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LIE SUPER-BIALGEBRAS ON GENERALIZED LOOP SUPER-VIRASORO ALGEBRAS

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Abstract. In this article we consider Lie super-bialgebra structures on the generalized loop super-Virasoro algebra $\mathcal{G}$. By proving that the first cohomology group $H^1(\mathcal{G}, \mathcal{G} \otimes \mathcal{G})$ is trivial, we obtain that all such Lie bialgebras are triangular coboundary.

1. Introduction

The Virasoro algebra and super-Virasoro algebra are closely related to the conformal field theory and the string theory, which are paid much attention by mathematics and physics (e.g., [1, 7, 9, 10]). The generalized forms associated to these algebras have been extensively investigated, such as the high rank Virasoro algebras, the loop-Virasoro algebra, generalized Virasoro algebras and generalized super-Virasoro algebras (e.g., [2, 7, 8, 11, 14–17, 19]).

The concept of Lie bialgebras was first introduced by Drinfeld in the framework of quantum group theory (see [4], [5]). Generally speaking, a Lie bialgebra is a Lie algebra provided with a Lie coalgebra structure which satisfies certain compatibility condition. A Lie bialgebra is a semiclassical structure of some quantum group. In recent decades, some articles about Lie bialgebras (super-bialgebra) appeared (e.g., [2, 3, 12, 13, 16, 18]). In [13], Lie bialgebra structures on the one-sided Witt algebra, the Witt algebra, and the Virasoro algebra are completely classified, which are shown to be triangular coboundary. Similar results were obtained for other Lie algebras, such as generalized Virasoro-like Lie algebras, Block Lie algebras, generalized Witt type Lie algebras, generalized loop Virasoro algebras, etc. In [19], the Lie super-bialgebra structures on generalized super-Virasoro algebras are investigated, where the Lie super-bialgebras are not all triangular coboundary.

In this paper, we consider the generalized loop super-Virasoro algebra $\mathcal{G}$ (cf. Definition 2.6), which is a tensor product of the centerless generalized super-Virasoro algebra $\mathfrak{sv}$ and the Laurent polynomial algebra $\mathbb{F}[t, t^{-1}]$. By proving

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that the first cohomology group $H^1(\mathcal{G}, \mathcal{G}\otimes\mathcal{G})$ is trivial, we show that all Lie super-bialgebra structures on the generalized loop super-Virasoro algebra are triangular coboundary. It should be pointed out that the classical techniques in [6] may not be directly applied to our case since $\mathcal{G}$ is not finitely generated as Lie algebras in general.

2. Preliminary and main result

We first recall some definitions related to Lie super-bialgebras. Let $U = U_{\bar{0}} + U_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. For $x \in U_{\bar{\alpha}}$ with $\bar{\alpha} \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, we say x is *homogeneous of degree* $\bar{\alpha}$, and write $[x] = \bar{\alpha}$. In what follows x is assumed to be homogeneous whenever $[x]$ occurs. Let $\mathcal{L}=\mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be a Lie superalgebra over an arbitrary field $\mathbb{F}$ of characteristic 0, i.e., there exists a bilinear map $[\ ,\ ] : \mathcal{L}\times\mathcal{L} \to \mathcal{L}$ such that

$$[\mathcal{L}_\sigma, \mathcal{L}_\tau] \subset \mathcal{L}_{\sigma+\tau}, \tag{1}$$

$$[x, y] = -(-1)^{|x||y|}[y, x], \tag{2}$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]], \tag{3}$$

where $\sigma, \tau \in \mathbb{Z}_2$ and $x, y, z \in \mathcal{L}$.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be an $\mathcal{L}$-module. A *homogenous derivation of degree* $[\mathfrak{d}] \in \mathbb{Z}_2$ is a $\mathbb{Z}_2$-homogenous linear map $\mathfrak{d} : \mathcal{L} \to V$, such that $\mathfrak{d}(\mathcal{L}_{\bar{i}}) \subset V_{\bar{i}+[\mathfrak{d}]}$ for $\bar{i} \in \mathbb{Z}_2$ and satisfying

$$\mathfrak{d}[x, y] = (-1)^{[\mathfrak{d}][x]}x \cdot \mathfrak{d}(y) - (-1)^{[y]([\mathfrak{d}]+[x])}y \cdot \mathfrak{d}(x) \text{ for } x, y \in \mathcal{L}.$$

A derivation $\mathfrak{d}$ is called *even* if $|\mathfrak{d}| = \bar{0}$, *odd* if $|\mathfrak{d}| = \bar{1}$. For a homogenous element $v \in V$, it is easy to see that the linear map $v_{inn} : \mathcal{L} \to V,\ x \to (-1)^{[v][x]}x \cdot v$, $\forall x \in \mathcal{L}$ is a derivation, and v_{inn} is called *inner*. Denote by $\mathrm{Der}_{\bar{\alpha}}(\mathcal{L}, V)$ the set of all derivations of homogenous of degree $\bar{\alpha}$. Then $\mathrm{Der}\,(\mathcal{L}, V) = \mathrm{Der}_{\,\bar{0}}(\mathcal{L}, V) \oplus \mathrm{Der}_{\,\bar{1}}(\mathcal{L}, V)$ is the derivation algebra on $\mathcal{L}$. Similarly, $\mathrm{Inn}\,(\mathcal{L}, V) = \mathrm{Inn}_{\,\bar{0}}(\mathcal{L}, V) \oplus \mathrm{Inn}_{\,\bar{1}}(\mathcal{L}, V)$ is the set of all inner derivations on $\mathcal{L}$. Let $H^1(\mathcal{L}, V)$ be the first cohomology group of $\mathcal{L}$ with coefficients in V. It is well known that

$$H^1(\mathcal{L}, V) \cong \mathrm{Der}(\mathcal{L}, V)/\mathrm{Inn}(\mathcal{L}, V).$$

