Bull. Korean Math. Soc. **53** (2016), No. 6, pp. 1651–1670
http://dx.doi.org/10.4134/BKMS.b150863
pISSN: 1015-8634 / eISSN: 2234-3016

SOME RESULTS OF EXPONENTIALLY BIHARMONIC MAPS INTO A NON-POSITIVELY CURVED MANIFOLD

Yingbo Han

Abstract. In this paper, we investigate exponentially biharmonic maps $u:(M,g)\to(N,h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. We obtain that if

$$\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g<\infty\ (p\geq 2),\ \int_M|\tau(u)|^2dv_g<\infty \text{ and}$$
$$\int_M|du|^2dv_g<\infty,$$

then u is harmonic. When u is an isometric immersion, we get that if $\int_M e^{\frac{pm^2|H|^2}{2}}|H|^q dv_g<\infty$ for $2\leq p<\infty$ and $0<q\leq p<\infty$, then u is minimal. We also obtain that any weakly convex exponentially biharmonic hypersurface in space form $N(c)$ with $c\leq 0$ is minimal. These results give affirmative partial answer to conjecture 3 (generalized Chen's conjecture for exponentially biharmonic submanifolds).

1. Introduction

Let (M^m,g) and (N^n,h) be Riemannian manifolds of dimensions m,n and $u:(M^m,g)\to(N^n,h)$ be a smooth map. The Dirichlet energy of u is defined by $E(u)=\int_M\frac{|du|^2}{2}dv_g$. The critical maps of $E(\cdot)$ are called harmonic maps. The Euler-Lagrange equation of harmonic maps is $\tau(u)=0$, where $\tau(u)$ is called the tension field of u. Extensions to the notions of p-harmonic maps, exponentially harmonic maps, F-harmonic maps and f-harmonic maps were introduced and many results have been carried out (for instance, see [1, 2, 3, 9, 19, 28]). In 1983, J. Eells and L. Lemaire [12] proposed the problem to consider the biharmonic maps: they are critical maps of the functional $E_2(u)=\int_M\frac{|\tau(u)|^2}{2}dv_g$. We see that harmonic maps are biharmonic maps and even more, minimizers of the bienergy functional. After G. Y. Jiang [18] studied the first and second variation formulas of the bienergy E_2, there have been extensive studies on biharmonic maps (for instance, see [10, 18, 20, 21, 26, 27]). Recently the author and S. X.

Received October 22, 2015; Revised February 19, 2016.
2010 *Mathematics Subject Classification.* Primary 58E20, 53C21.
Key words and phrases. exponentially biharmonic maps, exponentially biharmoinc submanifolds.

Feng in [15] introduced the following functional $E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2})dv_g$, where $\tau(u) = -\delta du = trace\widetilde{\nabla}(du)$. The map u is called an F-biharmonic map if it is a critical point of that F-bienergy $E_{F,2}(u)$, which is a generalization of biharmonic maps, p-biharmonic maps [17] or exponentially biharmonic maps. Notice that harmonic maps are always F-biharmonic by definition. When $F(t) = e^t$, we have exponential bienergy functional

$$E_{e,2}(u) = \int_M e^{\frac{|\tau(u)|^2}{2}} dv_g.$$

The Euler-Lagrange equation of $E_{e,2}$ is $\tau_{e,2}(u) = 0$, where $\tau_{e,2}(u)$ is given by (5). A map $u : (M, g) \to (N, h)$ is called an exponentially biharmonic map if $\tau_{e,2}(u) = 0$. When $u : (M, g) \to (N, h)$ is a exponentially biharmonic isometric immersion, then M is called an exponentially biharmonic submanifold in N.

Recently, N. Nakauchi, H. Urakawa and S. Gudmundsson [26] proved that biharmoic maps from a complete Riemannian manifold into a non-positive curved manifold with finite bienergy and energy are harmonic. S. Maeta [25] proved that biharmoic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a + 2)$-bienergy $\int_M |\tau(u)|^{a+2}dv_g < \infty$ $(a \geq 0)$ and energy are harmonic. The author and W. Zhang in [16] proved that p-biharmoinc maps from a complete manifold into a non-positive curved manifold with finite $a + p$-bienergy $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and energy are harmonic. In this paper, we first obtain the following result:

Theorem 1.1 (cf. Theorem 3.1)**.** *Let $u : (M^m, g) \to (N^n, h)$ be an exponentially biharmonic map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M e^{\frac{p|\tau(u)|^2}{2}} |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty \quad \textit{and} \quad \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $Vol(M, g) = \infty$, and*

$$\int_M e^{\frac{p|\tau(u)|^2}{2}} |\tau(u)|^p dv_g < \infty,$$

then u is harmonic.

