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A FAMILY OF FUNCTIONS ASSOCIATED WITH THREE TERM RELATIONS AND EISENSTEIN SERIES

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Abstract. In this paper, for $a \in \mathbb{C}$, we investigate functions g_a and ψ_a associated with three term relations. g_a is defined by means of function ψ_a. By using these functions, we obtain some functional equations related to the Eisenstein series and the Riemann zeta function. Also we find a generalized difference formula of function g_a.

1. Introduction

Recently, many authors has studied on period functions and three term relations. The period functions are real analytic functions $\psi(x)$ which satisfy three term relations, for $t \in \mathbb{R}$,

$$\psi(x) = \psi(x+1) + \frac{1}{(x+1)^{2s}}\psi\left(\frac{x}{x+1}\right),$$

where $s = \frac{1}{2} + it$ (cf. [4], [9]).

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The period function is also associated with a periodic and holomorphic function f defined by

$$f(z) = \psi(z) + \frac{1}{z^{2s}}\psi\left(-\frac{1}{z}\right),$$

where $z \in \mathbb{H}$.

In [4], Bettin and Conrey studied on the case of real analytic Eisenstein series. For these, the periodic function f turns out to be essentially

$$\sum_{n=1}^{\infty} \sigma_{2s-1}(n) e^{2\pi i n z},$$

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where, for $n \in \mathbb{N}$ and $a \in \mathbb{C}$,

$$\sigma_a(n) = \sum_{d|n} d^a.$$

A Maass wave form on the full modular group $\Gamma = PSL(2,\mathbb{Z})$ is a smooth Γ-invariant function u from the upper half plane $\mathbb{H}$ to $\mathbb{C}$ which is small as $y \to \infty$ and satisfies $\Delta u = \lambda u$ for some $\lambda \in \mathbb{C}$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the hyperbolic Laplacian (cf. [8], [9]).

Maass forms have many applications in a number of areas of mathematics such as number theory, dynamical systems and quantum chaos (cf. [9]).

In [7], Lewis showed that there exists a one-to-one correspondence between the space of even Maass wave forms with eigenvalue $\lambda = s(1-s)$ and the space of holomorphic functions on $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ satisfying three term relation together with a suitable growth condition.

In [9], Lewis and Zagier investigated the properties of general solutions of the three term relations, which is called periodlike functions. Also they were interested both in describing the totality of periodlike functions and in determining sufficient conditions for such a function to be the period function of a Maass form.

Let $u(z)$ be a Maass wave form with spectral parameter s. Then Lewis and Zagier defined the associated period function ψ in the upper and lower half-planes by the formula

$$\psi(z) \doteq \pm \sum_{n=1}^{\infty} n^{(s-1)/2} A_{\pm n} \left(e^{\pm 2\pi i n z} - z^{-2s} e^{\pm 2\pi i n/z} \right)$$

(Here, the symbol $\doteq$ denotes equality up to a factor depending only s). On the other hand, the original definition of the period function as given (in the even case) in [7] was represented by an integral transform; namely

$$\psi_1(z) \doteq \int_0^{\infty} z t^s (z^2 + t^2)^{-s-1} u(it) dt \ , \ \ (\mathrm{Re}(z) > 0)$$

where we have written "ψ_1" instead of "ψ" to avoid ambiguity.

In [5], Bruggeman gave a cohomological interpretation of theory of period functions and therefore the theory in the Maass context was developed.

In this paper, we focus on the results of Bettin and Conrey in [4]. By using these results, for $a \in \mathbb{C}$, we obtain some functional equations related to function g_a where g_a is associated with period function ψ_a. In final section, we give generalized difference formulas related to g_a.

In [4], for $a \in \mathbb{C}$, Bettin and Conrey gave a relation between extended Eisenstein series E_{a+1} and period function ψ_a. Therefore they obtained the function g_a associated with ψ_a. Also, for $\mathrm{Re}(\tau) > 0$ and $|z| < \tau$, they gave the Taylor series of $g_a(z)$ around τ.

For $a > 2$, the Eisenstein series are defined by (cf. [2], [10], [11])

$$G_a(z) = \sum_{\substack{m,n=-\infty \\ (m,n)\neq 0}}^{\infty} \frac{1}{(mz+n)^a}$$

and also the Eisenstein series has a Fourier expansion, for $k \geq 2$ and $a = 2k$, given by (cf. [2])

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z}.$$

In this expansion, for $\sigma > 1$ and $s = \sigma + it$, we define Riemann zeta function (cf. [3], [12])

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s = 1$, the Riemann zeta function is the harmonic series which diverges to ∞ and satisfies the following properties

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

and

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}, \tag{1.1}$$

where Bernoulli numbers and gamma function are denoted by B_n and $\Gamma(s)$, respectively (cf. [1], [12]).