Denote by τ the *super-twist map* of $\mathcal{L}\otimes\mathcal{L}$, namely,

$$\tau(x \otimes y) = (-1)^{[x][y]}y \otimes x, \quad \forall x, y \in \mathcal{L}.$$

Denote by ξ the *super-cyclic map* which cyclically permutes the coordinates of a element in $\mathcal{L}\otimes\mathcal{L}\otimes\mathcal{L}$, i.e.,

$$\xi(x_1 \otimes x_2 \otimes x_3) = (-1)^{[x_1]([x_2]+[x_3])}x_2 \otimes x_3 \otimes x_1, \quad \forall x_1, x_2, x_3 \in \mathcal{L}.$$

Then the definition of Lie superalgbra can be restated as follows.

Definition 2.1. A *Lie superalgebra* is a pair $(\mathcal{L}, \varphi)$ consisting of a super-vector space $\mathcal{L}$ and a bilinear map $\varphi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ (the super-bracket) satisfying the following conditions:

$$\mathrm{Ker}(1 \otimes 1 - \tau) \subset \mathrm{Ker}\varphi,$$
$$\varphi \cdot (1 \otimes \varphi) \cdot (1 \otimes 1 \otimes 1 + \xi + \xi^2) = 0,$$

where 1 denotes the identity map on $\mathcal{L}$.

Dually, we have the definition of Lie super-coalgebras.

Definition 2.2. A Lie super-coalgebra is a pair $(\mathcal{L}, \Delta)$ consisting of a super-vector space $\mathcal{L}$ and a linear map $\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ (the super-cobracket) satisfying the following conditions:

$$\mathrm{Im}\Delta \subset \mathrm{Im}(1 \otimes 1 - \tau),$$
$$(1 \otimes 1 \otimes 1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0.$$

Definition 2.3. A Lie super-bialgebra is a triple $(\mathcal{L}, \varphi, \Delta)$ satisfying the following conditions:

$$\begin{aligned}
&(\mathcal{L}, \varphi) \text{ is a Lie superalgebra,}\\
&(\mathcal{L}, \Delta) \text{ is a Lie super-coalgebra,}\\
&\Delta\varphi(x \otimes y) = x \cdot \Delta y - (-1)^{[x][y]} y \cdot \Delta x,
\end{aligned}$$

where $x, y \in \mathcal{L}$ and the symbol " $\cdot$ " means the adjoint diagonal action given by

$$x \cdot (a \otimes b) = [x, a] \otimes b + (-1)^{[x][a]} a \otimes [x, b], \quad x, a, b \in \mathcal{L},$$

and in general $[x, y] = \varphi(x \otimes y)$ for $x, y \in \mathcal{L}$.

Note that the third condition in above definition means $\Delta : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ is a derivation.

Definition 2.4. A coboundary Lie super-bialgebra is a quadruple $(\mathcal{L}, \varphi, \Delta, r)$, where $(\mathcal{L}, \varphi, \Delta)$ is a Lie super-bialgebra and $r \in \mathrm{Im}(1 \otimes 1 - \tau)$ such that $\Delta = \Delta_r$ is coboundary of r, i.e.,

$$\Delta_r(x) = (-1)^{[x][r]} x \cdot r, \quad \forall x \in \mathcal{L}.$$

Denote by $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$. For any $r = \sum_i a_i \otimes b_i \in \mathcal{L} \otimes \mathcal{L}$, define three elements in $\mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L})$ as follows:

$$r^{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r^{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r^{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

where 1 is the identity element of $\mathcal{U}(\mathcal{L})$.

Definition 2.5. A coboundary Lie super-bialgebra $(\mathcal{L}, \varphi, \Delta, r)$ is called triangular if r is a solution of the classical Yang-Baxter equation (CYBE):

$$c(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

Let Γ be a nonzero additive subgroup of $\mathbb{F}$ and $s \in \mathbb{F}$ such that $2s \in \Gamma$. The *centerless generalized super-Virasoro algebra* $\mathfrak{sv}$ is a Lie superalgebra with $\mathbb{F}$-basis $\{L_\alpha, G_\mu \mid \alpha \in \Gamma, \mu \in s+\Gamma\}$ satisfying the following bracket relations:

$$[L_\alpha, L_\beta] = (\beta-\alpha)L_{\alpha+\beta}, \quad [L_\alpha, G_\mu] = (\mu - \frac{\alpha}{2})G_{\alpha+\mu}, \quad [G_\mu, G_\nu] = 2L_{\mu+\nu},$$

where $\alpha, \beta \in \Gamma, \mu, \nu \in s+\Gamma$.

Definition 2.6. A *generalized loop super-Virasoro algebra* $\mathcal{G}$ is a Lie superalgebra which is a tensor product of the centerless generalized super-Virasoro algebra $\mathfrak{sv}$ and the Laurent polynomial algebra $\mathbb{F}[t,t^{-1}]$, i.e., $\mathcal{G} = \mathfrak{sv} \otimes \mathbb{F}[t,t^{-1}]$ with a basis $\{L_{\alpha,i}, G_{\mu,j} \mid \alpha \in \Gamma, \mu \in s+\Gamma, i,j \in \mathbb{Z}\}$ subject to the commutation relations:

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta-\alpha)L_{\alpha+\beta,i+j}, \tag{4}$$

$$[L_{\alpha,i}, G_{\mu,j}] = (\mu - \frac{\alpha}{2})G_{\alpha+\mu,i+j}, \tag{5}$$

$$[G_{\mu,i}, G_{\nu,j}] = 2L_{\mu+\nu,i+j}, \tag{6}$$

for $\alpha, \beta \in \Gamma$, $\mu, \nu \in \Gamma+s, i,j \in \mathbb{Z}$, where $L_{\alpha,i} = L_\alpha \otimes t^i$, $G_{\alpha,i} = G_\alpha \otimes t^i$.