One of the most interesting problems in the biharmonic theory is Chen's conjecture. In 1988, Chen raised the following problem:

Conjecture 1 ([8])**.** Any biharmonic submanifold in E^n is minimal.

There are many affirmative partial answers to Chen's conjecture.

On the other hand, Chen's conjecture was generalized as follows (cf. [6]): "Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal". There are also many affirmative partial answers to this conjecture.

(a) Any biharmonic submanifold in $H^3(-1)$ is minimal (cf. [5]).

(b) Any biharmonic hypersurfaces in $H^4(-1)$ is minimal (cf. [4]).

(c) Any weakly convex biharmonic hypersurfaces in space form $N^{m+1}(c)$ with $c \leq 0$ is minimal (cf. [22]).

(d) Any compact biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [18]).

(e) Any compact F-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [15]).

Motivated by Chen's conjecture, the author [14] proposed the following conjecture:

Conjecture 2 ([14])**.** Any p-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

Some partial affirmative answers to Conjecture 2 were proved in [7], [14], [16], and [24].

For exponentially biharmonic submanifolds, it is natural to consider the following problem.

Conjecture 3. Any exponentially biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

For exponentially biharmonic submanifolds, we obtain the following results:

Theorem 1.2 (cf. Theorem 4.1)**.** *Let $u : (M, g) \to (N, h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$.*

If

$$\int_M e^{\frac{pm^2|H|^2}{2}} |H|^q dv_g < \infty,$$

then u is minimal.

Theorem 1.3 (cf. Theorem 4.2)**.** *Let $u : (M, g) \to (N, h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature. If*

$$\int_{B_r(x_0)} e^{\frac{pm^2|H|^2}{2}} dv_g \leq C_0(1 + r)^s \tag{1}$$

for some positive integer s, C_0 independent of r and $p \geq 2$, then u is minimal.

Theorem 1.4 (cf. Theorem 4.3)**.** *Let $u : (M, g) \to (N, h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} e^{\frac{pm^2|H|^2}{2}} |H|^p dv_g$ $(p \geq 2)$ is of at most polynomial growth of r. Then u is minimal.*

In [29], G. Wheeler proposed a notion ε-super biharmonic submanifolds which is a generalization of submanifolds with harmonic mean curvature vector fields, as follows:

Definition 1.5 ([29]). Let M be a submanifold in N with the metric $\langle\cdot,\cdot\rangle$. Then we call M a ε-super biharmonic submanifold, if

$$\langle \triangle H, H\rangle \geq (\varepsilon-1)|\nabla H|^2, \tag{2}$$

where $\varepsilon\in[0,1]$ is a constant.

From the Definition 1.5, it is natural to consider the following definition.

Definition 1.6. Let M be a submanifold in N with the metric $\langle\cdot,\cdot\rangle$. Then we call M a ε-super exponentially biharmonic submanifold, if

$$\langle \triangle(e^{\frac{m^2|H|^2}{2}}H), e^{\frac{m^2|H|^2}{2}}H\rangle \geq (\varepsilon-1)|\nabla(e^{\frac{m^2|H|^2}{2}}H)|^2, \tag{3}$$

where $\varepsilon\in[0,1]$ is a constant.

In this note, we investigate the ε-super exponentially biharmonic submanifold, and get the following result:

Theorem 1.7 (cf. Theorem 4.4). *Let $u:(M,g)\to(N,h)$ be a complete ε-super exponentially biharmonic submanifold in N for $\varepsilon>0$. If*

$$\int_M e^{\frac{pm^2|H|^2}{2}}|H|^p dv_g<\infty, \tag{4}$$

then u is minimal, where $p\geq 2$.

In [22], Y. Luo investigate the weakly convex biharmonic hypersurfaces in a space form, and obtained the following result:

Theorem 1.8 ([22]). *Let $u:(M^m,g)\to(N^{m+1}(c),\langle,\rangle)$ be a weakly convex biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c\leq 0$. Then u is minimal.*

In this note, we investigate the weakly convex exponentially biharmonic hypersurface in a space form, and get the following result:

Theorem 1.9 (cf. Theorem 4.5). *Let $u:(M^m,g)\to(N^{m+1}(c),\langle,\rangle)$ be a weakly convex exponentially biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c\leq 0$. Then u is minimal.*

These results give affirmative partial answers to the generalized Chen's conjecture for exponentially biharmonic submanifold.

