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ON THE MINIMAL FREE RESOLUTION OF CURVES OF MAXIMAL REGULARITY

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Abstract. Let $C \subset \mathbb{P}^r$ be a nondegenerate projective curve of degree $d > r+1$ and of maximal regularity. Such curves are always contained in the threefold scroll $S(0,0,r-2)$. Also some of such curves are even contained in a rational normal surface scroll. In this paper we study the minimal free resolution of the homogeneous coordinate ring of C in the case where $d \leq 2r-2$ and C is contained in a rational normal surface scroll. Our main result provides all the graded Betti numbers of C explicitly.

1. Introduction

Let $C \subset \mathbb{P}^r$ $(r \geq 3)$ be a nondegenerate irreducible projective curve of degree d defined over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. We say that C is m-regular if its ideal sheaf $\mathcal{I}_C$ satisfies the following vanishing

$$H^i(\mathbb{P}^r, \mathcal{I}_C(m-i)) = 0 \quad \text{for all } i \geq 1.$$

The Castelnuovo-Mumford regularity (or simply the regularity) $\text{reg}(C)$ of C is defined as the smallest integer m such that C is m-regular. In the classical paper [6], the authors proved that $\text{reg}(C) \leq d-r+2$. They further classified all curves having the maximal regularity $\text{reg}(C) = d-r+2$. In particular, if $d \geq r+2$, then C is a curve of maximal regularity if and only if it is a smooth rational curve having a $(d-r+2)$-secant line, say $\mathbb{L}$. Later, M. Brodmann and P. Schenzel study algebraic properties of C by analyzing the union $C \cup \mathbb{L}$ (cf. [1] and [2]). Most of all, it is shown that if $d \leq 2r-2$, then $C \cup \mathbb{L}$ is arithmetically Cohen-Macaulay. In this case, they also describe the Hartshorn-Rao module and the shape of the minimal free resolution of the homogeneous ideal of C. However the problem to determine the graded Betti numbers explicitly is still open.

The present paper is devoted to the study of the graded Betti numbers of a curve $C \subset \mathbb{P}^r$ of degree $d \geq r+2$ and of maximal regularity. Let $\mathbb{L}$ be a $(d-r+2)$-secant line to C. Then the join of C with $\mathbb{L}$ is the singular threefold scroll $S(0,0,r-2)$ whose singular locus is exactly equal to $\mathbb{L}$. Note

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that $S(0,0,r-2)$ contains smooth rational normal surface scrolls projectively equivalent to $S(1,r-2)$. Thus we will say that C is *of type I* if it is contained in a rational normal surface scroll and *of type II* otherwise. Note that the graded Betti numbers of C of type I are uniquely determined by r and d (cf. Proposition 4.1 in [8]). On the other hand, curves of type I and curves of type II must have different Betti tables by Green's $K_{p,1}$ Theorem (cf. Example 5.3 in [1]). This means that for every $r \geq 3$ and $d \geq r+2$, there exist at least two different kinds of Betti tables of curves of maximal regularity.

Our main result in the present paper provides the explicit values of all the graded Betti numbers of C of type I with degree $d \leq 2r-2$.

Theorem 1.1. *Let $C \subset \mathbb{P}^r$ be a curve of degree $d \geq r+2$ and of maximal regularity of type I and let R and R_C denote respectively the homogeneous coordinate rings of $\mathbb{P}^r$ and C. If $d \leq 2r-2$, then*

$$\mathrm{Tor}_i^R(R_C,\mathbb{K}) \simeq \mathbb{K}(-1-i)^{\beta_{i,1}(C)} \oplus \mathbb{K}(-2-i)^{\beta_{i,2}(C)} \oplus \mathbb{K}(-d+r-1-i)^{\beta_{i,d-r+1}(C)}$$

and

$$\begin{cases} \beta_{i,1}(C) = \begin{cases} i\binom{r-1}{i+1} + (2r-1-d-i)\binom{r-1}{i-1} & \text{for } 1 \leq i \leq 2r-2-d, \\ i\binom{r-1}{i+1} & \text{for } 2r-1-d \leq i \leq r, \end{cases} \\ \beta_{i,2}(C) = \begin{cases} 0 & \text{for } 1 \leq i \leq 2r-2-d, \\ (i+d+2-2r)\binom{r-1}{i} & \text{for } 2r-1-d \leq i \leq r, \text{ and} \end{cases} \\ \beta_{i,d-r+1}(C) = \binom{r-1}{i-1}. \end{cases}$$

In particular, if $d = 2r-2$ and hence $\mathrm{reg}(C) = r$, then the homogeneous ideal I_C of C is minimally generated by $\binom{r-1}{2}$ quadratic equations, $(r-1)$ cubic equations and a form of degree r. Also if $d \leq 2r-3$, then I_C is minimally generated by $\left(\binom{r+1}{2} - d - 1\right)$ quadratic equations and a form of degree $d-r+2$.