It is clear that $\mathcal{G}$ contains $\mathfrak{sv}$ as a subalgebra. Let Γ' be an additive subgroup of $\mathbb{F}$ generated by $\Gamma \bigcup \{s\}$. For any $\alpha \in \Gamma'$, $i \in \mathbb{Z}$, we denote $L_{\alpha,i} = L_\alpha \otimes t^i$, $G_{\alpha,i} = G_\alpha \otimes t^i$. We use the convention that if an undefined notation appears in an expression, we treat it as zero; for instance, $L_{\alpha,i} = 0$, $G_{\beta,i} = 0$ if $\alpha \notin \Gamma$, $\beta \notin s+\Gamma$. Then $\mathcal{G} = \oplus_{\alpha\in\Gamma'}\mathcal{G}_\alpha$ is Γ'-graded with

$$\mathcal{G}_\alpha = \mathrm{span}\{L_{\alpha,i} \mid i \in \mathbb{Z}\} \oplus \mathrm{span}\{G_{\alpha,i} \mid i \in \mathbb{Z}\}, \alpha \in \Gamma'.$$

Obviously, $\mathcal{G}$ is also $\mathbb{Z}_2$-graded: $\mathcal{G} = \mathcal{G}_{\bar{0}} \oplus \mathcal{G}_{\bar{1}}$, with

$$\mathcal{G}_{\bar{0}} = \mathrm{span}\{L_{\alpha,i} \mid \alpha \in \Gamma, i \in \mathbb{Z}\}, \quad \mathcal{G}_{\bar{1}} = \mathrm{span}\{G_{\mu,i} \mid \mu \in s+\Gamma, i \in \mathbb{Z}\}.$$

The main result of this article is the following theorem.

Theorem 2.7. *Every Lie super-bialgebra structure on $\mathcal{G}$ is triangular coboundary* (cf. Definitions 2.3–2.6).

3. Proof of main result

Denote by V the tensor product $\mathcal{G}\otimes\mathcal{G}$ from now on. Note that V possesses a natural $\mathcal{G}$-module structure under the adjoint diagonal action of $\mathcal{G}$. For fixed i and j, let $V_{ij} = \mathrm{span}\{X_{\alpha,i} \otimes Y_{\beta,j} \mid \alpha,\beta \in \Gamma', X, Y \in \{L,G\}\}$, then $V = \oplus_{i,j\in\mathbb{Z}}V_{i,j}$. Note that $V_{i,j} \cong \mathfrak{sv}\otimes\mathfrak{sv}$ as a $\mathfrak{sv}$-module. As a vector space, we have $V = \mathcal{G}\otimes\mathcal{G} \cong (\mathfrak{sv}\otimes\mathfrak{sv})\otimes\mathbb{F}[x^{\pm1}, y^{\pm1}]$. The $\mathcal{G}$-module action on V can be written as follows:

$$X_{\alpha,i}\cdot(M_\beta\otimes N_\gamma x^j y^k) = [X_\alpha, M_\beta]\otimes N_\gamma x^{i+j}y^k + (-1)^{[\alpha][\beta]}M_\beta\otimes[X_\alpha, N_\gamma]x^j y^{i+k},$$

where $X, M, N \in \{L, G\}$.

The following result can be found in the references [4, 12, 19] or obtained by using the similar arguments as those given.

Lemma 3.1. *Let $\mathcal{L}$ be a Lie superalgebra and $r \in \mathrm{Im}(1 \otimes 1 - \tau) \subset \mathcal{L} \otimes \mathcal{L}$, then $(1 \otimes 1 \otimes 1 + \xi + \xi^2)(1 \otimes \Delta_r)\Delta_r(x) = x \cdot c(r), \forall x \in \mathcal{L}$. In particular, a triple $(\mathcal{L}, [\cdot,\cdot], \Delta_r)$ is a Lie super bialgebra if and only if r satisfies $x \cdot c(r) = 0$ for all $x \in \mathcal{L}$.*

Lemma 3.2. *Assume that an element $c \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ satisfies $x \cdot c = 0$ for all $x \in \mathcal{G}$. Then $c = 0$.*

Proof. Let $c = \sum_{\alpha,\beta,\gamma} M_\alpha \otimes N_\beta \otimes P_\gamma f^{M,N,P}_{\alpha,\beta,\gamma}(x,y,z)$, where $f^{M,N,P}_{\alpha,\beta,\gamma}(x,y,z) \in \mathbb{F}[x^{\pm}, y^{\pm}, z^{\pm}], M, N, P \in \{L, G\}$. Choose a total order on Γ which is compatible with the group structure of Γ. Naturally, this order induces a lexicographic order on $\Gamma \times \Gamma \times \Gamma$. Suppose that $c \neq 0$, then there exists a nonzero homogeneous term $M_{\alpha_0} \otimes N_{\beta_0} \otimes P_{\gamma_0} f^{M,N,P}_{\alpha_0,\beta_0,\gamma_0}(x,y,z)$. Moreover, we assume this term is maximal. Then we can choose an element $\delta \in \Gamma$ such that $[L_\delta, M_{\alpha_0}] = h(\delta,\alpha_0) M_{\alpha_0+\delta} \neq 0, h(\delta,\alpha_0) \in \mathbb{F}$. Thus $h(\delta,\alpha_0) M_{\alpha_0+\delta} \otimes N_{\beta_0} \otimes P_{\gamma_0} f^{M,N,P}_{\alpha_0,\beta_0,\gamma_0}(x,y,z)$ is a maximal term in $L_{\delta,0} \cdot c$, which is a contradiction with $L_{\delta,0} \cdot c = 0$. This completes the proof of the lemma. $\square$