2. Preliminaries

In this section we give more details for the definitions of harmonic maps, biharmonic maps, exponentially biharmonic maps and exponentially biharmonic submanifolds.

Let $u : (M, g) \to (N, h)$ be a map from an m-dimensional Riemannian manifold (M, g) to an n-dimensional Riemannian manifold (N, h). The energy of u is defined by

$$E(u) = \int_M \frac{|du|^2}{2} dv_g.$$

The Euler-Lagrange equation of E is

$$\tau(u) = \sum_{i=1}^{m} \{\widetilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)\} = 0,$$

where we denote by ∇ the Levi-Civita connection on (M, g) and $\widetilde{\nabla}$ the induced Levi-civita connection on $u^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame field on (M, g). $\tau(u)$ is called the tension field of u. A map $u : (M, g) \to (N, h)$ is called a harmonic map if $\tau(u) = 0$.

To generalize the notion of harmonic maps, in 1983 J. Eells and L. Lemaire [12] proposed considering the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$

In 1986, G. Y. Jiang [18] studied the first and second variation formulas of the bienergy E_2. The Euler-Lagrange equation of E_2 is

$$\tau_2(u) = -\widetilde{\triangle}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i))du(e_i) = 0,$$

where $\widetilde{\triangle} = \sum_i (\widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i} e_i})$ is the rough Laplacian on the section of $u^{-1}TN$ and $R^N(X, Y) = [{}^N\nabla_X, {}^N\nabla_Y] - {}^N\nabla_{[X,Y]}$ is the curvature operator on N. A map $u : (M, g) \to (N, h)$ is called a biharmonic map if $\tau_2(u) = 0$.

To generalize the notion of biharmoic maps, the author and S. X. Feng [15] introduced the F-bienergy functional

$$E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2}) dv_g,$$

where $F : [0, \infty) \to [0, \infty)$ is a C^3 function such that $F' > 0$ on $(0, \infty)$. The Euler-Lagrange equation of $E_{F,2}$ is

$$\tau_{F,2}(u) = -\widetilde{\triangle}(F'(\frac{|\tau(u)|^2}{2})\tau(u)) - \sum_i R^N(F'(\frac{|\tau(u)|^2}{2})\tau(u), du(e_i))du(e_i) = 0.$$

A map $u : (M, g) \to (N, h)$ is called a F-biharmonic map if $\tau_{F,2}(u) = 0$.

When $F(t)=e^t$, we have exponential bienergy functional

$$E_{e,2}(u)=\int_M e^{\frac{|\tau(u)|^2}{2}}dv_g.$$

The Euler-Lagrange equation of $E_{e,2}$ is

$$\tau_{e,2}(u)=-\widetilde{\triangle}(e^{\frac{|\tau(u)|^2}{2}}\tau(u))-\sum_i R^N(e^{\frac{|\tau(u)|^2}{2}}\tau(u),du(e_i))du(e_i)=0. \tag{5}$$

A map $u:(M,g)\to(N,h)$ is called an exponential biharmonic map if $\tau_{e,2}(u)=0$.

Now we introduce the definition of exponentially biharmonic submanifolds.

Let $u:(M,g)\to(N,h=\langle\cdot,\cdot\rangle)$ be an isometric immersion from an m-dimensional Riemannian manifold into an $m+t$-dimensional Riemannian manifold. We identify $du(X)$ with $X\in\Gamma(TM)$ for each $x\in M$. We also denote by $\langle\cdot,\cdot\rangle$ the induced metric $u^{-1}h$. The Gauss formula is given by

$${}^N\nabla_XY=\nabla_XY+B(X,Y),\quad X,Y\in\Gamma(TM),$$

where B is the second fundamental form of M in N. The Weingarten formula is give by

$${}^N\nabla_X\xi=-A_\xi X+\nabla^\perp_X\xi,\quad X\in\Gamma(TM),\ \xi\in\Gamma(T^\perp M),$$

