A PREDICTOR-CORRECTOR METHOD FOR FRACTIONAL EVOLUTION EQUATIONS

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Abstract. Numerical solutions for the evolutionary space fractional order differential equations are considered. A predictor corrector method is applied in order to obtain numerical solutions for the equation without solving nonlinear systems iteratively at every time step. Theoretical error estimates are performed and computational results are given to show the theoretical results.

1. Introduction

In this paper we discuss numerical approximate solutions for the fractional differential evolution equation with a nonlinear forcing term. The equation is described as

$$(1) \qquad \frac{\partial u(x,t)}{\partial t} = D_x^\alpha u(x,t) + f(u,x,t), \quad (x,t) \in \Omega \times (0,T]$$

with an initial condition

$$(2) \qquad u(x,0) = u_0(x), \quad x \in \bar{\Omega}$$

and Dirichlet boundary conditions

$$(3) \qquad u(x,t) = 0, \ x \in \partial\Omega, \quad t \in (0,T],$$

where $\partial\Omega$ is the boundary of the domain Ω and T is a positive real number as a terminal time. And the differential operator D_x^α is the Riemann-Liouville space fractional derivative of order α defined by

$$(4) \qquad D_x^\alpha \phi(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{\phi(\xi,t)}{(x-\xi)^{\alpha-n+1}} d\xi,$$

where n is an integer such that $n-1 < \alpha < n$ and $\Gamma(\cdot)$ is the gamma function

$$(5) \qquad \Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Throughout this paper, we will assume that the nonlinear source term $f(u, x, t)$ is bounded and globally Lipschitz continuous with a Lipschitz constant L with respect to u.

The fractional order diffusion equations have been discussed by many authors as generalizations of classical diffusion equation in order to treat sub- and super-diffusive processes. The values α of $0 < \alpha < 1$ and $1 < \alpha < 2$ model a sub-diffusive and a super-diffusive process, respectively, in the spatial direction and the model with $\alpha = 2$ describes the classical dispersion. The equation having these anomalous diffusion models in applications as fluid flow, finance, seeds dispersion and other biological sciences[1, 6, 9, 10, 14].

The analytical results on existence and uniqueness of the solution for (1) have been studied by Baeumer, Kovács and Meerschaert [2] using the semi-group theory when the source term $f(u, x, t)$ is globally Lipschitz continuous with respect to u. They have also shown that the solution exists uniquely by introducing the cut-off function when the function $f(u, x, t)$ is locally Lipschitz continuous. Moreover, they have shown that the solution is nonnegative when the initial condition $u_0(x)$ is nonnegative.

When the problem (1) is linear, that is, $f(u) = f(x, t)$, the finite difference numerical approximation to the problem (1)–(3) with Riemann-Liouville fractional derivative in one space dimension has been discussed by Meerschaert and Tadjeran [14]. It is known in [13] that the standard (unshifted) Grünwald formula is unstable regardless the finite difference scheme is either explicit or implicit but the right-shifted Grünwald formula allows the implicit Euler method is unconditionally stable. They used the Euler schemes with the shifted Grünwald estimate to the fractional derivative and showed that the explicit Euler scheme is conditionally stable. And they proved that the implicit Euler scheme is unconditionally stable by using Gerschigorin's circle theorem and obtained error estimates of $O(k + h)$. In order to obtain a second order of accuracy $O(k^2 + h^2)$, Tadjeran, Meerschaert and Scheffler [17] have applied the Crank-Nicolson difference scheme and the shifted Grünwald approximation with extrapolation technique.

For nonlinear problem of the type (1), numerical computations without any theoretical discussions are carried out by Lynch, Carreras, del-Castillo-Negrete, Ferreira-Mejias and Hicks [11] by using semi-implicit difference methods so called the L2 method and the L2C method defined in [15] and numerical results obtained by these methods are compared. Choi, Chung and Lee [4] have worked on numerical solutions for the problem (1) with a forcing term of Kolmogorov-Fisher type. They have studied existence, unconditional stability and error estimates with order of convergence $O(k + h)$ for the backward Euler method in time and the shifted Grünwald estimate in space of the equation (1). But they replaced the Riemann-Liouville fractional derivative $D_x^\alpha \phi$ by the Caputo

fractional derivative $D_*^\alpha \phi$ defined as

$$D_*^\alpha \phi(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\phi^{(n)}(\xi,t)}{(x-\xi)^{\alpha-n+1}} d\xi.$$

Choi and Chung [5] have studied finite element numerical solutions for the space fractional diffusion equation with a nonlinear source term.