Corollary 3.3. *Let $r \in \mathrm{Im}(1 \otimes 1 - \tau) \subset \mathcal{G} \otimes \mathcal{G}$. Then $(\mathcal{G}, [\cdot,\cdot], \Delta_r)$ is a Lie super-bialgebra if and only if r is a solution of* CYBE, *i.e., $c(r) = 0$.*

Lemma 3.4. *Let $\mathfrak{d} \in Der(\mathcal{G}, V)$. Then there exists an element $v \in V$ such that $\mathfrak{d}|_{\mathfrak{sv}} = \mathfrak{d}_v$, where $\mathfrak{d}_v$ is an inner derivation of $\mathfrak{sv}$.*

Proof. Clearly, $\mathfrak{d}|_{\mathfrak{sv}} \in \mathrm{Der}(\mathfrak{sv}, V)$. For convenience, we denote $\mathfrak{d}|_{\mathfrak{sv}}$ by $\mathfrak{d}$ in the following discussions. Since $V = \oplus_{i,j} V_{i,j}$ as a $\mathfrak{sv}$-module, where $V_{i,j} \cong \mathfrak{sv} \otimes \mathfrak{sv}$. Let $\mathfrak{d}_{i,j}$ belongs to $\mathrm{Der}(\mathfrak{sv}, V_{i,j}), i, j \in \mathbb{Z}$. Then we have $\mathfrak{d} = \sum_{i,j} \mathfrak{d}_{i,j}$, which holds in the sense that only finitely many terms $\mathfrak{d}_{i,j}(x) \neq 0$ appear when one applies $\mathfrak{d} = \sum_{i,j} \mathfrak{d}_{i,j}$ to any $x \in \mathcal{G}$. By Proposition 3.4 in [19], we have $H^1(\mathfrak{sv}, V_{i,j}) = H^1(\mathfrak{sv}, \mathfrak{sv} \otimes \mathfrak{sv}) = 0$, which means $\mathfrak{d}_{i,j}$ is a inner derivation for any $i, j \in \mathbb{Z}$. If the right hand of the expression $\mathfrak{d} = \sum_{i,j} \mathfrak{d}_{i,j}$ is a finite sum, then $\sum_{i,j} v_{i,j} \in V$. Clearly, $\mathfrak{d} = \mathfrak{d}_v$ is an inner derivation for such $v = \sum_{i,j} v_{i,j}$. This proves the lemma. In the rest of the proof we aim to show that $\sum_{i,j} \mathfrak{d}_{i,j}$ is a finite sum. The proof will be completed case by case.

Case 1: $\mathfrak{d}$ is even.

Considering the action $\mathfrak{d}$ on $L_{0,0}, L_{1,0}, L_{2,0}$, respectively, then we deduce that the set $S = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid L_{0,0} \cdot v_{i,j}, L_{1,0} \cdot v_{i,j}, L_{2,0} \cdot v_{i,j}$ not all is zero$\}$ is a finite set. Denote by $\bar{S}$ the complement of S in $\mathbb{Z} \times \mathbb{Z}$, i.e., $\bar{S} = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid L_{0,0} \cdot v_{i,j} = L_{1,0} \cdot v_{i,j} = L_{2,0} \cdot v_{i,j} = 0\}$.

Let $(i,j) \in \bar{S}$, i.e., $L_{0,0} \cdot v_{i,j} = L_{1,0} \cdot v_{i,j} = L_{2,0} \cdot v_{i,j} = 0$. Since $L_{0,0} \cdot v_{i,j} = 0$ and $\mathfrak{d}$ is even, we can assume

$$v_{i,j} = \sum_\alpha f(\alpha) L_\alpha \otimes L_{-\alpha} x^i y^j + \sum_\alpha \phi(\alpha) G_{\alpha+s} \otimes G_{-\alpha-s} x^i y^j.$$

Note that

$$\begin{aligned} L_{k,0}\cdot v_{i,j} = & \sum_\alpha \big((\alpha-2k)f(\alpha-k)-(\alpha+k)f(\alpha)\big)L_\alpha\otimes L_{k-\alpha}x^iy^j \\ & +\sum_\alpha \big((\alpha+s-\frac{3}{2}k)\phi(\alpha-k)-(\alpha+s+\frac{1}{2}k)\phi(\alpha)\big) \\ & G_{\alpha+s}\otimes G_{k-\alpha-s}x^iy^j. \end{aligned}$$

It follows from $L_{1,0}\cdot v_{i,j}=0$ that

$$(\alpha-2)f(\alpha-1)-(\alpha+1)f(\alpha)=0, \tag{7}$$

$$(\alpha+s-\frac{3}{2})\phi(\alpha-1)-(\alpha+s+\frac{1}{2})\phi(\alpha)=0. \tag{8}$$

Replacing α by $\alpha-1$ in the above formulas, we have

$$(\alpha-3)f(\alpha-2)-\alpha f(\alpha-1)=0, \tag{9}$$

$$(\alpha+s-\frac{5}{2})\phi(\alpha-2)-(\alpha+s-\frac{1}{2})\phi(\alpha-1)=0. \tag{10}$$

On the other hand, it follows from $L_{2,0}\cdot v_{i,j}=0$ that

$$(\alpha-4)f(\alpha-2)-(\alpha+2)f(\alpha)=0, \tag{11}$$

$$(\alpha+s-3)\phi(\alpha-2)-(\alpha+s+1)\phi(\alpha)=0. \tag{12}$$

The linear equations system consisting of (7), (9) and (11) $\big($rep., (8), (10) and (12)$\big)$ implies $f(\alpha)=0$ $\big($rep., $\phi(\alpha)=0\big)$ for all $\alpha\in\Gamma$. Then $v_{i,j}=0$ for all $(i,j)\in\bar{S}$, i.e., $\{(i,j)\mid v_{i,j}\neq 0\}\subset S$. We conclude that $\mathfrak{d}=\sum_{i,j}\mathfrak{d}_{i,j}$ is a finite sum since S is a finite set.