The Laurent series of $\zeta(s)$ in a neighborhood of its pole $s = 1$ has the form:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n, \tag{1.2}$$

where γ is the Euler-Mascheroni constant and γ_n is also expressed in (cf. [6]):

$$\gamma_n = \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} \frac{(\log k)^n}{k} - \frac{(\log m)^{n+1}}{n+1} \right\}.$$

In special case of $n = 0$, we have $\gamma_0 = \gamma$.

2. Some functional equations related to function g_a

Let $\mathrm{Im} z > 0$. We consider the function E_{a+1} defined by

$$E_{a+1}(z) = 1 + \frac{2}{\zeta(-a)} S_a(z),$$

where

$$S_a(z) = \sum_{n=1}^{\infty} \sigma_a(n) e^{2\pi i n z}.$$

For $2 \leq k \in \mathbb{N}$ and $a = 2k-1$, we arrive at the following well known property:

$$E_{2k}(z) - \frac{1}{z^{2k}} E_{2k}\left(-\frac{1}{z}\right) = 0. \tag{2.1}$$

If we extend $2k$ to any complex number $a+1$, the equation (2.1) is no longer true. However, Bettin and Conrey investigate the properties of the function

$$\psi_a(z) = E_{a+1}(z) - \frac{1}{z^{a+1}} E_{a+1}\left(-\frac{1}{z}\right). \tag{2.2}$$

In this section, we obtain some functional equations related to g_a by using the following theorem:

Theorem 2.1 (cf. [4]). *Let* $\mathrm{Im} z > 0$ *and* $a \in \mathbb{C}$. *Then* ψ_a *satisfies the three term relation*

$$\psi_a(z+1) - \psi_a(z) + \frac{1}{(z+1)^{1+a}} \psi_a\left(\frac{z}{z+1}\right) = 0$$

and extends to an analytic function on $\mathbb{C}' = \mathbb{C} \setminus (-\infty, 0]$ *via the representation*

$$\psi_a(z) = \frac{i}{\pi z} \frac{\zeta(1-a)}{\zeta(-a)} - i \frac{1}{z^{1+a}} \cot \frac{\pi a}{2} + i \frac{g_a(z)}{\zeta(-a)}, \tag{2.3}$$

where

$$\begin{aligned} g_a(z) = & -2 \sum_{1 \leq n \leq M} (-1)^n \frac{B_{2n}}{(2n)!} \zeta(1-2n-a)(2\pi z)^{2n-1} \\ & + \frac{1}{\pi i} \int_{(-\frac{1}{2}-2M)} \zeta(s)\zeta(s-a)\Gamma(s) \frac{\cos \frac{\pi a}{2}}{\sin \frac{\pi(s-a)}{2}} (2\pi z)^{-s} ds \end{aligned}$$

and M *is any integer greater or equal to* $-\frac{1}{2}\min(0, \mathrm{Re} a)$.

For $a \to 0^+$, we get

$$\psi_0(z) = -2 \frac{(-\gamma + \log 2\pi z)}{\pi i z} - 2i g_0(z), \tag{2.4}$$

where

$$g_0(z) = \frac{1}{\pi i} \int_{(-1/2)} \frac{\zeta(s)\zeta(1-s)}{\sin \pi s} z^{-s} ds.$$

In Theorem 2.1, Bettin and Conrey also showed that there exists a function $\psi_a \neq 0$ which satisfies the three term relation. Three term relation is a special case of the following functional equation:

$$\phi_a(z+1) - \phi_a(z) + \frac{1}{(z+1)^{1+a}} \phi_a\left(\frac{z}{z+1}\right) = f_a(z). \tag{2.5}$$

In the following theorem, we show the existence of functions $f_a \neq 0$ and $\phi_a \neq 0$ such that the equation (2.5) holds.