For a two-dimensional problem with fractional Riemann-Liouville derivative when $f(u) = f(x,y,t)$, Meerschaert, Scheffler and Tadjeran [12] used an implicit Euler difference scheme with the Grünwald estimate for the fractional derivative. They obtained consistency and convergence of order $O(k+h)$ and applied alternating-direction implicit (ADI) method for numerical computations. Finite element methods have been also applied to the fractional advection diffusion equations by Roop [16] and Ervin, Heuer and Roop [8] when $f(u) = f(x,y,t)$.

In Section 2, we apply a backward Euler scheme with the quadrature rule so called L2 method for the fractional derivative term in (1). This leads us an implicit finite difference scheme to the equation (1) with a nonlinear forcing term in u and we have to solve nonlinear systems at each time step to compute approximate solutions. This may be a time consuming job. In order to reduce computing time of solving nonlinear systems at every time step, we use the one-step predictor-corrector method. In Section 3, we discuss the consistency and convergence of the predictor-corrector method, which gives error estimates of $O\big((k+h^{3-\alpha})(1+\frac{k}{h^\alpha})\big)$. In Section 4, computational examples are carried out to see convergence of the proposed scheme.

2. A predictor–corrector finite difference approximation

When we solve the equation (1) by using an implicit finite difference scheme as in [4], we have to solve an almost fully nonlinear system of equations due to the fractional derivative. It takes time when we use the generalized Newton's method to solve the system numerically, in general. In order to reduce the computing time, we may use a predictor–corrector method for (1). A predictor-corrector method has been applied to solve a fractional differential equation of the Caputo type

$$D_*^\alpha y(t) = f(t, y(t))$$

by Diethlem, Ford and Freed [7]. They adopted an equivalent Volterra integral equation and the integral term was approximated by the trapezoidal quadrature formula.

We consider first a well known initial value problem

(6) $$y'(t) = f(t, y(t)), \quad y(0) = y_0.$$

We assume that the function f to be so that the problem (6) has a unique solution on some interval $[0,T]$. For a finite difference numerical approximation

scheme, define $t_n = nk$ with a temporal step size k. Then we may see that inductively

(7) $$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))ds.$$

If we approximate the integral term by the trapezoidal quadrature formula

$$\int_a^b g(s)ds \approx \frac{b-a}{2}(g(a) + g(b)),$$

then the equation(7) becomes

$$y(t_{n+1}) = y(t_n) + \frac{k}{2}\{f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\}.$$

Hence we obtain the implicit one-step Adams-Bashford-Moulton method by replacing $y(t_n)$ and $y(t_{n+1})$ by their approximations y_n and y_{n+1}, respectively,

(8) $$y_{n+1} = y_n + \frac{k}{2}\{f(t_n, y_n) + f(t_{n+1}, y_{n+1})\}.$$

But it is impossible to solve y_{n+1} directly because the equation (8) is nonlinear due to the nonlinearity of $f(t, y(t))$. Therefore we may have to replace y_{n+1} by a known approximate value in order to compute explicitly.

We introduce the predictor y_{n+1}^P for y_{n+1} in a similar way replacing the trapezoidal quadrature formula by the rectangular rule

$$\int_a^b g(s)ds \approx (b-a)g(a).$$

This gives an explicit Euler method

(9) $$y_{n+1}^P = y_n + kf(t_n, y_n).$$

Hence we get the one-step method

(10) $$y_{n+1} = y_n + \frac{k}{2}\{f(t_n, y_n) + f(t_{n+1}, y_{n+1}^P)\},$$

which is known to be convergent of order 2. That is, if y is sufficiently smooth, then

$$\max_{0 \le i \le N} |y(t_i) - y_i| = O(k^2).$$

Let h be the grid size in the spatial direction with $h = 1/N$ and $x_i = ih$ for $i = 0, 1, 2, \ldots, N$. We now use the central difference approximate scheme for $\phi_{xx}(x, t)$ and apply the rectangular quadrature rule which is called L2 method as in [15]. Since for $1 < \alpha < 2$,

$$D_x^\alpha \phi(x_i, t) = \frac{\phi(0, t)}{\Gamma(1-\alpha)}x_i^{-\alpha} + \frac{\phi_x(0, t)}{\Gamma(2-\alpha)}x_i^{-\alpha} + \frac{1}{\Gamma(2-\alpha)}\int_0^{x_i} \frac{\phi_{xx}(y, t)}{(x_i - y)^{\alpha-1}}dy,$$

it may have singular terms at the left boundary. If we assume that the left boundary conditions are given as

$$\phi(0, t) = 0 = \phi_x(0, t), \quad t \ge 0,$$

we obtain

$$D_x^\alpha \phi(x_i, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^{x_i} \frac{\phi_{xx}(y, t)}{(x_i - y)^{\alpha-1}} dy$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} \int_{x_j}^{x_{j+1}} \frac{\phi_{xx}(x_i - y, t)}{y^{\alpha-1}} dy.$$