Case 2: $\mathfrak{d}$ is odd.

Let $S,\bar{S}$ be defined as before. Since $\mathfrak{d}$ is odd, we can assume

$$v_{i,j}=\sum_\alpha g(\alpha)L_\alpha\otimes G_{-\alpha}x^iy^j+\sum_\alpha h(\alpha)G_{\alpha+s}\otimes L_{-\alpha-s}x^iy^j.$$

Note that

$$\begin{aligned} L_{k,0}\cdot v_{i,j} = & \sum_\alpha \big((\alpha-2k)g(\alpha-k)-(\alpha+\frac{k}{2})g(\alpha)\big)L_\alpha\otimes G_{k-\alpha}x^iy^j \\ & +\sum_\alpha \big((\alpha+s-\frac{3}{2}k)h(\alpha-k)-(\alpha+s+k)h(\alpha)\big) \\ & G_{\alpha+s}\otimes L_{-\alpha-s+k}x^iy^j. \end{aligned}$$

It follows from $L_{1,0}\cdot v_{i,j}=0$ that

$$(\alpha-2)g(\alpha-1)-(\alpha+\frac{1}{2})g(\alpha)=0, \tag{13}$$

$$(\alpha+s-\frac{3}{2})h(\alpha-1)-(\alpha+s+1)h(\alpha)=0. \tag{14}$$

Replacing α by $\alpha - 1$ in the above formulas, we have

$$(\alpha-3)g(\alpha-2)-(\alpha-\frac{1}{2})g(\alpha-1)=0, \tag{15}$$

$$(\alpha+s-\frac{5}{2})h(\alpha-2)-(\alpha+s)h(\alpha-1)=0. \tag{16}$$

On the other hand, it follows from $L_{2,0}\cdot v_{i,j}=0$ that

$$(\alpha-4)g(\alpha-2)-(\alpha+1)g(\alpha)=0, \tag{17}$$

$$(\alpha+s-3)h(\alpha-2)-(\alpha+s+2)h(\alpha)=0. \tag{18}$$

From the linear equations system consisting of (13), (15) and (17), by the elimination method we have $(\frac{5}{4}\alpha+5)g(\alpha)=(\frac{5}{4}\alpha+5)g(\alpha-2)=0$. Thus $g(\alpha)=0$ for all $\alpha\in\Gamma$. Similarly, by the linear equations system consisting of (14), (16) and (18), we have $(\alpha+s-6)h(\alpha)=(\alpha+s-6)h(\alpha-2)=0$. Then $h(\alpha)=0$, $\alpha\in\Gamma$. By the similar discussions as those in the case 1, we deduce that $\mathfrak{d}=\sum_{i,j}\mathfrak{d}_{i,j}$ is a finite sum. □

Proposition 3.5. $\mathrm{Der}(\mathcal{G},V)=\mathrm{Inn}(\mathcal{G},V), i.e., H^1(\mathcal{G},V)=0$.

Proof. Since $\mathcal{G}$ is Γ'-graded, then V can be equipped with a Γ'-grading: $V=\oplus_{\alpha\in\Gamma'}V_\alpha$ with $V_\alpha=\sum_{\beta+\gamma=\alpha}\mathcal{G}_\beta\otimes\mathcal{G}_\gamma,\alpha,\beta,\gamma\in\Gamma'$. A derivation $\mathfrak{d}$ is called *homogeneous of degree* α, $\alpha\in\Gamma'$, if $\mathfrak{d}(V_\beta)\subset V_{\alpha+\beta}$ for all $\beta\in\Gamma'$. Denote by $\mathrm{Der}(\mathcal{G},V)_\alpha$ the set consisting of all derivation of the degree α. Let $\mathfrak{d}\in\mathrm{Der}(\mathcal{G},V)$. For any $x\in\mathcal{G}_\beta$ with $\beta\in\Gamma'$, $\mathfrak{d}(x)$ can be written into $\mathfrak{d}(x)=\sum_{\gamma\in\Gamma'}v_\gamma$ with $v_\gamma\in V_\gamma$, then we define $\mathfrak{d}_\alpha$ by $\mathfrak{d}_\alpha(x)=v_{\alpha+\beta}$. It is easy to check $\mathfrak{d}_\alpha\in\mathrm{Der}(\mathcal{G},V)_\alpha$, and we have

$$\mathfrak{d}=\sum_{\alpha\in\Gamma'}\mathfrak{d}_\alpha,\ \text{where } \mathfrak{d}_\alpha\in\mathrm{Der}(\mathcal{G},V)_\alpha,$$

which holds in the sense that for every $x\in\mathcal{G}$ only finitely many nonzero terms $\mathfrak{d}_\alpha(x)$ appeared in $\mathfrak{d}(x)=\sum_{\alpha\in\Gamma'}\mathfrak{d}_\alpha(x)$.