Theorem 2.2. *Let* $\mathrm{Im} z > 0$ *and* $a \in \mathbb{C}$. *Then, we have*

$$\begin{aligned}&\frac{1}{(z+1)^{1+a}} g_a\left(\frac{z}{z+1}\right) + g_a(z+1) - g_a(z)\\&= \frac{\zeta(1-a)}{\pi z(z+1)} + \frac{\zeta(-a)\cot\frac{\pi a}{2}}{(z+1)^{1+a}} - \frac{\zeta(1-a)}{\pi z(z+1)^a}.\end{aligned}$$

Proof. From equation (2.3), we know that

$$\psi_a(z) = \frac{i}{\pi z}\frac{\zeta(1-a)}{\zeta(-a)} - i\frac{1}{z^{1+a}}\cot\frac{\pi a}{2} + i\frac{g_a(z)}{\zeta(-a)},$$

$$\begin{aligned}\frac{1}{(z+1)^{1+a}}\psi_a\left(\frac{z}{z+1}\right) &= \frac{i}{\pi z(z+1)^a}\frac{\zeta(1-a)}{\zeta(-a)} - i\frac{1}{z^{1+a}}\cot\frac{\pi a}{2}\\&\quad + \frac{i}{\zeta(-a)}\frac{1}{(z+1)^{1+a}} g_a\left(\frac{z}{z+1}\right)\end{aligned}$$

and

$$\psi_a(z+1) = \frac{i}{\pi(z+1)}\frac{\zeta(1-a)}{\zeta(-a)} - i\frac{1}{(z+1)^{1+a}}\cot\frac{\pi a}{2} + i\frac{g_a(z+1)}{\zeta(-a)},$$

where

$$\psi_a(z+1) - \psi_a(z) + \frac{1}{(z+1)^{1+a}}\psi_a\left(\frac{z}{z+1}\right) = 0. \tag{2.6}$$

By using the above equations, we arrange (2.6) as a functional equation represented by g_a. ☐

Remark 2.3. In Theorem 2.2, consider $z \to z-1$. Then,

$$\begin{aligned}&\lim_{a\to 0}\frac{1}{\zeta(-a)}\left\{\frac{1}{z^{a+1}} g_a\left(\frac{z-1}{z}\right) + g_a(z) - g_a(z-1)\right\}\\&= \lim_{a\to 0}\frac{\zeta(1-a)}{\zeta(-a)}\left\{\frac{1}{\pi(z-1)} - \frac{1}{\pi z}\right\} + \lim_{a\to 0}\frac{\cot\frac{\pi a}{2}}{z^{1+a}} - \lim_{a\to 0}\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi(z-1)z^a}\\&= -\lim_{a\to 0}\left\{\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi z} - \frac{1}{z^{1+a}}\cot\frac{\pi a}{2}\right\} + \frac{1}{\pi(z-1)}\lim_{a\to 0}\frac{\zeta(1-a)}{\zeta(-a)}\frac{(z^a-1)}{z^a}.\end{aligned}$$

By using the properties of (1.1) and (1.2) related to zeta function, we get

$$\lim_{a\to 0}\left\{\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi z} - \frac{1}{z^{1+a}}\cot\frac{\pi a}{2}\right\} = 2\frac{(-\gamma + \log 2\pi z)}{\pi z} \tag{2.7}$$

and

$$\lim_{a\to 0}\zeta(1-a)(z^a-1) = -\log z. \tag{2.8}$$

From (2.7) and (2.8), we arrive at

$$\frac{1}{z}g_0\left(\frac{z-1}{z}\right)+g_0(z)-g_0(z-1)=\frac{-\gamma+\log 2\pi z}{\pi z}-\frac{1}{\pi(z-1)}\log z.$$

Lemma 2.4. *Let* $\mathrm{Im}z>0$ *and* $a\in\mathbb{C}$. *Then, we have*

$$\psi_a(z)+\frac{1}{z^{a+1}}\psi_a\left(-\frac{1}{z}\right)=\left(1+(-1)^a\right)E_{a+1}(z).$$

Proof. If we consider $z\to -1/z$ in equation (2.2), we get

$$\frac{1}{z^{a+1}}\psi_a\left(-\frac{1}{z}\right)=\frac{1}{z^{a+1}}E_{a+1}\left(-\frac{1}{z}\right)+(-1)^aE_{a+1}(z). \tag{2.9}$$

By combining the equations (2.2) and (2.9), we arrive at the desired result. □

Theorem 2.5. *Let* $\mathrm{Im}z>0$ *and* $a\in\mathbb{C}$. *Then, we have*

$$\begin{aligned}g_a(z)+\frac{1}{z^{a+1}}g_a\left(-\frac{1}{z}\right)=&-i\zeta(-a)(1+(-1)^a)E_{a+1}(z)-\frac{\zeta(1-a)}{\pi}\left(\frac{1}{z}-\frac{1}{z^a}\right)\\&-\zeta(-a)\left((-1)^a-\frac{1}{z^{1+a}}\right)\cot\frac{\pi a}{2}.\end{aligned}$$