In this case the Riemann-Liouville fractional derivative $D_x^\alpha \phi(x_i, t)$ becomes the Caputo fractional derivative $D_*^\alpha \phi(x_i, t)$. Then the problem (1) becomes

$$(11) \qquad \frac{\partial u(x, t)}{\partial t} = D_*^\alpha u(x, t) + f(u(x, t)), \quad (x, t) \in \Omega \times (0, T].$$

We now introduce $\nabla_*^\alpha \phi(x_i, t)$ as an approximation to $D_*^\alpha \phi(x_i, t)$, which is defined as

$$\nabla_*^\alpha \phi(x_i, t)$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} \frac{\phi(x_i - x_{j-1}) - 2\phi(x_i - x_j) + \phi(x_i - x_{j+1})}{h^2} \int_{x_j}^{x_{j+1}} \frac{dy}{y^{\alpha-1}}$$

$$= \frac{1}{\Gamma(3-\alpha)h^\alpha} \sum_{j=0}^{i-1} [(j+1)^{2-\alpha} - j^{2-\alpha}]$$

$$\times [\phi(x_i - x_{j-1}) - 2\phi(x_i - x_j) + \phi(x_i - x_{j+1})]$$

$$= \sum_{j=-1}^{i} w_j(\alpha)\phi_{i-j},$$

where the weights $w_j(\alpha)$ are

$$w_{-1}(\alpha) = \frac{1}{\Gamma(3-\alpha)h^\alpha}, \quad w_0(\alpha) = \frac{2^{2-\alpha} - 3}{\Gamma(3-\alpha)h^\alpha},$$

$$w_j(\alpha) = \frac{(j+2)^{2-\alpha} - 3(j+1)^{2-\alpha} + 3j^{2-\alpha} - (j-1)^{2-\alpha}}{\Gamma(3-\alpha)h^\alpha}, \quad 1 \le j \le i-2,$$

$$w_{i-1}(\alpha) = \frac{-2i^{2-\alpha} + 3(i-1)^{2-\alpha} - (i-2)^{2-\alpha}}{\Gamma(3-\alpha)h^\alpha},$$

$$w_i(\alpha) = \frac{i^{2-\alpha} - (i-1)^{2-\alpha}}{\Gamma(3-\alpha)h^\alpha}.$$

In this case, the discretization error is $O(h^{3-\alpha})$. If we define a super diagonal matrix W as

$$W = \begin{bmatrix} w_0 & w_{-1} & 0 & 0 & \cdots & 0 \\ w_1 & w_0 & w_{-1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ w_{i-1} & \cdot & \cdot & \cdot & \cdots & w_{-1} \\ w_i & w_{i-1} & w_{i-2} & \cdots & w_1 & w_0 \end{bmatrix}.$$

Then we obtain the following lemma by simple calculation.

Lemma 2.1. *The sum of absolute values of entries of the matrix W is bounded. In fact,*

$$\sum_{j=-1}^{i} |w_j(\alpha)| \leq \frac{6 - 2 \cdot 2^{2-\alpha}}{\Gamma(3-\alpha)h^\alpha}, \quad 0 \leq i \leq N.$$

It follows from Lemma 2.1 that the matrix W is bounded.

Corollary 2.1. *Let $\|W\|_\infty = \sup_{x \neq 0} \frac{\|Wx\|_\infty}{\|x\|_\infty}$. Then there is a constant C such that $\|W\|_\infty \leq C$.*

Remark 2.1. Instead of using L2 approximation to the fractional derivative of Caputo type, we may use the right-shifted Grünwald formula defined as, for $1 < \alpha < 2$,

$$\nabla_*^\alpha \phi(x_i, t^n) = \frac{1}{h^\alpha} \sum_{m=0}^{i+1} g_m(\alpha)\phi_{i-m+1}^n,$$

where

$$g_m(\alpha) = \frac{\Gamma(m-\alpha)}{\Gamma(-\alpha)\Gamma(m+1)} = (-1)^m \binom{\alpha}{m} = (-1)^m \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}.$$

In this case, it is clear that Lemma 2.1 holds in much simpler form. That is, for the matrix derived W from the right-shifted Grünwald estimates

$$\sum_{j=-1}^{i} |w_j(\alpha)| \leq \frac{2\alpha}{h^\alpha}.$$

Since the right-shifted Grünwald formula produces a local truncation error of $O(h)$, the authors in [17] applied an extrapolation technique in order to obtain a second order $O(h^2)$.