Replacing $\mathfrak{d}$ by $\mathfrak{d}-\mathfrak{d}_v$ for some $v\in V$, we can assume $\mathfrak{d}(\mathfrak{sv})=0$ since Lemma 3.4. In particular, we have $\mathfrak{d}(L_{0,0})=0$. For any $x_\beta\in\mathcal{G}_\beta,\beta\in\Gamma'$, applying $\mathfrak{d}_\alpha$ to $[L_{0,0},x_\beta]=\beta x_\beta$ and using the facts $\mathfrak{d}_\alpha(x_\beta)\in V_{\alpha+\beta}$ and $L_{0,0}|_{V_{\alpha+\beta}}=\alpha+\beta$, we have $\mathfrak{d}_\alpha(x_\beta)=(-1)^{[\alpha][\beta]}x_\beta\cdot\alpha^{-1}\mathfrak{d}_\alpha(L_{0,0})$ if $\alpha\neq0$. Then $\mathfrak{d}_\alpha\in\mathrm{Inn}(\mathcal{G},\mathrm{V})$ if $\alpha\neq0$.

From now on we aim to investigate the derivation of degree 0. We need to consider two cases: $\mathfrak{d}$ is even and $\mathfrak{d}$ is odd.

Case 1: If $\mathfrak{d}\in(\mathrm{Der}(\mathcal{G},V))_0$ is even, then $\mathfrak{d}=0$.

Assume

$$\mathfrak{d}(L_{\alpha,i})=\sum_{\gamma\in\Gamma}(G_\gamma\otimes G_{\alpha-\gamma})a_{\alpha,i,\gamma}+\sum_{\gamma\in\Gamma}(L_\gamma\otimes L_{\alpha-\gamma})b_{\alpha,i,\gamma},$$

$$\mathfrak{d}(G_{\mu,i})=\sum_{\gamma\in\Gamma}(L_\gamma\otimes G_{\mu-\gamma})c_{\mu,i,\gamma}+\sum_{\gamma\in\Gamma}(G_\gamma\otimes L_{\mu-\gamma})d_{\mu,i,\gamma}.$$

Claim 1: $\mathfrak{d}(L_{\alpha,i}) = 0, \alpha \in \Gamma, i \in \mathbb{Z}$.

Applying $\mathfrak{d}$ to the equation $[L_{\alpha,i}, L_{\beta,i}] = (\beta - \alpha)L_{\alpha+\beta,i+j}$, we have

$$(\beta - \alpha)a_{\alpha+\beta,i+j,\gamma} = (\gamma - \frac{3}{2}\alpha)x^i a_{\beta,j,\gamma-\alpha} + (\beta - \gamma - \frac{\alpha}{2})y^i a_{\beta,j,\gamma} - (\gamma - \frac{3}{2}\beta)x^j a_{\alpha,i,\gamma-\beta} - (\alpha - \gamma - \frac{\beta}{2})y^j a_{\alpha,i,\gamma}. \tag{19}$$

Letting $\alpha = \beta = 0$ in (19), we obtain

$$(x - y)a_{0,i,\gamma} = (x^i - y^i)a_{0,1,\gamma} \text{ if } \gamma \neq 0. \tag{20}$$

Setting $\beta = i = 0$ in (19), one has

$$-\alpha a_{\alpha,j,\gamma} = (\gamma - \frac{3}{2}\alpha)x^i a_{0,j,\gamma-\alpha} - (\gamma + \frac{\alpha}{2})y^i a_{0,j,\gamma}. \tag{21}$$

Substituting (21) into (19) and letting $\beta = \alpha$, we have

$$\begin{aligned} &(\frac{\alpha}{2} - \gamma)y^i\big((\gamma - \frac{3}{2}\alpha)a_{0,j,\gamma-\alpha} - (\gamma + \frac{\alpha}{2})a_{0,j,\gamma}\big) \\ &+(\gamma - \frac{3}{2}\alpha)x^i\big((\gamma - \frac{5}{2}\alpha)a_{0,j,\gamma-2\alpha} - (\gamma - \frac{1}{2}\alpha)a_{0,j,\gamma-\alpha}\big) \\ &-(\frac{\alpha}{2} - \gamma)y^j\big((\gamma - \frac{3}{2}\alpha)a_{0,j,\gamma-\alpha} - (\gamma + \frac{\alpha}{2})a_{0,j,\gamma}\big) \\ &-(\gamma - \frac{3}{2}\alpha)x^j\big((\gamma - \frac{5}{2}\alpha)a_{0,j,\gamma-2\alpha} - (\gamma - \frac{1}{2}\alpha)a_{0,j,\gamma-\alpha}\big) = 0. \end{aligned} \tag{22}$$

Multiplying (22) by $x - y$ and using (20), we have

$$\begin{aligned} &2(\gamma - \frac{\alpha}{2})(\gamma - \frac{3}{2}\alpha)a_{0,1,\gamma-\alpha} \\ &= (\gamma - \frac{3}{2}\alpha)(\gamma - \frac{5}{2}\alpha)a_{0,1,\gamma-2\alpha} + (\gamma - \frac{\alpha}{2})(\gamma + \frac{\alpha}{2})a_{0,1,\gamma}. \end{aligned} \tag{23}$$

Note that $\mathfrak{d}(L_{0,1}) = \sum_\gamma G_\gamma \otimes G_{-\gamma}a_{0,1,\gamma} + \sum_\gamma L_\gamma \otimes L_{-\gamma}b_{0,1,\gamma}$, in which every sum is finite. This fact together with the equation (23) means there exists some α such that $\gamma \neq \pm\frac{\alpha}{2}$ and $a_{0,1,\gamma-2\alpha} = a_{0,1,\gamma-\alpha} = 0$. Then we deduce that $a_{0,1,\gamma} = 0$ if $\gamma \neq 0$. This together with (20) implies $a_{0,i,\gamma} = 0$ if $\gamma \neq 0$. Similarly, we have $b_{0,i,\gamma} = 0$ for $\gamma \neq 0$.

