Bull. Korean Math. Soc. **53** (2016), No. 6, pp. 1715–1723
http://dx.doi.org/10.4134/BKMS.b150897
pISSN: 1015-8634 / eISSN: 2234-3016

THE UNIT TANGENT SPHERE BUNDLE WHOSE CHARACTERISTIC JACOBI OPERATOR IS PSEUDO-PARALLEL

Jong Taek Cho and Sun Hyang Chun

Abstract. We study the characteristic Jacobi operator $\ell = \bar{R}(\cdot,\xi)\xi$ (along the Reeb flow ξ) on the unit tangent sphere bundle T_1M over a Riemannian manifold (M^n, g). We prove that if ℓ is pseudo-parallel, i.e., $\bar{R}\cdot\ell = L\mathcal{Q}(\bar{g},\ell)$, by a non-positive function L, then M is locally flat. Moreover, when L is a constant and $n \neq 16$, M is of constant curvature 0 or 1.

1. Introduction

It is intriguing to study the interplay between Riemannian manifolds and their unit tangent sphere bundles. In particular, we are interested in the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$ of a unit tangent sphere bundle T_1M over a given Riemannian manifold (M, g). It is remarkable that the characteristic vector field ξ on T_1M contains a crucial information about M. In fact, all the geodesics in M are controlled by the geodesic flow on T_1M which is precisely given by ξ. Apart from the defining structure tensors η, $\bar{g}$, ϕ and ξ, the so-called characteristic Jacobi operator $\ell = \bar{R}(\cdot,\xi)\xi$ plays a fundamental role in contact Riemannian geometry, especially in the unit tangent sphere bundle (cf. [2]). Here, $\bar{R}$ denotes the Riemannian curvature tensor determined by $\bar{g}$. In Section 3, we prove that the characteristic Jacobi operator ℓ vanishes if and only if M is locally flat (Proposition 2).

On the other hand, for a Riemannian manifold $(\bar{M}, \bar{g})$ a tensor field F of type $(1,3)$;

$$F : \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$$

is said to be *curvature-like* provided that F has the symmetric properties of $\bar{R}$. Here $\mathfrak{X}(\bar{M})$ is the Lie algebra of all vector fields on $\bar{M}$. For example,

Received November 2, 2015.
2010 *Mathematics Subject Classification.* 53C15, 53C25, 53D10.
Key words and phrases. unit tangent sphere bundle, contact metric structure, characteristic Jacobi operator.
The second author was supported by research fund from Chosun University, 2014.

$(\bar{X} \wedge \bar{Y})\bar{Z} = \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{Z}, \bar{X})\bar{Y}$, $\bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\bar{M})$, defines a curvature-like tensor field on $\bar{M}$. Note that a Riemannian manifold $(\bar{M}, \bar{g})$ of constant curvature c satisfies the formula $\bar{R}(\bar{X}, \bar{Y}) = c(\bar{X} \wedge \bar{Y})$.

As is well-known, a curvature-like tensor field F acts on the algebra $\mathcal{T}_s^1(\bar{M})$ of all tensor fields on $\bar{M}$ of type $(1, s)$ as a derivation (cf. [5]). Then P is said to be *semi-parallel* if $\bar{R} \cdot P = 0$, where $\cdot$ means that $\bar{R}$ acts as a derivation on P. Pseudo-parallelism is defined as the natural generalization. Namely, P is said to be *pseudo-parallel* if $\bar{R} \cdot P = L\mathcal{Q}(\bar{g}, P)$ for some function L, where $\mathcal{Q}(\bar{g}, P)$ is defined by

$$\mathcal{Q}(\bar{g}, P)(X_1, \ldots, X_s; Y, X) = (X \wedge Y)P(X_1, \ldots, X_s) - \sum_{j=1}^{s} P(X_1, \ldots, (X \wedge Y)X_j, \ldots, X_s).$$

In the present paper, we study pseudo-parallelism of the characteristic Jacobi operator ℓ on the unit tangent sphere bundle T_1M: $\bar{R} \cdot \ell = L\mathcal{Q}(\bar{g}, \ell)$ for a function L on T_1M. Then we easily see that vanishing ℓ implies pseudo-parallel ℓ. Moreover, pseudo-parallel ℓ includes the case of semi-parallel ℓ $(L = 0)$. The main purpose of the present paper is to prove the following.