For numerical solutions of (11), we first adopt the method of lines such as

$$(12) \qquad \frac{\partial u(x_i, t)}{\partial t} = D_*^\alpha u(x_i, t) + f(u(x_i, t)), \quad t \in (0, T].$$

We now apply the one-step predictor-corrector method to (12). Then we obtain a system of equations

$$(13) \qquad U^P(x_i, t_{n+1}) = U(x_i, t_n) + k\{\nabla_*^\alpha U(x_i, t_n) + f(U(x_i, t_n))\},$$

$$(14) \qquad \begin{aligned} U(x_i, t_{n+1}) = U(x_i, t_n) + \frac{k}{2}\{&\nabla_*^\alpha U(x_i, t_n) + f(U(x_i, t_n)) \\ &+ \nabla_*^\alpha U^P(x_i, t_{n+1}) + f(U^P(x_i, t_{n+1}))\}. \end{aligned}$$

Then the predictor-corrector method (13)–(14) can be rewritten by a matrix form as

$$(15) \qquad U^{n+1,P} = (I + kW)U^n + kf(U^n),$$

$$(16) \qquad U^{n+1} = (I + \frac{k}{2}W)U^n + \frac{k}{2}WU^{n+1,P} + \frac{k}{2}\{f(U^n) + f(U^{n+1,P})\}.$$

3. Convergence of the predictor-corrector method

The finite difference method (15)–(16) defined above has a local truncation error of $O(k^2 + h^{3-\alpha})$, since the approximation of the fractional derivative term by the L2 method is locally of $O(h^{3-\alpha})$ and that of the predictor-corrector method is of $O(k^2)$. It is clear that the method is consistent.

In order to show stability and convergence of approximate solutions for (11), we use the mathematical induction. Let $u(x_i, t_n)$ be the exact solution of (11) and $U(x_i, t_n)$ be the solution of (13)–(14). For the error estimates, we define the maximum errors as

$$\|e^{n,P}\|_\infty = \max_{0 \le i \le N} |u(x_i, t_n) - U^P(x_i, t_n)|$$

and

$$\|e^n\|_\infty = \max_{0 \le i \le N} |u(x_i, t_n) - U(x_i, t_n)|.$$

Before we show that the numerical scheme (13)–(14) is stable, we introduce the discrete Gronwall's inequality which will be used judiciously [3].

Lemma 3.1 (Gronwall's inequality). *Assume that $G(n)$, $a(n)$, and $w(n)$ are three sequences of real nonnegative numbers such that*

$$G(n) \le a(n) + \sum_{i=0}^{n-1} w(i)G(i), \ n = 1, 2, \dots.$$

Furthermore, assume that $a(n)$ is nondecreasing. Then

$$G(n) \le a(n) \exp \left(\sum_{i=0}^{n-1} w(i) \right).$$

Theorem 3.1. *Let $U(x_i, t_n)$ be the solution of (13)–(14). Then there is a positive constant C such that*

$$\|U^n\|_\infty \le C(\alpha, T, \|u\|_\infty).$$

Proof. Substituting (15) into (16), we obtain

$$(17) \qquad U^{n+1} = (\frac{k^2}{2}W^2 + kW + I)U^n + \frac{k}{2}(I + kW)f(U^n) + \frac{k}{2}f(U^{n+1,P}).$$

It follows from (17) that, summing from $n = 0$ to n,

$$U^{n+1} = U^0 + k(\frac{k}{2}W^2 + W)\sum_{m=0}^{n} U^m + \frac{k}{2}(kW + I)\sum_{m=0}^{n} f(U^m) + \frac{k}{2}\sum_{m=0}^{n} f(U^{m+1,P}).$$

Applying Corollary 2.1 and boundedness of f by M, we obtain

$$\|U^{n+1}\|_\infty \le \|U^0\|_\infty + k(\frac{k}{2}\|W\|_\infty^2 + \|W\|_\infty)\sum_{m=1}^{n} \|U^m\|_\infty + TM.$$

An application of the discrete Gronwall's inequality completes the proof. $\quad\square$

Then we obtain the following error estimates for the case $n = 1$.

