ and $\widetilde{R}_k = R_k - \rho_k \leq R_k - p_k \cdot 2h_k$, where $p_k := [\rho_k/2h_k]$. Since an integer part $[a] \geq \max\{1, a/2\} \geq a/2$ for any real number $a \geq 1$ and $h_k := \varepsilon_0 \rho_k^{1/\alpha} < 2^{-1}\rho_k^{1/\alpha} < 2^{-1}\rho_k$, $p_k := [\rho_k/2h_k] \geq \rho_k/4h_k = \rho_k^{1-1/\alpha}/4\varepsilon_0$. Hence, by iterating (44) p_k times with $\rho = R_k$, we have

$$\widetilde{m}_k := m(\widetilde{R}_k) \leq m\left(R_k - p_k 2h_k\right) \leq \beta_1^{p_k} m\left(R_k\right) \leq \exp(-c_1\, \varepsilon_0^{-1}\, \rho_k^{1-1/\alpha}) m_k, \tag{47}$$

where $c_1 = c_1(n, \nu, S, \mu) = -\ln \beta_1/4 > 0$. Similarly, $m_{k+1} \leq \xi_k \widetilde{m}_k$ can be derived, where $\xi_k := \exp(-c_1\, \varepsilon_0^{-1}\, \rho_k^{1-1/\alpha})$.

Now we will prove the following:

$$\inf_{T_{k+1}^+} w_k \geq \eta_k \inf_{T_k^+} w_k, \tag{48}$$

where $w_k := w + \widetilde{m}_k$ and $\eta_k = \exp(-c_2 \varepsilon_0^{\alpha-1} \rho_k^{1-1/\alpha})$ with $c_2 = c_2(n, \nu, S, \Lambda, \alpha, R) > 0$. Note that each function $w_k \geq 0$ and $Lw_k = 0$ in $\widetilde{T}_k := \Omega_{\widetilde{R}_k} \cap \{d(x) < h_k\}$. We will apply the interior Harnack inequality to a function w_k in $\widetilde{T}_k$ to get the lower estimate for M_{k+1} in terms of M_k. Choose an arbitrary point $a = (x, y) \in T_{k+1}^+ = \Omega_{R_{k+1}} \cap \{h_{k+2} \leq d(a) < h_{k+1}\}$, where $x \in \mathbb{R}^{n-1}$ and

$y \in \mathbb{R}$. Let $\hat{a} = (x, \hat{y})$, where $d(\hat{a}) = h_{k+1} = \varepsilon_0 \rho_{k+1}^{1/\alpha}$. Let a rectifiable line $\gamma(a, \hat{a})$ be a vertical line connecting from the point a to the point $\hat{a}$. Take $s_i := (x, y_i) \in \gamma(a, \hat{a})$, where $a = s_0$, $\hat{a} = s_{q_k}$, and $y_i = y + \sum_{j=0}^{i-1} r_j$ with $r_j \leq d(s_j)/16$ for $j = 0, 1, 2, \ldots, q_k$. Here a constant q_k will be specified later. Note that $2^{-4} h_{k+2} \leq r_0$ and $r_j \leq r_{j+1}$ for all j. In addition, from (6) for $\hat{a} \in \Omega_{3R/2}$, there exists a constant K depending only on α, Λ, and R such that $\delta(\hat{a}) \leq K d^\alpha(\hat{a}) = K h_{k+1}^\alpha$. Now, we will have another constrained condition $\varepsilon_0 \leq K^{-1/\alpha}$. Then, we have

$$|\gamma(a, \hat{a})| \leq \delta(\hat{a}) \leq K \varepsilon_0^\alpha \rho_{k+1} \leq \rho_{k+1}. \tag{49}$$

Therefore, for each $s_i \in \gamma(a, \hat{a})$,

$$|s_i| + 8r_i \leq |s_0| + |s_i - s_0| + 8r_i \leq R_{k+1} + \rho_{k+1} + 2^{-1} h_{k+1} < \widetilde{R}_k. \tag{50}$$

This implies $B_{8r_i}(s_i) \subset \widetilde{T}_k := \Omega_{\widetilde{R}_k} \cap \{d(x) \leq h_k\}$ for all s_i. Since $|\gamma(a, \hat{a})| \leq K \varepsilon_0^\alpha \rho_{k+1}$ and $|s_i - s_{i-1}| \geq 2^{-4} \varepsilon_0 \rho_{k+2}^{1/\alpha}$ for all i, we can choose a finite number of points $s_0, s_1, \ldots, s_{q_k} \in \gamma(a, \hat{a})$ such that $s_0 = a \in T_{k+1}^+$ and $s_{q_k} = \hat{a} \in T_k^+$ with $q_k \leq |\gamma(a, \hat{a})| / 2^{-4} h_{k+2} = c\, \varepsilon_0^{\alpha-1} \rho_k^{1-1/\alpha}$, where a constant c depends on α, Λ, and R. Since $B_{8r_i}(s_i) \subset \widetilde{T}_k$, we can apply the interior Harnack inequality (2.2) to the function w_k in each ball $B_{8r_i}(s_i)$:

$$w_k(\hat{a}) = w_k(s_{q_k}) \leq N_0 w_k(s_{q_k - 1}) \leq \cdots \leq N_0^{q_k} w_k(s_0) = N_0^{q_k} w_k(a). \tag{51}$$

Therefore, $w_k(\hat{a}) \leq \exp(c_2 \varepsilon_0^{\alpha-1} \rho_k^{1-1/\alpha}) w_k(a)$, where $c_2 := c \ln N_0 > 0$ depends only on n, ν, S, α, Λ, and R. Since a is an arbitrary point in T_{k+1}^+,

$$\inf_{T_k^+} w_k \leq w_k(\hat{a}) \leq \exp(c_2 \varepsilon_0^{\alpha-1} \rho_k^{1-1/\alpha}) \inf_{T_{k+1}^+} w_k. \tag{52}$$

Lastly, let's assume $\varepsilon_0 \leq (c_1/c_2)^{1/\alpha}$, where c_1 and c_2 are constants in (47) and (52), which guarantee that $\xi_k \leq \eta_k$ for all k. Finally, by taking $\varepsilon_0 = \min\{2^{-1/\alpha}, K^{-1/\alpha}, (c_1/c_2)^{1/\alpha}\}$, which depends only on n, ν, S, Λ, α, and μ, from (47) and (52), we have

$$\begin{aligned} m_{k+1} + \widetilde{m}_k &\leq \eta_k \left(m_k + \widetilde{m}_k\right), \\ M_{k+1} + \widetilde{m}_k &\geq \eta_k \left(M_k + \widetilde{m}_k\right). \end{aligned}$$

This implies $M_{k+1} - m_{k+1} \geq \eta_k (M_k - m_k)$. Therefore, by the principle of mathematical induction, (41) is true for all k. In conclusion, the estimate (37) is true with $N = N(\varepsilon_0) := \max\{N_2^2, 2N_2N_1\}$, which depends on n, ν, S, μ, α, Λ, R, R_0, and $d(x_0)$. The proof is complete. □

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Hyejin Kim
Department of Mathematics and Statistics
University of Michigan–Dearborn
Dearborn, Michigan 48128, USA
E-mail address: khyejin@umich.edu