Main Theorem. *Let (M, g) be an n-dimensional Riemannian manifold and T_1M be the unit tangent sphere bundle over M with the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. Suppose that the characteristic Jacobi operator ℓ of T_1M is pseudo-parallel by a function L on T_1M. Then we have the following results*:

(i) *if $L \leq 0$, then M is locally flat,*

(ii) *if L is constant and $n \neq 16$, then M is of constant curvature* 0 *or* 1.

Conversely, for the unit tangent sphere bundle over a space of constant curvature $c = 0$ or $c = 1$, the characteristic Jacobi operator ℓ is pseudo-parallel with $L = 0$ or $L = 1$, respectively.

2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. We start by collecting some fundamental material about contact metric geometry. We refer to [1] for further details. A $(2n+1)$-dimensional manifold $\bar{M}^{2n+1}$ is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, the *characteristic vector field,* satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \bar{X}) = 0$ for any vector field $\bar{X}$ on $\bar{M}$. It is well-known that there exists a Riemannian metric $\bar{g}$ on $\bar{M}$ and a $(1,1)$-tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{1}$$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $\bar{M}$. From (1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi\bar{X}, \phi\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}). \tag{2}$$

A Riemannian manifold $\bar{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (1) is said to be a contact metric manifold and is denoted by $\bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi)$. Given a contact metric manifold $\bar{M}$, we define the *structural operator* h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{3}$$

$$\bar{\nabla}_{\bar{X}}\xi = -\phi\bar{X} - \phi h\bar{X}, \tag{4}$$

where $\bar{\nabla}$ is the Levi-Civita connection. From (3) and (4) we see that each trajectory of ξ is a geodesic. We denote by $\bar{R}$ the Riemannian curvature tensor defined by

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \bar{\nabla}_{\bar{X}}(\bar{\nabla}_{\bar{Y}}\bar{Z}) - \bar{\nabla}_{\bar{Y}}(\bar{\nabla}_{\bar{X}}\bar{Z}) - \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}$$

for all vector fields $\bar{X}, \bar{Y}$ and $\bar{Z}$. Along a trajectory of ξ, the Jacobi operator $\ell = \bar{R}(\cdot, \xi)\xi$ is a symmetric $(1,1)$-tensor field. We call it *the characteristic Jacobi operator*. A contact metric manifold for which ξ is Killing is called a *K-contact manifold.* For a contact Riemannian manifold M one may define naturally an almost complex structure J on $\bar{M} \times \mathbb{R}$:

$$J(\bar{X}, f\frac{d}{dt}) = (\varphi\bar{X} - f\xi, \eta(\bar{X})\frac{d}{dt}),$$

where $\bar{X}$ is a vector field tangent to $\bar{M}$, t the coordinate on $\mathbb{R}$ and f a function on $\bar{M} \times \mathbb{R}$. If the almost complex structure J is integrable, $\bar{M}$ is said to be *normal* or *Sasakian.* It is known that a contact metric manifold $\bar{M}$ is normal if and only if $\bar{M}$ satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is also characterized by the condition $(\bar{\nabla}_{\bar{X}}\varphi)\bar{Y} = \bar{g}(\bar{X}, \bar{Y})\xi - \eta(\bar{Y})\bar{X}$ and this is equivalent to

$$\bar{R}(\bar{X}, \bar{Y})\xi = \eta(\bar{Y})\bar{X} - \eta(\bar{X})\bar{Y} \tag{5}$$

for all vector fields $\bar{X}$ and $\bar{Y}$.