Lemma 3.2. *There is a positive constant C such that*
$$\|e^{1,P}\|_\infty \le C(k^2 + kh^{3-\alpha}).$$

Proof. It follows from (1.1) and (13) that, for each i,
$$u(x_i, t_1) - U^P(x_i, t_1)$$
$$= u(x_i, t_0) - U(x_i, t_0) + \int_0^{t_1} \{D_*^\alpha u(x_i, s) - \nabla_*^\alpha U(x_i, t_0)\} ds$$
$$+ \int_0^{t_1} \{f(u(x_i, s)) - f(U(x_i, t_0))\} ds.$$

Since $U(x_i, t_0) = u(x_i, t_0)$, we obtain
$$|u(x_i, t_1) - U^P(x_i, t_1)|$$
$$\le \int_0^{t_1} \|D_*^\alpha u_t(x_i, \cdot)\|_\infty |s - t_0| ds$$
$$+ \int_0^{t_1} |D_*^\alpha u(x_i, t_0) - \nabla_*^\alpha u(x_i, t_0)| ds$$
$$+ \int_0^{t_1} |\nabla_*^\alpha u(x_i, t_0) - \nabla_*^\alpha U(x_i, t_0)| ds$$
$$+ L \int_0^{t_1} \{|u(x_i, s) - u(x_i, t_0)| + |u(x_i, t_0) - U(x_i, t_0)|\} ds$$
$$\le \frac{k^2}{2}(\|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty) + kh^{3-\alpha}.$$

This completes the proof. $\quad\square$

Lemma 3.3. *There is a positive constant C such that*
$$\|e^1\|_\infty \le C(k^2 + kh^{3-\alpha})\left(1 + \frac{k}{h^\alpha}\right).$$

Proof. It follows from again (1), (13) and (14) that
$$u(x_i, t_1) - U(x_i, t_1)$$
$$= u(x_i, t_0) + \int_0^{t_1} \{D_*^\alpha u(x_i, s) + f(u(x_i, s))\} ds$$
$$- \left\{ U(x_i, t_0) + \frac{k}{2}\{\nabla_*^\alpha U(x_i, t_0) + f(U(x_i, t_0)) \right.$$
$$\left. + \nabla_*^\alpha U^P(x_i, t_1) + f(U^P(x_i, t_1))\} \right\}$$
$$= u(x_i, t_0) - U(x_i, t_0)$$

$$+ \frac{1}{2} \int_0^{t_1} \left\{ D_*^\alpha u(x_i, s) - \nabla_*^\alpha U(x_i, t_0) + f(u(x_i, s)) - f(U(x_i, t_0)) \right\} ds$$

$$+ \frac{1}{2} \int_0^{t_1} \left\{ D_*^\alpha u(x_i, s) - \nabla_*^\alpha U^P(x_i, t_1) + f(u(x_i, s)) - f(U^P(x_i, t_1)) \right\} ds.$$

Since $u(x_i, t_0) = U(x_i, t_0)$, it follows from the proof of Lemma 3.2 that

$$|u(x_i, t_1) - U(x_i, t_1)|$$

$$\leq \frac{1}{2} |u(x_i, t_1) - U^P(x_i, t_1)|$$

$$+ \frac{1}{2} \int_0^{t_1} \left\{ |D_*^\alpha u(x_i, s) - \nabla_*^\alpha U^P(x_i, t_1)| + |f(u(x_i, s)) - f(U^P(x_i, s))| \right\} ds$$

$$\leq \frac{1}{2} |u(x_i, t_1) - U^P(x_i, t_1)|$$

$$+ \frac{1}{2} \int_0^{t_1} \left\{ |D_*^\alpha u(x_i, s) - D_*^\alpha u(x_i, t_1) + D_*^\alpha u(x_i, t_1) - \nabla_*^\alpha u(x_i, t_1) \right.$$

$$+ \nabla_*^\alpha u(x_i, t_1) - \nabla_*^\alpha U^P(x_i, t_1)| \right\} ds$$

$$+ \frac{1}{2} \int_0^{t_1} |f(u(x_i, s)) - f(u(x_i, t_1)) + f(u(x_i, t_1)) - f(U^P(x_i, t_1))| ds$$

$$\leq \frac{1}{2} |u(x_i, t_1) - U^P(x_i, t_1)| + \frac{1}{2} \int_0^{t_1} |D_*^\alpha u_t(x_i, \xi)(s - t_1)| ds$$

$$+ \frac{1}{2} \int_0^{t_1} \left\{ |D_*^\alpha u(x_i, t_1) - \nabla_*^\alpha u(x_i, t_1)| + |\nabla_*^\alpha u(x_i, t_1) - \nabla_*^\alpha U^P(x_i, t_1)| \right\} ds$$