We should mention that the authors in [2] prove the vanishing $\beta_{i,2}(C) = 0$ for $1 \leq i \leq 2r-2-d$ in our theorem without assuming that C is of type I.

At the end of Section 3, we provide an example of two curves C_1 and C_2 in $\mathbb{P}^{10}$ of maximal regularity such that C_1 is of type I and C_2 is of type II.

Remark 1.2. According to Theorem 1.1 and [8, Proposition 4.1], it is natural to ask how many different Betti diagrams of curves of maximal regularity of type II exist for each pair (r,d) with $r \geq 3$ and $d \geq r+2$.

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2. Preliminaries

Throughout this section, let $C \subset \mathbb{P}^r$ be a curve of degree $d \geq r+2$ and of maximal regularity. Also let Z be the scheme-theoretic union of C and a $(d-r+2)$-secant line $\mathbb{L}$ to C. That is, the ideal sheaf $\mathcal{I}_Z$ of Z in $\mathbb{P}^r$ is equal to the intersection $\mathcal{I}_C \cap \mathcal{I}_{\mathbb{L}}$ of the ideal sheaves $\mathcal{I}_C$ and $\mathcal{I}_{\mathbb{L}}$ of C and $\mathbb{L}$.

Lemma 2.1. *Suppose that C is of type I and let $S \subset \mathbb{P}^r$ be a rational normal surface scroll which contains C. Then $S = S(1, r-2)$ and $\mathbb{L}$ is exactly the line section $S(1)$ of S. Furthermore, C is linearly equivalent to $H + (d-r+1)F$ as a divisor of S where H and F are respectively a hyperplane section and a ruling line of S.*

Proof. Let $S = S(a_1, a_2)$ be a rational normal surface scroll that contains C. Assume that the divisor class of C is $aH + bF$ for some $a \geq 1$. First we will show that $a_1 > 0$ and $a = 1$. Indeed if $a_1 = 0$, then C should be arithmetically Cohen-Macaulay(see [4]). On the other hand, our curve C is an isomorphic projection of a rational normal curve and hence it is not linearly normal. Now suppose that $a \geq 2$ and consider the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-aH - bF) \to 0.$$

Then since S is arithmetically Cohen-Macaulay, we have the exact sequence

$$0 \to H^1(\mathbb{P}^r, \mathcal{I}_C(1)) \to H^1(S, \mathcal{O}_S((1-a)H - bF)) \to \cdots .$$

Also $H^1(S, \mathcal{O}_S((1-a)H - bF)) = 0$ for $a \geq 2$. Thus C is linearly normal which is a contradiction. This completes the proof that $a = 1$. Now, by using the degree of C, it can be shown that $b = d-r+1$. Finally, it remains to show that $a_1 = 1$ and a line section $S(1)$ of S is a line $\mathbb{L}$. Indeed, the line $\mathbb{L}$ is contained in S since S is cut out by quadratic equations and $\sharp(S \cap \mathbb{L}) \geq \sharp(C \cap \mathbb{L}) \geq 3$. Also the last inequality implies that $\mathbb{L}$ cannot be a ruling of S. In consequence, $\mathbb{L}$ must be a line section of S and hence $a_1 = 1$. $\square$

Remark 2.2. Lemma 2.1 implies that the surface scroll S that contains C of type I is already contained in the threefold $S(0, 0, r-2)$ which is given as the join of C and $\mathbb{L}$.

We use the convention that $\binom{a}{b} = 0$ if $a < b$.

Lemma 2.3. *Suppose that C is of type I and $d \leq 2r-2$. Then*

(1) *Z is arithmetically Cohen-Macaulay and* $\operatorname{reg}(Z) = 3$.

(2) $h^0(\mathbb{P}^r, \mathcal{I}_C(j)) = h^0(\mathbb{P}^r, \mathcal{I}_Z(j)) + \binom{d-r+1-j}{1}$ *for all $j \geq 0$.*

(3) $h^1(\mathbb{P}^r, \mathcal{I}_C(j)) = \binom{d-r+1-j}{1}$ *for all $j \geq 1$.*

Proof. Note that $\mathbb{L} = S(1)$ by Lemma 2.1 and hence it is linearly equivalent to $H - (r-2)F$. Also $\Gamma = C \cap \mathbb{L}$ is a finite scheme of length $(d-r+2)$ on the line $\mathbb{L}$.