Proposition 1. *For a Sasakian manifold, the characteristic Jacobi operator ℓ is pseudo-parallel with $L = 1$.*

Proof. Let $\bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi)$ be a Sasakian manifold. Then, from (5) we get

$$\ell\bar{X} = \bar{X} - \eta(\bar{X})\xi \tag{6}$$

for any vector field $\bar{X}$ on $\bar{M}$. Using (6) we compute

$$\begin{aligned} &(\bar{R}(\bar{X}, \bar{Y}) \cdot \ell)\bar{Z} \\ &= \bar{R}(\bar{X}, \bar{Y})\ell\bar{Z} - \ell(\bar{R}(\bar{X}, \bar{Y})\bar{Z}) \\ &= \eta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\xi - \eta(\bar{Y})\bar{g}(\bar{X}, \bar{Z})\xi + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X}, \end{aligned} \tag{7}$$

$$
\begin{aligned}
&L((\bar{X} \wedge \bar{Y}) \cdot \ell)\bar{Z} \\
&= L\{(\bar{X} \wedge \bar{Y})\ell\bar{Z} - \ell((\bar{X} \wedge \bar{Y})\bar{Z})\} \\
&= L\{\bar{g}(\bar{Y}, \ell\bar{Z})\bar{X} - \bar{g}(\bar{X}, \ell\bar{Z})\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})\ell\bar{X} + \bar{g}(\bar{X}, \bar{Z})\ell\bar{Y}\} \\
&= L\{\eta(\bar{X})\bar{g}(\bar{Y}, \bar{Z})\xi - \eta(\bar{Y})\bar{g}(\bar{X}, \bar{Z})\xi + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X}\}.
\end{aligned} \tag{8}
$$

Then from (7) and (8), we can see that ℓ is pseudo-parallel and $L = 1$. □

3. The contact metric structure of the unit tangent sphere bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [6], [9], [14]). We only briefly review some notations and definitions. Let $M = (M, g)$ be an n-dimensional Riemannian manifold and let TM denote its tangent bundle with the projection $\pi : TM \to M$, $\pi(p, u) = p$. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X) \circ \pi$, where ω is a 1-form on M. For the Levi Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h, Y^h) = \tilde{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \tilde{g}(X^h, Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then $\tilde{g}$ is a Hermitian metric for the almost complex structure J.

The unit tangent sphere bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u, u) = 1$. Note that $\bar{\pi} = \pi \circ i$, where i is the immersion of T_1M into TM. A unit normal vector field $N = u^v$ to T_1M is given by the vertical lift of u for (p, u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of a vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p, u) \in T_1M$ by

$$X^t_{(p,u)} = (X - g(X, u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the standard contact metric structure of the unit tangent sphere bundle T_1M over a Riemannian manifold (M, g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Since $g'(\bar{X}, \phi'\bar{Y}) = 2d\eta'(\bar{X}, \bar{Y})$, (η', g', ϕ', ξ') is not a contact metric structure. If we rescale this structure by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

we get the standard contact metric structure $(\eta, \bar{g}, \phi, \xi)$. The tensors ξ and ϕ are explicitly given by

$$\xi = 2u^h, \quad \phi X^t = -X^h + \frac{1}{2}\, g(X,u)\xi, \quad \phi X^h = X^t, \tag{9}$$

where X and Y are vector fields on M.

From now on, we consider $T_1M = (T_1M; \eta, \bar{g}, \phi, \xi)$ with the standard contact metric structure. Then the Levi-Civita connection $\bar{\nabla}$ of T_1M is described by

$$\begin{aligned}
\bar{\nabla}_{X^t} Y^t &= -g(Y,u)X^t, \\
\bar{\nabla}_{X^t} Y^h &= \frac{1}{2}\,(R(u,X)Y)^h, \\
\bar{\nabla}_{X^h} Y^t &= (\nabla_X Y)^t + \frac{1}{2}\,(R(u,Y)X)^h, \\
\bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2}\,(R(X,Y)u)^t
\end{aligned} \tag{10}$$

for all vector fields X and Y on M.