$$+ \frac{1}{2} \int_0^{t_1} L|u_t(x_i, \eta)(s - t_1)| ds + \frac{1}{2} \int_0^{t_1} L|u(x_i, t_1) - U^P(x_i, t_1)| ds$$

$$\leq \frac{1}{2} |u(x_i, t_1) - U^P(x_i, t_1)| + \frac{k^2}{4} \|D_*^\alpha u_t(x_i, \cdot)\|_\infty$$

$$+ \frac{k}{2} |D_*^\alpha u(x_i, t_1) - \nabla_*^\alpha u(x_i, t_1)| + \frac{k}{2} |\nabla_*^\alpha u(x_i, t_1) - \nabla_*^\alpha U^P(x_i, t_1)|$$

$$+ \frac{k^2}{4} L \|u_t(x_i, \cdot)\|_\infty + \frac{k}{2} L|u(x_i, t_1) - U^P(x_i, t_1)|.$$

It follows from Lemma 3.2 that

$$|u(x_i, t_1) - U(x_i, t_1)|$$

$$\leq \frac{1}{2} (1 + kL)|u(x_i, t_1) - U^P(x_i, t_1)|$$

$$+ \frac{k^2}{4} \left\{ \|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty \right\}$$

$$+ \frac{k}{2} \left\{ |D_*^\alpha u(x_i, t_1) - \nabla_*^\alpha u(x_i, t_1)| + |\nabla_*^\alpha u(x_i, t_1) - \nabla_*^\alpha U^P(x_i, t_1)| \right\}.$$

Furthermore, since for some positive generic constant C

$$|D_*^\alpha u(x_i, t_1) - \nabla_*^\alpha u(x_i, t_1)| \leq Ch^{3-\alpha}$$

and

$$|\nabla_*^\alpha u(x_i, t_1) - \nabla_*^\alpha U^P(x_i, t_1)| \leq \frac{C}{h^\alpha}\|e^{1,P}\|_\infty,$$

there is a generic positive constant C such that

$$|u(x_i, t_1) - U(x_i, t_1)| \leq C(k^2 + kh^{3-\alpha})\big(1 + \frac{k}{h^\alpha}\big).$$

This completes the proof. $\square$

In order to use the mathematical induction, we assume that Lemmas 3.2–3.3 hold for t_n. Then we may obtain error estimates of the predictor-corrector method at t_{n+1}.

Lemma 3.4. *There is a positive constant C such that*

$$\|e^{n+1,P}\|_\infty \leq C(k^2 + kh^{3-\alpha})\big(1 + \frac{k}{h^\alpha}\big).$$

Proof. It follows from (1) and (13) that

$$u(x_i, t_{n+1}) = u(x_i, t_n) + \int_{t_n}^{t_{n+1}} \{D_*^\alpha u(x_i, s) + f(u(x_i, s))\}ds$$

and

$$U^P(x_i, t_{n+1}) = U(x_i, t_n) + k\{\nabla_*^\alpha U(x_i, t_n) + f(U(x_i, t_n))\}.$$

Following the proof of Lemma 3.2, we obtain

$$|u(x_i, t_{n+1}) - U^P(x_i, t_{n+1})|$$

$$\leq |u(x_i, t_n) - U(x_i, t_n)| + \int_{t_n}^{t_{n+1}} |D_*^\alpha u(x_i, s) - \nabla_*^\alpha U(x_i, t_n)|ds$$

$$+ \int_{t_n}^{t_{n+1}} L|u(x_i, s) - U(x_i, t_n)|ds$$

$$\leq (1 + kL)|u(x_i, t_n) - U(x_i, t_n)|$$

$$+ \frac{k^2}{2}\{\|D_*^\alpha u_t(x_i, \cdot)\|_\infty + \|u_t(x_i, \cdot)\|_\infty\}$$

$$+ k\{|D_*^\alpha u(x_i, t_n) - \nabla_*^\alpha u(x_i, t_n)| + |\nabla_*^\alpha u(x_i, t_n) - \nabla_*^\alpha U(x_i, t_n)|\}$$

$$\leq C(1 + kL + \frac{k}{h^\alpha})|u(x_i, t_n) - U(x_i, t_n)| + C(kh^{3-\alpha} + k^2).$$