Proof. If we consider $z\to -1/z$ in Theorem 2.1, we get

$$\psi_a\left(-\frac{1}{z}\right)=-\frac{iz}{\pi}\frac{\zeta(1-a)}{\zeta(-a)}+i(-1)^az^{1+a}\cot\frac{\pi a}{2}+\frac{i}{\zeta(-a)}g_a\left(-\frac{1}{z}\right)$$

or

$$\frac{1}{z^{1+a}}\psi_a\left(-\frac{1}{z}\right)=-\frac{i}{\pi z^a}\frac{\zeta(1-a)}{\zeta(-a)}+i(-1)^a\cot\frac{\pi a}{2}+\frac{i}{\zeta(-a)}\frac{1}{z^{a+1}}g_a\left(-\frac{1}{z}\right). \tag{2.10}$$

By combining equations (2.3) and (2.10), we have

$$\begin{aligned}(1+(-1)^a)E_{a+1}(z)=&\frac{i}{\pi}\frac{\zeta(1-a)}{\zeta(-a)}\left(\frac{1}{z}-\frac{1}{z^a}\right)+i\left((-1)^a-\frac{1}{z^{1+a}}\right)\cot\frac{\pi a}{2}\\&+\frac{i}{\zeta(-a)}\left\{g_a(z)+\frac{1}{z^{a+1}}g_a\left(-\frac{1}{z}\right)\right\}.\end{aligned}$$

After some elementary calculation, we arrive at the desired result. □

In Theorem 2.5, we also show that there exist the functions $\phi^*\neq 0$ and $f^*\neq 0$ such that they satisfies the following functional equation:

$$\phi_a^*(z)-\frac{1}{z^{a+1}}\phi_a^*\left(-\frac{1}{z}\right)=f_a^*(z).$$

Remark 2.6. By using Theorem 2.5, we get

$$\begin{aligned}&\lim_{a\to 0}\frac{1}{\zeta(-a)}\left\{g_a(z)+\frac{1}{z^{a+1}}g_a\left(-\frac{1}{z}\right)\right\}\\ =&-i\lim_{a\to 0}(1+(-1)^a)E_{a+1}(z)-\lim_{a\to 0}\left\{\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi z}-\frac{1}{z^{a+1}}\cot\frac{\pi a}{2}\right\}\\ &-\lim_{a\to 0}\left\{\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi z^a}-(-1)^a\cot\frac{\pi a}{2}\right\}\end{aligned}$$

or

$$\begin{aligned}&-2\left\{g_0(z)+\frac{1}{z}g_0\left(-\frac{1}{z}\right)\right\}\\ =&-2iE_1(z)-2\frac{(-\gamma+\log 2\pi z)}{\pi z}+\lim_{a\to 0}\left\{\frac{\zeta(1-a)}{\zeta(-a)}\frac{1}{\pi z^a}-(-1)^a\cot\frac{\pi a}{2}\right\}.\end{aligned}$$

By using the properties of (1.1) and (1.2) related to zeta function, we get

$$\begin{aligned}\lim_{a\to 0}\left\{-\frac{2}{\pi}\zeta(1-a)-(-1)^a\cot\frac{\pi a}{2}\right\}&=-\frac{2\gamma}{\pi}+\frac{2}{\pi}\lim_{a\to 0}\frac{(1-(-1)^a)}{a}\\ &=-\frac{2\gamma}{\pi}-2i.\end{aligned}$$

Therefore, we arrive at the following result:

$$g_0(z)+\frac{1}{z}g_0\left(-\frac{1}{z}\right)=\frac{\gamma}{\pi}+i(1+E_1(z))+\frac{-\gamma+\log 2\pi z}{\pi z}.$$

In the following theorem, we consider

$$f_a(z)=\frac{1}{(z+1)^{1+a}}g_a\left(\frac{z}{z+1}\right)+g_a(z+1)-g_a(z).$$

Theorem 2.7. *The function* f_a *has Taylor series as follows*:

$$f_a(z)=\sum_{n=0}^{\infty}(-1)^n\left\{\zeta(-a)\binom{n+a}{a}\cot\frac{\pi a}{2}-\frac{\zeta(1-a)}{\pi}\left(1-\binom{n+a}{a-1}\right)\right\}z^n,$$

where $|z|<1$.

Proof. We know that

$$\sum_{n=0}^{\infty}(-1)^n z^n=\frac{1}{z+1}, \tag{2.11}$$

where $|z|<1$.

If we take the a-th derivative of the above series, we have

$$\sum_{n=0}^{\infty}(-1)^n\binom{n+a}{a}z^n=\frac{1}{(z+1)^{1+a}}, \tag{2.12}$$

where $|z|<1$.