Also the Riemann curvature tensor $\bar{R}$ of T_1M is given by

$$\begin{aligned}
\bar{R}(X^t,Y^t)Z^t &= -(g(X,Z) - g(X,u)g(Z,u))Y^t \\
&\quad + (g(Y,Z) - g(Y,u)g(Z,u))X^t, \\
\bar{R}(X^t,Y^t)Z^h &= \big\{R(X - g(X,u)u, Y - g(Y,u)u)Z\big\}^h \\
&\quad + \frac{1}{4}\big\{[R(u,X), R(u,Y)]Z\big\}^h, \\
\bar{R}(X^h,Y^t)Z^t &= -\frac{1}{2}\big\{R(Y - g(Y,u)u, Z - g(Z,u)u)X\big\}^h \\
&\quad - \frac{1}{4}\big\{R(u,Y)R(u,Z)X\big\}^h, \\
\bar{R}(X^h,Y^t)Z^h &= \frac{1}{2}\big\{R(X,Z)(Y - g(Y,u)u)\big\}^t - \frac{1}{4}\big\{R(X,R(u,Y)Z)u\big\}^t \\
&\quad + \frac{1}{2}\big\{(\nabla_X R)(u,Y)Z\big\}^h, \\
\bar{R}(X^h,Y^h)Z^t &= \big\{R(X,Y)(Z - g(Z,u)u)\big\}^t \\
&\quad + \frac{1}{4}\big\{R(Y,R(u,Z)X)u - R(X,R(u,Z)Y)u\big\}^t \\
&\quad + \frac{1}{2}\big\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\big\}^h, \\
\bar{R}(X^h,Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}\big\{R(u,R(X,Y)u)Z\big\}^h \\
&\quad - \frac{1}{4}\big\{R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y\big\}^h \\
&\quad + \frac{1}{2}\big\{(\nabla_Z R)(X,Y)u\big\}^t
\end{aligned} \tag{11}$$

for all vector fields X, Y and Z on M. Using the formulae (11), we get

$$\begin{aligned} \ell X^t &= (R_u^2 X)^t + 2(R'_u X)^h, \\ \ell X^h &= 4(R_u X)^h - 3(R_u^2 X)^h + 2(R'_u X)^t, \end{aligned} \tag{12}$$

where $R_u = R(\cdot, u)u$, $R'_u = (\nabla_u R)(\cdot, u)u$ and $R_u^2 = R(R(\cdot, u)u, u)u$. We can refer to [2, 3, 4] for the formulas (10) $\sim$ (12). From (12), we have the following proposition.

Proposition 2. *The characteristic Jacobi operator ℓ of T_1M vanishes if and only if M is locally flat.*

Proof. Suppose that the characteristic Jacobi operator ℓ vanishes. Then we get from (12) $R'_u X = 0$ and $R_u^2 X = 0$. The former implies that (M, G) is a locally symmetric space ([8], [13]) and the latter does that the eigenvalues of R_u are constant and equal to 0, i.e., (M, G) is a globally Osserman space (i.e., the eigenvalues of R_u do not depend on the point p and not on the choice of unit vector u at p). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([7]). Therefore, we conclude that M is a space of constant curvature 0. □

4. Proof of Main Theorem

Suppose that the characteristic Jacobi operator ℓ of T_1M is pseudo-parallel by a function L on T_1M. Then T_1M satisfies

$$\begin{aligned} &\bar{R}(\bar{X}, \bar{Y})\ell\bar{Z} - \ell(\bar{R}(\bar{X}, \bar{Y})\bar{Z}) \\ &= L\{\bar{g}(\bar{Y}, \ell\bar{Z})\bar{X} - \bar{g}(\bar{X}, \ell\bar{Z})\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})\ell\bar{X} + \bar{g}(\bar{X}, \bar{Z})\ell\bar{Y}\}. \end{aligned} \tag{13}$$

We put $\bar{Y} = \xi$ in (13). Then we have

$$\bar{R}(\bar{X}, \xi)\ell\bar{Z} - \ell(\bar{R}(\bar{X}, \xi)\bar{Z}) = L\{-\bar{g}(\bar{X}, \ell\bar{Z})\xi - \eta(\bar{Z})\ell\bar{X}\}. \tag{14}$$