(1) Note that Z is linearly equivalent to $2H + (d-2r+3)F$. For $d \leq 2r-2$, it follows by [8, Theorem 4.3] that Z is arithmetically Cohen-Macaulay and $\operatorname{reg}(Z) = 3$.

(2) Observe that

$$\mathcal{I}_C/\mathcal{I}_Z = \big(\mathcal{I}_C + \mathcal{I}_{\mathbb{L}}\big)/\mathcal{I}_{\mathbb{L}} \cong \mathcal{O}_{\mathbb{L}}(-\Gamma) = \mathcal{O}_{\mathbb{P}^1}(-d+r-2).$$

Thus we have the exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{I}_C \to \mathcal{O}_{\mathbb{P}^1}(-d+r-2) \to 0. \tag{2.1}$$

This sequence yields the exact sequence

(2.2) $0 \to H^0(\mathbb{P}^r, \mathcal{I}_Z(j)) \to H^0(\mathbb{P}^r, \mathcal{I}_C(j)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-d+r-2)) \to 0$

for all $j \geq 0$. Then the assertion comes immediately from the sequence (2.2).
(3) Since Z is arithmetically Cohen-Macaulay, we get the following exact sequence:

$$0 \to H^1(\mathbb{P}^r, \mathcal{I}_C(j)) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-d+r-2)) \to H^2(\mathbb{P}^r, \mathcal{I}_Z(j)) \to 0.$$

Also $H^2(\mathbb{P}^r, \mathcal{I}_Z(j)) = 0$ for $j \geq 1$ since Z is 3-regular. Therefore $H^1(\mathbb{P}^r, \mathcal{I}_C(j)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-d+r-2))$ which completes the proof. □

Remark 2.4. By combining Lemma 2.3 with the fact that $\operatorname{reg}(C) = d-r+2$ and $\operatorname{reg}(Z) = 3$, one can see that the homogeneous ideal I_C of C is generated by the homogeneous ideal I_Z of Z and a form of degree $d-r+2$.

3. Graded Betti numbers

Our aim of this section is to study the graded Betti numbers of curves of maximal regularity of type I.

Notation and Remarks 3.1. Let $C \subset \mathbb{P}^r$ ($r \geq 4$) be a curve of degree $d \leq 2r-2$ and of type I. Let $\mathbb{L}$ be a $(d-r+2)$-secant line to C. Also let $S = S(1, r-2)$ be a rational normal surface scroll which contains C (cf. Lemma 2.1).

(1) The union $Z = C \cup \mathbb{L}$ is arithmetically Cohen-Macaulay. Also C and Z are respectively linearly equivalent to $H + (d-r+1)F$ and $2H + (d-2r+3)F$ where H and F are respectively a hyperplane section and a ruling line of S. The finite subscheme $\Gamma = C \cap \mathbb{L}$ on $\mathbb{L}$ is of length $d-r+2$.

(2) Let I_C and I_Z be respectively the homogeneous ideals of C and Z in $\mathbb{P}^r$. Then since Z is arithmetically Cohen-Macaulay, the sequence (2.1) induces an exact sequence

$$0 \to I_Z \to I_C \to E \to 0,$$

where $E = \bigoplus_{j \in Z} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(j-d+r-2))$.

(3) Since E has no element of degree $\leq d-r+1$ and since $\operatorname{reg}(E) = d-r+2$, we know that

$$\operatorname{Tor}^S_{i-1}(E, K)_{i+j} = 0 \quad \text{for all } j \neq d-r+1.$$

Notation and Remarks 3.2. For a closed subscheme $X \subset \mathbb{P}^r$, let R and R_X be respectively the homogeneous coordinate rings of $\mathbb{P}^r$ and X. Also let I_X be the homogeneous ideal of X.

(1) The graded Betti numbers $\beta_{i,j}(X)$ of X are obtained as

$$\beta_{i,j}(X) = \dim_{\mathbb{K}} \operatorname{Tor}^R_i(R_X, \mathbb{K})_{i+j}.$$

Note that X is m-regular if and only if $\beta_{i,j}(X) = 0$ for all $j \geq m$.

(2) For $p \geq 1$, X is said to satisfy property $N_{2,p}$ if I_X is generated by quadrics and the k-th syzygies among them are generated by linear syzygies up to $(p-1)$th step. Also X is said to satisfy property N_p if it is arithmetically normal and satisfies property $N_{2,p}$.