Now we assume $\mathfrak{d}(L_{0,i}) = G_0 \otimes G_0 a_i + L_0 \otimes L_0 b_i$, where $a_i = a_{0,i,0}, b_i = b_{0,i,0}$. Noting the fact $\mathfrak{d}(L_{\alpha,0}) = 0$ and applying $\mathfrak{d}$ to the equation $[L_{0,i}, L_{\alpha,0}] = \alpha L_{\alpha,i}$, one has

$$\mathfrak{d}(L_{\alpha,i}) = \frac{1}{2}(G_\alpha \otimes G_0)a_i + (L_\alpha \otimes L_0 + L_0 \otimes L_\alpha)b_i \quad \text{for } \alpha \neq 0.$$

Setting $\gamma = \alpha$ in (22), we obtain $a_j x^i + a_i x^j = 0$ for all $i, j \in \mathbb{Z}$. Hence $a_i = 0$ for all $i \in \mathbb{Z}$. Similarly, $b_i = 0, \forall i \in \mathbb{Z}$. We conclude that $\mathfrak{d}(L_{\alpha,i}) = 0$.

Claim 2: $\mathfrak{d}(G_{\mu,i}) = 0, \mu \in s + \Gamma, i \in \mathbb{Z}$.

Applying $\mathfrak{d}$ to the equation $[L_{\alpha,i}, G_{\mu,j}] = (\mu - \frac{\alpha}{2})G_{\alpha+\mu}$ and using the fact $\mathfrak{d}(L_{\alpha,i}) = 0$, we have

$$(\gamma - 2\alpha)x^i c_{\mu,j,\gamma-\alpha} + (\mu - \gamma - \frac{\alpha}{2})y^i c_{\mu,j,\gamma} = (\mu - \frac{\alpha}{2})c_{\mu+\alpha,i+j,\gamma}, \tag{24}$$

$$(\gamma - \frac{3}{2}\alpha)x^i d_{\mu,j,\gamma-\alpha} + (\mu - \gamma - \alpha)y^i d_{\mu,j,\gamma} = (\mu - \frac{\alpha}{2})d_{\mu+\alpha,i+j,\gamma}. \tag{25}$$

Letting $\alpha = 0$ in (24), one has

$$(\mu - \gamma)y^i c_{\mu,j,\gamma} + \gamma x^i c_{\mu,j,\gamma} = \mu c_{\mu,i+j,\gamma}. \tag{26}$$

Setting $j = 0$ (rep. $\mu = 0$) in (26), we have

$$c_{\mu,i,\gamma} = 0 \text{ if } (\mu, \gamma) \neq (0, 0). \tag{27}$$

Similarly, we deduce from (25) that

$$d_{\mu,i,\gamma} = 0 \text{ if } (\mu, \gamma) \neq (0, 0). \tag{28}$$

We can assume $\mathfrak{d}(G_{0,i}) = L_0 \otimes G_0 c_i + G_0 \otimes L_0 d_i$, where $c_i = c_{0,i,0}, d_i = d_{0,i,0}$. Applying $\mathfrak{d}$ to the equation $[L_{\mu,0}, G_{0,i}] = -\frac{\mu}{2}G_{\mu,i}$, one has

$$\mathfrak{d}(G_{\mu,i}) = c_i(2L_\mu \otimes G_0 + L_0 \otimes G_\mu) + d_i(G_\mu \otimes L_0 + 2G_0 \otimes L_\mu), \mu \neq 0.$$

Letting $\mu = 0$ and $\gamma = \alpha \neq 0$ in (24), we have $c_i = 0$ for all $i \in \mathbb{Z}$. Similarly, $d_i = 0$. Then $\mathfrak{d}(G_{\mu,i}) = 0, \forall \mu \in s + \Gamma$. This proves Case 1.

Case 2: If $\mathfrak{d}$ is odd, then we also have $\mathfrak{d} = 0$.

Using the same techniques as those in Case 1, one can prove Case 2. □

Lemma 3.6. *Assume $r \in V$ such that $x \cdot r \in \mathrm{Im}(1 \otimes 1 - \tau)$ for all $x \in \mathcal{G}$. Then $r \in \mathrm{Im}(1 \otimes 1 - \tau)$.*

Proof. Write $r = \sum_{\alpha \in S} r_\alpha, r_\alpha \in V_\alpha$, where S is a finite subset of Γ'. Obviously,

$$x \cdot r \in \mathrm{Im}(1 \otimes 1 - \tau) \text{ if and only if } x \cdot r_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau) \text{ for all } \alpha \in S.$$

Thus we can suppose that $r = r_\alpha$ is homogeneous.

Since $L_{0,0} \cdot r_\alpha = \alpha r_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau)$, we have $r_\alpha \in \mathrm{Im}(1 \otimes 1 - \tau)$ if $\alpha \neq 0$. If $\alpha = 0$, we assume that $r_0 = \sum_\alpha M_\alpha \otimes N_{-\alpha} f_\alpha^{M,N}(x, y)$, where $M, N \in \{L, G\}, f_\alpha^{M,N}(x, y) \in \mathbb{F}[x^\pm, y^\pm]$. Note that $L_{0,i} \cdot r_0 = \sum_\alpha M_\alpha \otimes N_{-\alpha}(x^i - y^i)\alpha f_\alpha^{M,N}(x, y) \in \mathrm{Im}(1 \otimes 1 - \tau)$ and $\tau(M_\alpha \otimes N_{-\alpha} f_\alpha^{M,N}(x, y) = N_{-\alpha} \otimes M_\alpha f_\alpha^{M,N}(y, x)$. Then we have $f_\alpha^{M,N}(x, y) + f_{-\alpha}^{M,N}(y, x) = 0$ if $\alpha \neq 0$. Hence $r'_0 = \sum_{\alpha \neq 0} M_\alpha \otimes N_{-\alpha} f_\alpha^{M,N}(x, y) \in \mathrm{Im}(1 \otimes 1 - \tau)$.