Then, by using the series (2.11) and (2.12), we get

$$\begin{aligned} f_a(z) = & \frac{\zeta(1-a)}{\pi z}\sum_{n=0}^{\infty}(-1)^n z^n + \zeta(-a)\cot\frac{\pi a}{2}\sum_{n=0}^{\infty}(-1)^n \binom{n+a}{a} z^n \\ & - \frac{\zeta(1-a)}{\pi z}\sum_{n=0}^{\infty}(-1)^n \binom{n+a-1}{a-1} z^n \\ = & \frac{\zeta(1-a)}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(1-\binom{n+a-1}{a-1}\right) z^{n-1} \\ & + \zeta(-a)\cot\frac{\pi a}{2}\sum_{n=0}^{\infty}(-1)^n \binom{n+a}{a} z^n \\ = & -\frac{\zeta(1-a)}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(1-\binom{n+a}{a-1}\right) z^n \\ & + \zeta(-a)\cot\frac{\pi a}{2}\sum_{n=0}^{\infty}(-1)^n \binom{n+a}{a} z^n. \end{aligned}$$

Therefore, we obtain the Taylor series of function f_a around $z_0 = 0$. □

3. Generalized difference formulas related to function g_a

In this section, by using the properties of the function ψ_a, we obtain some generalized difference formulas related to the function g_a.

Lemma 3.1. *Let $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then we have*

$$\psi_a(z) - \psi_a(z+N+1) = \sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} \psi_a\left(\frac{z+k-1}{z+k}\right). \tag{3.1}$$

Proof. We use the iteration $z \to z+1$ as follows:

$$\psi_a(z) - \psi_a(z+1) = \frac{1}{(z+1)^{1+a}}\psi_a\left(\frac{z}{z+1}\right),$$

$$\psi_a(z+1) - \psi_a(z+2) = \frac{1}{(z+2)^{1+a}}\psi_a\left(\frac{z+1}{z+2}\right),$$

$$\psi_a(z+2) - \psi_a(z+3) = \frac{1}{(z+3)^{1+a}}\psi_a\left(\frac{z+2}{z+3}\right),$$

$$\vdots$$

$$\psi_a(z+N) - \psi_a(z+N+1) = \frac{1}{(z+N+1)^{1+a}}\psi_a\left(\frac{z+N}{z+N+1}\right).$$

By combining the above equations, we arrive at the desired result. □

Theorem 3.2. *Let $N \in \mathbb{N}$ and $a \in \mathbb{C}$. Then we have*

$$\begin{aligned}
&\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} g_a\left(\frac{z+k-1}{z+k}\right) \\
&= \frac{(N+1)\zeta(1-a)}{\pi z(z+N+1)} + g_a(z) - g_a(z+N+1) \\
&\quad + \sum_{k=1}^{N+1} \left\{ \frac{\zeta(-a)\cot\frac{\pi a}{2}}{(z+k)} - \frac{\zeta(1-a)}{\pi(z+k-1)} \right\} \frac{1}{(z+k)^a}.
\end{aligned}$$

Proof. By using the equations (2.3) and (3.1), we have

$$\begin{aligned}
&\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} \left\{ \frac{i(z+k)\zeta(1-a)}{\pi(z+k-1)\zeta(-a)} - \frac{i(z+k)^{1+a}}{(z+k-1)^{1+a}} \cot\frac{\pi a}{2} \right. \\
&\qquad\qquad \left. + \frac{i}{\zeta(-a)} g_a\left(\frac{z+k-1}{z+k}\right) \right\} \\
&= \frac{i(N+1)\zeta(1-a)}{\pi z(z+N+1)\zeta(-a)} + i\left\{ \frac{1}{(z+N+1)^{1+a}} - \frac{1}{z^{1+a}} \right\} \\
&\quad + \frac{i}{\zeta(-a)} \left\{ g_a(z) - g_a(z+N+1) \right\}
\end{aligned}$$

or

$$\begin{aligned}
&\sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}} g_a\left(\frac{z+k-1}{z+k}\right) \\
&= g_a(z) - g_a(z+N+1) \\
&\quad + \frac{(N+1)\zeta(1-a)}{\pi z(z+N+1)} + \zeta(-a)\left\{ \frac{1}{(z+N+1)^{1+a}} - \frac{1}{z^{1+a}} \right\} \cot\frac{\pi a}{2} \\
&\quad + \sum_{k=1}^{N+1} \left\{ \frac{\zeta(-a)\cot\frac{\pi a}{2}}{(z+k-1)^{1+a}} - \frac{\zeta(1-a)}{\pi(z+k)^a(z+k-1)} \right\}
\end{aligned}$$

because of

$$\frac{1}{(z+N+1)^{1+a}} - \frac{1}{z^{1+a}} + \sum_{k=1}^{N+1} \frac{1}{(z+k-1)^{1+a}} = \sum_{k=1}^{N+1} \frac{1}{(z+k)^{1+a}}.$$