Setting $\bar{X} = X^t$, $\bar{Z} = Z^t$ in (14), and applying the Riemmanian metric $\bar{g}$ on T_1M for Y^h on both sides, then we have the following equation:

$$\begin{aligned} &\frac{1}{2}g(R(X, R_u^2 Z)u, Y) + \frac{1}{2}g(X, u)g(R_u^3 Z, Y) + \frac{1}{4}g(R(X, u)R_u^3 Z, Y) \\ &- g((\nabla_u R)(u, X)R'_u Z, Y) = -\frac{1}{4}Lg(X, R_u^2 Z)g(Y, u). \end{aligned} \tag{15}$$

We put $Y = u$ in (15). Then we have

$$g(-\frac{1}{4}R_u^4 X - {R'_u}^2 X, Z) = -\frac{1}{4}Lg(R_u^2 X, Z)$$

for any vector fields X and Z on M, that is, it holds

$$R_u^4 X + 4{R'_u}^2 X = L R_u^2 X. \tag{16}$$

Since R_u is symmetric operator, if $L \leq 0$, from (16) we have $R'_u = 0$ and $R_u = 0$. Therefore, using the similar arguments in the proof of Proposition 2 we see that M is locally flat. This completes the proof of (i).

Next, in order to prove the second part of Main Theorem we prepare the following lemma.

Lemma 3. *Let (M, g) be a locally symmetric space. Then the characteristic Jacobi operator ℓ of T_1M is pseudo-parallel by a function L on T_1M if and only if M is of constant curvature* 0 *or* 1.

Proof. If we set $\bar{X} = X^h$, $\bar{Z} = Z^h$ in (14), and apply the Riemmanian metric $\bar{g}$ on T_1M for Y^h on both sides, then we have the following equation:

(17)

$$\begin{aligned}
&4g(R(X,u)R_uZ,Y) + 2g(R(u,R_uX)R_uZ,Y) - g(R(R_u^2Z,u)X,Y)\\
&- g(R(X,R_uZ)u,R_uY) - 3g(R(X,u)R_u^2Z,Y) - \frac{3}{2}g(R(u,R_uX)R_u^2Z,Y)\\
&+ \frac{3}{4}g(R(R_u^3Z,u)X,Y) - \frac{3}{4}g(R(R_u^2Z,X)u,R_uY) + g((\nabla_X R)(u,R_u'Z)u,Y)\\
&- g((\nabla_u R)(u,R_u'Z)X,Y) - 4g(R(X,u)Z,R_uY) + 3g(R(X,u)Z,R_u^2Y)\\
&- 2g(R(u,R_uX)Z,R_uY) + \frac{3}{2}g(R(u,R_uX)Z,R_u^2Y) + g(R(R_uZ,u)X,R_uY)\\
&- \frac{3}{4}g(R(R_uZ,u)X,R_u^2Y) + g(R(X,Z)u,R_u^2Y) - \frac{3}{4}g(R(X,Z)u,R_u^3Y)\\
&- g((\nabla_Z R)(X,u)u,R_u'Y)\\
= {}& \frac{1}{4}L\{-4g(X,R_uZ)g(Y,u) + 3g(X,R_u^2Z)g(Y,u) - 4g(R_uX,Y)g(Z,u)\\
&+ 3g(R_u^2X,Y)g(Z,u)\}.
\end{aligned}$$

Putting $Y = u$ in (17), we have

$$-\frac{9}{4}R_u^4X + 6R_u^3X - 4R_u^2X - {R_u'}^2X = \frac{1}{4}L(-4R_uX + 3R_u^2X). \tag{18}$$

We suppose that M is locally symmetric. Then from (16) and (18), we obtain

$$R_u^4X = LR_u^2X, \tag{19}$$

$$-9R_u^4X + 24R_u^3X - 16R_u^2X = L(-4R_uX + 3R_u^2X). \tag{20}$$

We assume that $R_uX = \lambda X$ for a function λ on M. Then from (19) and (20), we have

$$\lambda^4 = L\lambda^2, \tag{21}$$

$$9\lambda^4 - 24\lambda^3 + 16\lambda^2 - 4L\lambda + 3L\lambda^2 = 0. \tag{22}$$