where A_ξ is the shape operator for a unit normal vector field ξ on M, and $\nabla^\perp$ denotes the normal connection on the normal bundle of M in N. For any $x\in M$, the mean curvature vector field H of M at x is given by

$$H=\frac{1}{m}\sum_{i=1}^m B(e_i,e_i).$$

If an isometric $u:(M,g)\to(N,h)$ is exponentially biharmonic, then M is called an exponentially biharmonic submanifold in N. In this case, we remark that the tension field $\tau(u)$ of u is written $\tau(u)=mH$, where H is the mean curvature vector field of M. The necessary and sufficient condition for M in N to be exponentially biharmonic is the following:

$$-\widetilde{\triangle}(e^{\frac{m^2|H|^2}{2}}H)-\sum_i R^N(e^{\frac{m^2|H|^2}{2}}H,e_i)e_i=0. \tag{6}$$

From (6), we obtain the necessary and sufficient condition for M in N to be exponentially biharmonic as follows:

(7)

$$\triangle^\perp(e^{\frac{m^2|H|^2}{2}}H)-\sum_{i=1}^m B(e_i,A_{e^{\frac{m^2|H|^2}{2}}H}(e_i))+[\sum_{i=1}^m R^N(e^{\frac{m^2|H|^2}{2}}H,e_i)e_i]^\perp=0,$$

(8)

$$Tr_g(\nabla A_{e^{\frac{m^2|H|^2}{2}}H})+Tr_g[A_{\nabla^\perp(e^{\frac{m^2|H|^2}{2}}H)}(\cdot)]-[\sum_{i=1}^m R^N(e^{\frac{m^2|H|^2}{2}}H,e_i)e_i]^\top=0,$$

where $\triangle^{\perp} = \sum_{i=1}^{m}(\nabla^{\perp}_{e_i}\nabla^{\perp}_{e_i} - \nabla^{\perp}_{\nabla_{e_i}e_i})$ is the Laplace operator associated with the normal connection $\nabla^{\perp}$.

We also need the following lemma.

Lemma 2.1 (Gaffney, [13])**.** *Let (M, g) be a complete Riemannian manifold. If a C^1 a-form α satisfies that $\int_M |\alpha| dv_g < \infty$ and $\int_M (\delta\alpha) dv_g < \infty$, or equivalently, a C^1 vector X defined by $\alpha(Y) = \langle X, Y\rangle$ $(\forall Y \in \Gamma(TM))$ satisfies that $\int_M |X| dv_g < \infty$ and $\int_M div(X) dv_g < \infty$, then*

$$\int_M (-\delta\alpha) dv_g = \int_M div(X) dv_g = 0. \tag{9}$$

3. Exponentially biharmonic maps into non-positively curved manifolds

In this section, we obtain the following result.

Theorem 3.1. *Let $u : (M^m, g) \to (N^n, h)$ be an exponentially biharmonic map from a Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M e^{\frac{p|\tau(u)|^2}{2}} |\tau(u)|^p dv_g < \infty, \quad \int_M |\tau(u)|^2 dv_g < \infty \quad \text{and} \quad \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $Vol(M, g) = \infty$, and*

$$\int_M e^{\frac{p|\tau(u)|^2}{2}} |\tau(u)|^p dv_g < \infty,$$

then u is harmonic.

Proof. Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$\begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla\lambda| \leq \frac{C}{r}, & x \in M, \end{cases} \tag{10}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of (M, g). From (5), we have

$$\begin{aligned} &\int_M \langle -\widetilde{\triangle}(e^{\frac{|\tau(u)|^2}{2}}\tau(u)), \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g \\ &= \int_M \lambda^2 e^{\frac{p|\tau(u)|^2}{2}} |\tau(u)|^{p-2} \sum_{i=1}^{m} \langle R^N(\tau(u), du(e_i))du(e_i), \tau(u)\rangle dv_g \leq 0, \end{aligned} \tag{11}$$