This completes the proof. $\square$

Theorem 3.2. *There is a positive constant C such that*

$$\|e^{n+1}\|_\infty \leq C(k + h^{3-\alpha})\big(1 + \frac{k}{h^\alpha}\big).$$

Proof. It follows from (1) and (14) that

$$u(x_i, t_{n+1}) - U(x_i, t_{n+1})$$
$$= u(x_i, t_n) - U(x_i, t_n)$$
$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \left\{ D_*^\alpha u(x_i, s) - \nabla_*^\alpha U(x_i, t_n) + f(u(x_i, s)) - f(U(x_i, t_n)) \right\} ds$$
$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \left\{ D_*^\alpha u(x_i, s) - \nabla_*^\alpha U^P(x_i, t_{n+1}) + f(u(x_i, s)) - f(U^P(x_i, t_{n+1})) \right\} ds.$$

Thus, following the idea of proof in Lemma 3.3, we obtain

$$|u(x_i, t_{n+1}) - U(x_i, t_{n+1})|$$
$$\leq |u(x_i, t_n) - U(x_i, t_n)| + \frac{k^2}{2} \{ \|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty \}$$
$$+ \frac{kL}{2} |u(x_i, t_n) - U(x_i, t_n)| + \frac{kL}{2} |u(x_i, t_{n+1}) - U^P(x_i, t_{n+1})|$$
$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \{ |D_*^\alpha u(x_i, t_n) - \nabla_*^\alpha u(x_i, t_n)| + |\nabla_*^\alpha u(x_i, t_n) - \nabla_*^\alpha U(x_i, t_n)| \} ds$$
$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \left\{ |D_*^\alpha u(x_i, t_{n+1}) - \nabla_*^\alpha u(x_i, t_{n+1})| \right.$$
$$\left. + |\nabla_*^\alpha u(x_i, t_{n+1}) - \nabla_*^\alpha U^P(x_i, t_{n+1})| \right\} ds$$
$$\leq (1 + \frac{kL}{2}) |u(x_i, t_n) - U(x_i, t_n)| + \frac{k^2}{2} \{ \|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty \}$$
$$+ \frac{kL}{2} |u(x_i, t_n) - U^P(x_i, t_n)|$$
$$+ \frac{k}{2} \{ |D_*^\alpha u(x_i, t_n) - \nabla_*^\alpha u(x_i, t_n)| + |\nabla_*^\alpha u(x_i, t_n) - \nabla_*^\alpha U(x_i, t_n)| \}$$
$$+ \frac{k}{2} \{ |D_*^\alpha u(x_i, t_{n+1}) - \nabla_*^\alpha u(x_i, t_{n+1})| + |\nabla_*^\alpha u(x_i, t_{n+1}) - \nabla_*^\alpha U^P(x_i, t_{n+1})| \}$$
$$\leq (1 + \frac{kL}{2}) \|e^n\|_\infty + \frac{k^2}{2} \{ \|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty \}$$
$$+ \frac{kL}{2} \|e^{n,P}\|_\infty + \frac{k}{2} (Ch^{3-\alpha} + \frac{C}{h^\alpha} \|e^n\|_\infty) + \frac{k}{2} (Ch^{3-\alpha} + \frac{C}{h^\alpha} \|e^{n+1,P}\|_\infty).$$

It follows from Lemma 3.4 that we obtain, summing from $n = 0$ to n,

$$\|e^{n+1}\|_\infty \leq \|e^0\|_\infty + \frac{k^2}{2} \sum_{m=0}^{n} \{ \|D_*^\alpha u_t(x_i, \cdot)\|_\infty + L\|u_t(x_i, \cdot)\|_\infty \}$$
$$+ Ck \sum_{m=0}^{n} \{ h^{3-\alpha} + \|e^{m,P}\|_\infty \} + Ck\|e^{n+1,P}\|_\infty + Ck \sum_{m=0}^{n} \|e^m\|_\infty.$$

Now we apply Gronwall's inequality. Then we obtain

$$\|e^{n+1}\|_\infty \leq C(k + h^{3-\alpha})\big(1 + \frac{k}{h^\alpha}\big).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. Numerical experiments

In order to see the implementation of the previous theoretical results, we consider two examples.

Example 4.1. We first consider a space fractional linear diffusion equation

(18)
$$\frac{\partial u(x,t)}{\partial t} = D_*^\alpha u(x,t) + \frac{2t}{t^2+1}u(x,t)$$
$$- 2(t^2+1)\Big(\frac{x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{6x^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{12x^{4-\alpha}}{\Gamma(5-\alpha)}\Big)$$

with an initial condition

(19) $$u(x,0) = x^2(1-x)^2, \ x \in [0,1]$$

and boundary conditions

(20) $$u(0,t) = u(1,t) = 0.$$

When the diffusion coefficient $\alpha = 1.8$, the exact solution of the equation is known as

$$u(x,t) = (t^2+1)x^2(1-x)^2.$$

Table 1 shows the order of convergence and the maximum error between the exact solution and the approximate solution obtained by predictor-corrector method for (18)–(20) when $\alpha = 1.8$. For numerical computation, the temporal step size $k = 0.001$ is used. As seen in Table 1, order of convergence is close to $3 - \alpha$ but it decreases as h becomes smaller.