Then, we arrive at the desired result. □

In Theorem 3.2, taking the limit $a \to 0$, we have

$$\begin{aligned}
&\sum_{k=1}^{N+1} \frac{1}{(z+k)} g_0\left(\frac{z+k-1}{z+k}\right) \\
&= g_0(z) - g_0(z+N+1) + \lim_{a\to 0} \frac{(N+1)\zeta(1-a)}{\pi z(z+N+1)}
\end{aligned}$$

$$+\lim_{a\to 0}\sum_{k=1}^{N+1}\left\{\frac{\zeta(-a)\cot\frac{\pi a}{2}}{(z+k)}-\frac{\zeta(1-a)}{\pi(z+k-1)}\right\}$$

$$=g_0(z)-g_0(z+N+1)-\lim_{a\to 0}\sum_{k=1}^{N+1}\frac{1}{(z+k)}\left\{\frac{\cot\frac{\pi a}{2}}{2}+\frac{\zeta(1-a)}{\pi}\right\}$$

because of

$$\frac{1}{z}-\frac{1}{(z+N+1)}-\sum_{k=1}^{N+1}\frac{1}{(z+k-1)}=-\sum_{k=1}^{N+1}\frac{1}{(z+k)}$$

and $\zeta(0)=-1/2$.

We know that

$$\lim_{a\to 0}\left\{\frac{\cot\frac{\pi a}{2}}{2}+\frac{\zeta(1-a)}{\pi}\right\}=\frac{\gamma}{\pi}. \tag{3.2}$$

Therefore, we obtain the following corollary:

Corollary 3.3. *Let $N\in\mathbb{N}$. Then, we have*

$$g_0(z)-g_0(z+N+1)=\sum_{k=1}^{N+1}\frac{1}{(z+k)}\left\{\frac{\gamma}{\pi}+g_0\left(\frac{z+k-1}{z+k}\right)\right\}. \tag{3.3}$$

By using the equation (2.4), for $k\in\{1,2,\dots,N+1\}$, we have

$$\frac{1}{z+k}\psi_0\left(\frac{z+k-1}{z+k}\right)$$

$$=\frac{1}{z+k}\left\{-2\frac{\left\{-\gamma+\log\left(2\pi\left(\frac{z+k-1}{z+k}\right)\right)\right\}}{\pi i\left(\frac{z+k-1}{z+k}\right)}-2ig_0\left(\frac{z+k-1}{z+k}\right)\right\}.$$

By summing from 1 to $N+1$, we deduce from the above that

$$\sum_{k=1}^{N+1}\frac{1}{z+k}\psi_0\left(\frac{z+k-1}{z+k}\right)$$

$$=-2\sum_{k=1}^{N+1}\frac{\left\{-\gamma+\log\left(2\pi\left(\frac{z+k-1}{z+k}\right)\right)\right\}}{\pi i(z+k-1)}-2i\sum_{k=1}^{N+1}\frac{1}{z+k}g_0\left(\frac{z+k-1}{z+k}\right).$$

By using the equations (3.1) and (3.3), we get

$$\psi_0(z)-\psi_0(z+N+1)$$

$$=-2i\left\{g_0(z)-g_0(z+N+1)\right\}+\frac{2i}{\pi}\sum_{k=1}^{N+1}\frac{\log\left(2\pi\left(\frac{z+k-1}{z+k}\right)\right)}{(z+k-1)}$$

$$-\frac{2i\gamma}{\pi}\sum_{k=1}^{N+1}\left(\frac{1}{z+k-1}-\frac{1}{z+k}\right).$$

After some elementary calculations, we arrive at the following corollary:

Corollary 3.4. *Let $N \in \mathbb{N}$. Then, we have*

$$\begin{aligned}&\sum_{k=1}^{N+1} \frac{1}{z+k-1} \log\left(2\pi\left(\frac{z+k-1}{z+k}\right)\right)\\ &= \gamma\left(\frac{1}{z} - \frac{1}{z+N+1}\right) + \pi\left\{g_0(z) - g_0(z+N+1)\right\}\\ &\quad - \frac{\pi i}{2}\left\{\psi_0(z) - \psi_0(z+N+1)\right\}.\end{aligned}$$