Lemma 3.3. *Keep the notations in Notation and Remarks 3.1. Let R_Z be the homogeneous coordinate ring of Z. Then for each $i \geq 1$, we have*

$$\mathrm{Tor}_i^R(R_Z, \mathbb{K}) \cong \mathbb{K}(-1-i)^{\beta_{i,1}(Z)} \oplus \mathbb{K}(-2-i)^{\beta_{i,2}(Z)}$$

where

(i) $\beta_{i,1}(Z) = \begin{cases} i\binom{r-1}{i+1} + (2r-1-d-i)\binom{r-1}{i-1} & \text{for } 1 \leq i \leq 2r-2-d, \\ i\binom{r-1}{i+1} & \text{for } 2r-1-d \leq i \leq r-1, \end{cases}$

(ii) $\beta_{i,2}(Z) = \begin{cases} 0 & \text{for } 1 \leq i \leq 2r-2-d, \text{ and} \\ (i+d+2-2r)\binom{r-1}{i} & \text{for } 2r-1-d \leq i \leq r-1. \end{cases}$

In particular, Z satisfies property $N_{2,2r-2-d}$.

Proof. Since Z is an arithmetically Cohen-Macaulay divisor on the rational normal surface scroll $S(1, r-2)$, it follows that Z and its general hyperplane section $Z \cap H$ have the same graded Betti numbers. Also $Z \cap H$ is a finite subscheme of the rational normal curve $S(1, r-2) \cap H$ of degree $r-1$. Moreover, the length of $Z \cap H$ is equal to $d+1$. Hence the assertion comes immediately from [7, Theorm 2.4]. □

Lemma 3.4. *Let C be as in Lemma 3.3. Then we have for all $i \geq 1$*

$$\mathrm{Tor}_i^R(R_C, \mathbb{K})_{i+j} = \begin{cases} \mathrm{Tor}_i^R(R_Z, \mathbb{K})_{i+j}, & \text{for } \quad 1 \leq j \leq 2, \\ 0, & \text{for } \quad 3 \leq j \leq d-r, \\ \mathrm{Tor}_{i-1}^R(E, \mathbb{K})_{i+d-r+1}, & \text{for } \quad j = d-r+1. \end{cases}$$

In particular, $\dim_{\mathbb{K}} \mathrm{Tor}_{i-1}^R(E, \mathbb{K})_{i+d-r+1} = \binom{r-1}{i-1}$.

Proof. From the sequence in Notation and Remarks 3.1(2), we have a short exact sequence

$$0 \to E \to R_Z \to R_C \to 0,$$

and a long exact sequence

$$\cdots \to \mathrm{Tor}_i^R(E, \mathbb{K})_{i+j} \to \mathrm{Tor}_i^R(R_Z, \mathbb{K})_{i+j} \to \mathrm{Tor}_i^R(R_C, \mathbb{K})_{i+j}$$
$$\to \mathrm{Tor}_{i-1}^R(E, \mathbb{K})_{i+j} \to \mathrm{Tor}_{i-1}^R(R_Z, \mathbb{K})_{i+j} \to \cdots .$$

Then we obtain our assertion since $\mathrm{reg}(E) = d-r+2 \geq 4$ and $\mathrm{reg}(Z) = 3$. Also one can easily obtain that

$$\dim_{\mathrm{K}} \mathrm{Tor}_{i-1}^S(E, K)_{i+d-r+1}$$
$$= h^1(\mathbb{P}^r, \bigwedge^i \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}^1}(-d+r-2) \otimes \mathcal{O}_{\mathbb{P}^1}(d-r+1))$$

$$
\begin{aligned}
&= h^1(\mathbb{P}^r, \bigwedge^i \mathcal{M} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \\
&= h^1(\mathbb{P}^1, \bigwedge^i (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{r-1}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \\
&= \binom{r-1}{i-1} \quad \text{for} \quad i \geq 1
\end{aligned}
$$

(for details, see Theorem (1.b.4) in [5] or Theorem 5.8 in [3]). □

Now we are ready to provide:

Proof of Theorem 1.1. Since C is linearly equivalent to a divisor $H+(d-r+1)F$ on $S(1, r-2)$, the proof is obtained by combining Lemma 3.3 and Lemma 3.4. Finally, the generating structure of the homogeneous ideal I_C of C follows immediately from the values of the graded Betti numbers of C. □

We finish this section with some remarks and examples.