TABLE 1. Maximum error and orders of convergence.

h	$\|\mathbf{u} - \mathbf{u}_h\|_\infty$	
	Error	Order
1/4	2.00466e-02	-
1/8	7.20539e-03	1.476
1/16	2.93931e-03	1.294
1/32	1.29116e-03	1.187

According to Table 1, we may find that the order of convergence is close to $O(h^{3-\alpha})$ for this linear fractional diffusion problem (18)–(20) when $\alpha = 1.8$. But the order of convergence slightly decreases as h becomes smaller. This implies that numerical computations confirm the theoretical results.

We plot the exact solution and approximate solution obtained by the predictor-corrector method (2.10)–(2.11) using $h = 1/32$ and $k = 1/1000$ for (18)–(20) with $\alpha = 1.8$. Figure 1 shows the contour plots of an exact solution and numerical solution at $t = 1$.
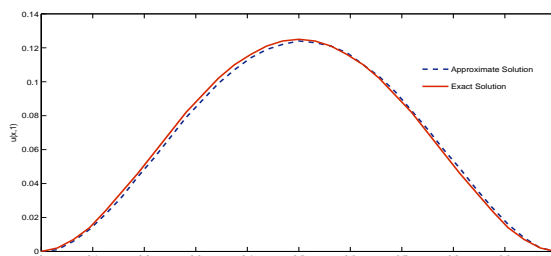

FIGURE 1. Exact and numerical solutions with $\alpha = 1.8$.

Example 4.2. We consider a space fractional diffusion equation with a nonlinear Fisher type source term which is described as

$$(21) \qquad \frac{\partial u(x,t)}{\partial t} = \kappa_\alpha D_*^\alpha u(x,t) + \lambda u(x,t)(1 - \beta u(x,t))$$

with an initial condition

$$u(x,0) = u_0(x)$$

and boundary conditions

$$(22) \qquad u(0,t) = u(10,t) = 0.$$

In fact, we will consider the case of $\kappa_\alpha = 0.1$, $\lambda = 0.25$, $\beta = 1$ in (21) with an initial condition

$$(23) \qquad u_0(x) = \begin{cases} e^{-10(x-5)}, & x \geq 5, \\ e^{10(x-5)}, & x < 5. \end{cases}$$

Choi and Chung [5] have obtained computational solutions for (21) with initial conditions

$$u_0(x) = \begin{cases} e^{-10x}, & x \geq 0, \\ e^{10x}, & x < 0. \end{cases}$$

using a Galerkin finite element method. We obtain computational results using the method as in Example 4.1. Figure 2 shows contour plots of numerical solutions at $t = 1$ for (21)–(23). In case, step sizes $h = 0.05$ and $k = 0.002$ are used for numerical computation. From the numerical results, we may find that numerical solutions converge to the solution of classical diffusion equation as α approaches to 2.
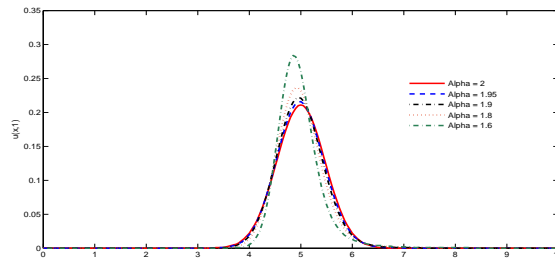
FIGURE 2. Numerical solutions for (21) with (23) with α.

5. Concluding remarks

Since the fractional derivative is defined as an weakly singular integral form, we have to solve an almost fully nonlinear system of equations at each time step when we apply an implicit finite difference method to the fractional evolution equations. It may need lot of computing time to do that. We have considered the one-step predictor-corrector method in order to avoid solving nonlinear systems. We discussed order of convergence for the predictor-corrector method and obtained order of $O\big((k + h^{3-\alpha})(1 + \frac{k}{h^{\alpha}})\big)$. This may give computational limitation for large α close to 2. Computational implementations are performed to a linear problem as well as a nonlinear problem. We see that computational results follow the theoretical ones.

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