Let $r''_0 = r_0 - r'_0 = \sum_{M,N} M_0 \otimes N_0 f_0^{M,N}(x, y)$. Clearly, $L_{\alpha,0} \cdot r''_0 \in \mathrm{Im}(1 \otimes 1 - \tau)$. It follows that $f_0^{M,N}(x, y) + f_0^{M,N}(y, x) = 0$ for $M, N \in \{L, G\}$. Then $r''_0 = \sum_{M,N} M_0 \otimes N_0 f_0^{M,N}(x, y) \in \mathrm{Im}(1 \otimes 1 - \tau)$. Therefore, $r_0 = r'_0 + r''_0 \in \mathrm{Im}(1 \otimes 1 - \tau)$. This proves the lemma. □

Proof of Theorem 2.7. Let $(\mathcal{G}, \varphi, \Delta)$ be a Lie super-bialgebra structure on $\mathcal{G}$. Then $\Delta = \Delta_r$ for some $r \in \mathcal{G} \otimes \mathcal{G}$ by Proposition 3.5, and $x \cdot r \in \mathrm{Im}(1 \otimes 1 - \tau)$ for any $x \in \mathcal{G}$ since $\mathrm{Im}\Delta \subset \mathrm{Im}(1 \otimes 1 - \tau)$. By Lemma 3.6, we have $r \in \mathrm{Im}(1 \otimes 1 - \tau)$. We also have $c(r) = 0$ by Corollary 3.3. We conclude that $(\mathcal{G}, \varphi, \Delta)$ is a triangular coboundary Lie super-bialgebra by Definitions 2.3, 2.4 and 2.5. $\square$

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References

[1] I. Bakas, *Conformal invariance, the KdV equation and coadjoint orbits of the Virasoro algebra*, Nuclear Phys. B **302** (1988), no. 2, 189–203.
[2] M. Chaichian and P. Prešnajder, *Discrete-time quantum field theory and the deformed super-Virasoro algebra*, Phys. Lett. A **322** (2004), no. 3-4, 156–165.
[3] Y.-S. Cheng, G.-A. Song, and B. Xin, *Lie bialgebra structures on Lie algebras of block type*, Algebra Colloq. **16** (2009), no. 4, 677–690.
[4] V. G. Drinfeld, *Constant quasiclassical solutions of the Yang-Baxter quantum equation*, Soviet Math. Dokl. **28** (1983), no. 3, 667–671.
[5] ______, *Quantum groups*, In: Proceeding of the International Congress of Mathematicians, Vol. 1, 2, pp. 798–820, Berkeley, Calif. Providence, RI: Amer. Math. Soc., 1986.
[6] R. Farnsteiner, *Derivations and central extensions of finitely generated graded Lie algebras*, J. Algebra **118** (1988), no. 1, 33–45.
[7] P. Goddard, A. Kent, and D. Olive, *Unitary representations of the Virasoro and super-Virasoro algebras*, Commun. Math. Phys. **103** (1986), no. 1, 105–119.
[8] X. Guo, R. Lu, and K. Zhao, *Simple Harish-Chandra modules, intermediate series modules, and Verma modules over the loop-Virasoro algebra*, Forum Math. **23** (2011), no. 5, 1029–1052.
[9] V. G. Kac, *Superconformal algebras and transitive groups actions on quadrics*, Commun. Math Phys. **186** (1997), no. 1, 233–252.
[10] V. G. Kac and J. W. V. D. Leur, *On Classification of Superconformal Algebras*, Strings **88**, Sinapore: World Scientific, 1988.
[11] I. Kenji and K. Yoshiyuki, *Representation theory of $N=2$ super Virasoro algebra: twisted sector*, J. Funct. Anal. **214** (2004), no. 2, 450–518.
[12] W. Michaelis, *A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras*, Adv. Math. **107** (1994), no. 2, 365–392.
[13] S.-H. Ng and E. J. Taft, *Classification of the Lie bialgebra structures on the Witt and Virasoro algebras*, J. Pure Appl. Algebra **151** (2000), no. 1, 67–88.
[14] J. Patera and H. Zassenhaus, *The higher rank Virasoro algebras*, Commun. Math. Phys. **36** (1991), no. 3, 1–14.
[15] H. T. Sato, *Deformation of super Virasoro algebra in noncommutative quantum superspace*, Phys. Lett. B **415** (1997), no. 2, 170–174.
[16] G.-A. Song and Y.-C. Su, *Lie bialgebras of generalized Witt type*, Sci. China Math. **49** (2006), no. 4, 533–544.
[17] Y.-C. Su and K. Zhao, *Generalized Virasoro and super-Virasoro algebras and modules of the intermediate series*, J. Algebra **252** (2002), no. 2, 1–19.

[18] H.-N. Wu, S. Wang, and X.-Q. Yue, *Lie bialgebras of generalized loop Virasoro algebra*, Chin. Ann. Math. Ser. B **3**, (2015), no. 3, 437–446.

[19] H.-Y. Yang and Y.-C. Su, *Lie super-bialgebra structures on generalized super-virasoro algebras*, Acta Math. Sci. Ser. B Engl. Ed. **1** (2010), no. 1, 225–239.

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