Remark 3.5. Let $C \subset \mathbb{P}^r$ be a curve of maximal regularity whose degree d is at most $2r-2$. Then by Lemma 3.3 and Lemma 3.4 (cf. Theorem 3.3 and Proposition 4.1 in [1]), the graded Betti diagram of C is of the following shape:

$\beta_{1,1}$	$\beta_{2,1}$	$\cdots$	$\beta_{r-3,1}$	$\beta_{r-2,1}$	0	0
$\beta_{1,2}$	$\beta_{2,2}$	$\cdots$	$\beta_{r-3,2}$	$\beta_{r-2,2}$	$\beta_{r-1,2}$	0
0	0	$\cdots$	0	0	0	0
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\cdots$	0	0	0	0
$\beta_{1,d-r+1}$	$\beta_{2,d-r+1}$	$\cdots$	$\beta_{r-3,d-r+1}$	$\beta_{r-2,d-r+1}$	$\beta_{r-1,d-r+1}$	$\beta_{r,d-r+1}$

Using Green's $K_{p,1}$ Theorem ([5, Theorem (3.c.1)]), it holds that C is of type I if $\beta_{r-2,1} > 0$ and it is of type II if $\beta_{r-2,1} = 0$. Indeed, Theorem 1.1 says that $\beta_{r-2,1} = r-2$ when C is of type I.

Example 3.6. We consider two smooth rational curves $C_k \subset \mathbb{P}^{10}$ of degree 16 $(k = 1, 2)$ which are given by the maps $\phi_1, \phi_2 : \mathbb{P}^1 \to \mathbb{P}^{10}$ defined respectively by

$$\phi_1 : [s:t] \mapsto [s^{16}:s^{15}t:s^8t^8:s^7t^9:s^6t^{10}:s^5t^{11}:s^4t^{12}:s^3t^{13}:s^2t^{14}:st^{15}:t^{16}]$$

and

$$\phi_2 : [s:t] \mapsto [s^{16}:s^{15}t+s^{14}t^2:s^8t^8:s^7t^9:s^6t^{10}:s^5t^{11}:s^4t^{12}:s^3t^{13}:s^2t^{14}:st^{15}:t^{16}].$$

By computer algebra system "Singular", we have the graded Betti numbers of C_1 and C_2 as follows:

TABLE 1. Betti Table of $C_1 \subset \mathbb{P}^{10}$

i	$\beta_{1,i}$	$\beta_{2,i}$	$\beta_{3,i}$	$\beta_{4,i}$	$\beta_{5,i}$	$\beta_{6,i}$	$\beta_{7,i}$	$\beta_{8,i}$	$\beta_{9,i}$	$\beta_{10,i}$
1	38	177	378	504	420	216	63	8	0	0
2	0	0	84	252	378	336	180	54	7	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	1	9	36	84	126	126	84	36	9	1

TABLE 2. Betti Table of $C_2 \subset \mathbb{P}^{10}$

i	$\beta_{1,i}$	$\beta_{2,i}$	$\beta_{3,i}$	$\beta_{4,i}$	$\beta_{5,i}$	$\beta_{6,i}$	$\beta_{7,i}$	$\beta_{8,i}$	$\beta_{9,i}$	$\beta_{10,i}$
1	38	177	378	448	350	160	35	0	0	0
2	0	0	28	182	322	308	172	54	7	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	1	9	36	84	126	126	84	36	9	1

Observe that C_1 and C_2 are both curves of maximal regularity. Furthermore, $\beta_{8,1}(C_1) = 8$ and hence C_1 is of type I. One can see that the first table agrees with Theorem 1.1. On the other hand, $\beta_{8,1}(C_1) = 0$ and hence C_2 is of type II. That is, there exists no rational surface scroll which contains C_2.

References

[1] M. Brodmann and P. Schenzel, *On projective curves of maximal regularity*, Math. Z. **244** (2003), no. 2, 271–289.

[2] M. Brodmann and P. Schenzel, *Projective curves with maximal regularity and applications to syzygies and surfaces*, Manuscripta Math. **135** (2011), no. 3-4, 469–495.

[3] D. Eisenbud, *The Geometry of syzygies*, Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

[4] R. Ferraro, *Weil divisors on rational normal scrolls*, Geometric and combinatorial aspects of commutative algebra (Messina, 1999), 183–197, Lecture Notes in Pure and Appl. Math., 217, Dekker, New York, 2001.

[5] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), no. 1, 125–171.

[6] L. Gruson, R. Lazarsfeld, and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. **72** (1983), no. 3, 491–506.

[7] U. Nagel, *Arithmetically Buchsbaum divisors on varieties of minimal degree*, Trans. Amer. Math. Soc. **351** (1999), no. 11, 4381–4409.

[8] E. Park, *On syzygies of divisors on rational normal scrolls*, Math. Nachr. **287** (2014), no. 11-12, 1383–1393.

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