Lemma 3.5. *Let $a, \lambda \in \mathbb{C}$ and $\lambda \neq 0$. Then, we have*

$$\frac{\psi_a\left(\frac{z}{\lambda}\right) - \psi_a\left(\frac{z+\lambda}{\lambda}\right)}{\psi_a\left(\frac{z}{z+\lambda}\right)} = \lambda^{a+1}\left(\frac{\psi_a\left(z+\lambda-1\right) - \psi_a\left(z+\lambda\right)}{\psi_a\left(\frac{z+\lambda-1}{z+\lambda}\right)}\right). \tag{3.4}$$

Proof. From equation (2.6), we know

$$\frac{\psi_a\left(z\right) - \psi_a\left(z+1\right)}{\psi_a\left(\frac{z}{z+1}\right)} = \frac{1}{(z+1)^{a+1}}.$$

For $z \to z/\lambda$ $(0 \neq \lambda \in \mathbb{C})$, we get

$$\begin{aligned}\frac{\psi_a\left(\frac{z}{\lambda}\right) - \psi_a\left(\frac{z}{\lambda}+1\right)}{\psi_a\left(\frac{z}{z+\lambda}\right)} &= \frac{\lambda^{a+1}}{(z+\lambda)^{a+1}}\\ &= \frac{\lambda^{a+1}}{(z+\lambda-1+1)^{a+1}}\\ &= \lambda^{a+1}\left(\frac{\psi_a\left(z+\lambda-1\right) - \psi_a\left(z+\lambda\right)}{\psi_a\left(\frac{z+\lambda-1}{z+\lambda}\right)}\right).\end{aligned}$$

Then, we have the desired result. □

Theorem 3.6. *Let $a, \lambda \in \mathbb{C}$ and $N \in \mathbb{N}$. Then, we have*

$$\begin{aligned}&\sum_{k=1}^{N}\left\{\begin{array}{c}\left(\frac{z+k\lambda-1}{z+(k-1)\lambda}\right)^{1+a} \times \left(\frac{(z+(k-1)\lambda)^a(z+k\lambda)\zeta(1-a) - \pi(z+k\lambda)^{1+a}\zeta(-a)\cot\frac{\pi a}{2} + \pi(z+(k-1)\lambda)^{1+a} g_a\left(\frac{z+(k-1)\lambda}{z+k\lambda}\right)}{(z+k\lambda-1)^a(z+k\lambda)\zeta(1-a) - \pi(z+k\lambda)^{1+a}\zeta(-a)\cot\frac{\pi a}{2} + \pi(z+k\lambda-1)^{1+a} g_a\left(\frac{z+k\lambda-1}{z+k\lambda}\right)}\right)\\ \times\left(\frac{\zeta(1-a)}{\pi(z+k\lambda)(z+k\lambda-1)} - \left(\frac{1}{(z+k\lambda-1)^{1+a}} - \frac{1}{(z+k\lambda)^{1+a}}\right)\zeta(-a)\cot\frac{\pi a}{2} + g_a\left(z+k\lambda-1\right) - g_a\left(z+k\lambda\right)\right)\end{array}\right\}\\ &= \frac{N\lambda^{1-a}\zeta(1-a)}{\pi z(z+N\lambda)} + \left(\frac{1}{(z+N\lambda)^{1+a}} - \frac{1}{z^{1+a}}\right)\zeta(-a)\cot\frac{\pi a}{2}\\ &\quad + \frac{1}{\lambda^{1+a}}\left(g_a\left(\frac{z}{\lambda}\right) - g_a\left(\frac{z+N\lambda}{\lambda}\right)\right).\end{aligned}$$

Proof. From equation (3.4), we use the iteration $z \to z+\lambda$ as follows:

$$\psi_a\left(\frac{z}{\lambda}\right) - \psi_a\left(\frac{z+\lambda}{\lambda}\right) = \lambda^{a+1}\frac{\psi_a\left(\frac{z}{z+\lambda}\right)}{\psi_a\left(\frac{z+\lambda-1}{z+\lambda}\right)}\left\{\psi_a\left(z+\lambda-1\right) - \psi_a\left(z+\lambda\right)\right\},$$

$$\psi_a\left(\frac{z+\lambda}{\lambda}\right)-\psi_a\left(\frac{z+2\lambda}{\lambda}\right)=\lambda^{a+1}\frac{\psi_a\left(\frac{z+\lambda}{z+2\lambda}\right)}{\psi_a\left(\frac{z+2\lambda-1}{z+2\lambda}\right)}\left\{\psi_a\left(z+2\lambda-1\right)-\psi_a\left(z+2\lambda\right)\right\},$$