From (21), we have $\lambda = 0$ or $L = \lambda^2$. If $L = \lambda^2$ and $\lambda \neq 0$, from (22), we have

$$(3\lambda - 4)(\lambda - 1) = 0.$$

Hence, $\lambda = 0, 1$ or $\frac{4}{3}$, and then (M, g) is a globally Osserman space. But, it is also locally symmetric, and then it is locally isometric to a rank one symmetric space. However, we can easily check that T_1M of a space of constant curvature

$\frac{4}{3}$ does not satisfy pseudo-parallelism of ℓ. Therefore, we conclude that (M, g) is of constant curvature 0 or 1. By Propositions 1 and 2, the converse is easily proved. □

Now we assume that L is constant. Then, from (16) and (18), we have

$$2R_u^4X - 6R_u^3X + 4R_u^2X = L(R_uX - R_u^2X). \tag{23}$$

If we put $R_uX = \lambda X$, we get

$$\lambda(\lambda - 1)(2\lambda^2 - 4\lambda + L) = 0. \tag{24}$$

Here, we use Nikolayevsky's results ([10, 11, 12]) on the Osserman conjecture. Then we find that (M^n, g) is locally isometric to a rank one symmetric space, when $n \neq 16$. Thus, by Lemma 3 we conclude that (M, g) is of constant curvature 0 or 1, when $n \neq 16$. Conversely, by Propositions 1 and 2, we see that for the unit tangent sphere bundle over a space of constant curvature $c = 0$ or $c = 1$, the characteristic Jacobi operator ℓ is pseudo-parallel with $L = 0$ or $L = 1$, respectively. This completes the proof of Main Theorem.

Corollary 4. *If ℓ of T_1M is semi-parallel, that is, $L = 0$, then M is locally flat.*

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Second edition, Progr. Math. 203, Birkhäuser Boston, Inc., Boston, MA, 2010.
[2] E. Boeckx, J. T. Cho, and S. H. Chun, *Flow-invariant structures on unit tangent bundles*, Publ. Math. Debrecen **70** (2007), no. 1-2, 167–178.
[3] E. Boeckx and L. Vanhecke, *Characteristic reflections on unit tangent sphere bundles*, Houston J. Math. **23** (1997), no. 3, 427–448.
[4] J. T. Cho and S. H. Chun, *On the classification of contact Riemannian manifolds satisfying the condition* (C), Glasg. Math. J. **45** (2003), no. 3, 475–492.
[5] J. T. Cho and J.-I. Inoguchi, *Pseudo-symmetric contact 3-manifolds*, J. Korean Math. Soc. **42** (2005), no. 5, 913–932.
[6] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88.
[7] P. Gilkey, A. Swann, and L. Vanhecke, *Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator*, Quart. J. Math. Oxford Ser. (2) **46** (1995), no. 183, 299–320.
[8] A. Gray, *Classification des variétés approximativement kählériennes de courbure sectionelle holomorphe constante*, J. Reine Angew. Math. **279** (1974), 797–800.
[9] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. **250** (1971), 124–129.
[10] Y. Nikolayevsky, *Osserman manifolds of dimension 8*, Manuscr. Math. **115** (2004), no. 1, 31–53.
[11] ———, *Osserman conjecture in dimension $n \neq 8, 16$*, Math. Ann. **331** (2005), no. 3, 505–522.
[12] ———, *On Osserman manifolds of dimension 16*, Contemporary geometry and related topics, 379–398, Univ. Belgrade Fac. Math., Belgrade, 2006.
[13] L. Vanhecke and T. J. Willmore, *Interactions of tubes and spheres*, Math. Anal. **21** (1983), no. 1, 31–42.
[14] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc., 1973.

Jong Taek Cho
Department of Mathematics
Chonnam National University
Gwangju 61186, Korea
E-mail address: jtcho@chonnam.ac.kr

Sun Hyang Chun
Department of Mathematics
Chosun University
Gwangju 61452, Korea
E-mail address: shchun@chosun.ac.kr