$$\psi_a\left(\frac{z+2\lambda}{\lambda}\right)-\psi_a\left(\frac{z+3\lambda}{\lambda}\right)=\lambda^{a+1}\frac{\psi_a\left(\frac{z+2\lambda}{z+3\lambda}\right)}{\psi_a\left(\frac{z+3\lambda-1}{z+3\lambda}\right)}\left\{\psi_a\left(z+3\lambda-1\right)-\psi_a\left(z+3\lambda\right)\right\},$$

$$\vdots$$

$$\psi_a\left(\frac{z+(N-1)\lambda}{\lambda}\right)-\psi_a\left(\frac{z+N\lambda}{\lambda}\right)=\lambda^{a+1}\frac{\psi_a\left(\frac{z+(N-1)\lambda}{z+N\lambda}\right)}{\psi_a\left(\frac{z+N\lambda-1}{z+N\lambda}\right)}\left\{\psi_a\left(z+N\lambda-1\right)-\psi_a\left(z+N\lambda\right)\right\}.$$

By combining the above equations, we get

(3.5)
$$\frac{\psi_a\left(\frac{z}{\lambda}\right)-\psi_a\left(\frac{z+N\lambda}{\lambda}\right)}{\lambda^{a+1}}=\sum_{k=1}^{N}\frac{\psi_a\left(\frac{z+(k-1)\lambda}{z+k\lambda}\right)}{\psi_a\left(\frac{z+k\lambda-1}{z+k\lambda}\right)}\left\{\psi_a\left(z+k\lambda-1\right)-\psi_a\left(z+k\lambda\right)\right\}.$$

By using the equation (2.3), we arrange the equation (3.5). Then, we arrive at the desired result. □

In Theorem 3.6, taking the limit $a\to 0$, we have

$$\sum_{k=1}^{N}\left\{\begin{array}{c}\left(\frac{z+k\lambda-1}{z+(k-1)\lambda}\right)\times\left(\frac{(z+k\lambda)\left(\lim_{a\to 0}(\zeta(1-a)+\frac{\pi}{2}\cot\frac{\pi a}{2})\right)+\pi(z+(k-1)\lambda)g_0\left(\frac{z+(k-1)\lambda}{z+k\lambda}\right)}{(z+k\lambda)\left(\lim_{a\to 0}(\zeta(1-a)+\frac{\pi}{2}\cot\frac{\pi a}{2})\right)+\pi(z+k\lambda-1)g_0\left(\frac{z+k\lambda-1}{z+k\lambda}\right)}\right)\\ \times\left(\frac{1}{(z+k\lambda)(z+k\lambda-1)}\lim_{a\to 0}\left(\frac{\zeta(1-a)}{\pi}+\frac{1}{2}\cot\frac{\pi a}{2}\right)+g_0\left(z+k\lambda-1\right)-g_0\left(z+k\lambda\right)\right)\end{array}\right\}$$
$$=\frac{N\lambda}{z(z+N\lambda)}\lim_{a\to 0}\left(\frac{\zeta(1-a)}{\pi}+\frac{1}{2}\cot\frac{\pi a}{2}\right)+\frac{1}{\lambda}\left(g_0\left(\frac{z}{\lambda}\right)-g_0\left(\frac{z+N\lambda}{\lambda}\right)\right).$$

By using (3.2), we arrive at the following corollary:

Corollary 3.7. *Let $\lambda\in\mathbb{C}$ and $N\in\mathbb{N}$. Then, we have*

$$\sum_{k=1}^{N}\left\{\begin{array}{c}\left(\frac{z+k\lambda-1}{z+(k-1)\lambda}\right)\times\left(\frac{\gamma(z+k\lambda)+\pi(z+(k-1)\lambda)g_0\left(\frac{z+(k-1)\lambda}{z+k\lambda}\right)}{\gamma(z+k\lambda)+\pi(z+k\lambda-1)g_0\left(\frac{z+k\lambda-1}{z+k\lambda}\right)}\right)\\ \times\left(\frac{\gamma}{\pi(z+k\lambda)(z+k\lambda-1)}+g_0\left(z+k\lambda-1\right)-g_0\left(z+k\lambda\right)\right)\end{array}\right\}$$
$$=\frac{\gamma N\lambda}{\pi z(z+N\lambda)}+\frac{1}{\lambda}\left(g_0\left(\frac{z}{\lambda}\right)-g_0\left(\frac{z+N\lambda}{\lambda}\right)\right).$$

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