since the sectional curvature of (N,h) is non-positive. From (11), we have

$$
\begin{aligned}
0 \geq & \int_M \langle -\widetilde{\triangle}(e^{\frac{|\tau(u)|^2}{2}}\tau(u)), \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g \\
= & \int_M \langle \widetilde{\nabla}(e^{\frac{|\tau(u)|^2}{2}}\tau(u)), \widetilde{\nabla}(\lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} e^{\frac{|\tau(u)|^2}{2}}\tau(u))\rangle dv_g \\
= & \int_M \sum_{i=1}^m \langle \widetilde{\nabla}_{e_i}(e^{\frac{|\tau(u)|^2}{2}}\tau(u)), \widetilde{\nabla}_{e_i}(\lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} e^{\frac{|\tau(u)|^2}{2}}\tau(u))\rangle dv_g \\
= & \int_M \sum_{i=1}^m [\langle \widetilde{\nabla}_{e_i}(e^{\frac{|\tau(u)|^2}{2}}\tau(u)), 2\lambda e_i(\lambda) |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} e^{\frac{|\tau(u)|^2}{2}}\tau(u) \\
& + \lambda^2 e_i(|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2}) e^{\frac{|\tau(u)|^2}{2}}\tau(u) \\
& + \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau|^{p-2} \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle] dv_g \\
= & \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g \\
& + \int_M \sum_{i=1}^m (p-2)\lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-4} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle^2 dv_g \\
& + \int_M \sum_{i=1}^m \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle dv_g \\
\geq & \int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g \\
& + \int_M \sum_{i=1}^m \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle dv_g, \qquad (12)
\end{aligned}
$$

where the inequality follows from

$$\lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-4} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle^2 \geq 0.$$

From (12), we have

$$
\begin{aligned}
& \int_M \sum_{i=1}^m \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle dv_g \\
\leq & -\int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g. \qquad (13)
\end{aligned}
$$

By using Young's inequality, we have

$$-\int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2} \langle \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle dv_g$$

$$\leq \frac{1}{2}\int_M \sum_{i=1}^m \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2}|\widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]|^2 dv_g$$

$$+ 2\int_M |\nabla\lambda|^2 e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g. \tag{14}$$

From (13) and (14), we have

$$\int_M \sum_{i=1}^m \lambda^2 |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{p-2}\langle\widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle dv_g$$

$$\leq 4\int_M |\nabla\lambda|^2 e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g$$

$$\leq \frac{4C^2}{r^2}\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g. \tag{15}$$

By assumption $\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g < \infty$, letting $r \to \infty$ in (15), we have

$$\int_M \sum_{i=1}^m e^{\frac{(p-2)|\tau(u)|^2}{2}}|\tau(u)|^{p-2}\langle\widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)], \widetilde{\nabla}_{e_i}[e^{\frac{|\tau(u)|^2}{2}}\tau(u)]\rangle dv_g = 0.$$

Therefore, we obtain that $e^{\frac{|\tau(u)|^2}{2}}|\tau(u)|$ is constant and $\widetilde{\nabla}_X[e^{\frac{|\tau(u)|^2}{2}}\tau(u)] = 0$, that is $\langle\widetilde{\nabla}_X\tau(u), \tau(u)\rangle\tau(u) + \widetilde{\nabla}_X\tau(u) = 0$ for any vector field X on M.

Therefore, if $Vol(M) = \infty$ and $|\tau(u)| \neq 0$, then

$$\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g = |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^p Vol(M) = \infty,$$

which yields a contradiction. Thus, we have $|\tau(u)| = 0$, i.e., u is harmonic. We have (ii).

For (i), assume $\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g < \infty$, $\int_M |\tau(u)|^2 dv_g < \infty$ and $\int_M |du|^2 dv_g < \infty$. Define a 1-from α on M defined by

$$\alpha(X) = |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}-1}\langle du(X), e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle \tag{16}$$

for any vector $X \in \Gamma(TM)$.

Note here that

$$\int_M |\alpha|^2 dv_g = \int_M [\sum_{i=1}^m |\alpha(e_i)|^2]^{\frac{1}{2}} dv_g$$

$$= \int_M [\sum_{i=1}^m [|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}-1}\langle du(e_i), e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle]^2]^{\frac{1}{2}} dv_g$$

$$\leq \int_M |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}}|du| dv_g$$

$$\leq [\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g]^{\frac{1}{2}}[\int_M |du|^2 dv_g]^{\frac{1}{2}} < \infty. \tag{17}$$

Now we compute

$$\begin{aligned}
-\delta\alpha &= \sum_{i=1}^{m}(\nabla_{e_i}\alpha)(e_i) = \sum_{i=1}^{m}[\nabla_{e_i}\alpha(e_i) - \alpha(\nabla_{e_i}e_i)] \\
&= \sum_{i=1}^{m}\nabla_{e_i}[|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}-1}\langle du(e_i), e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle] \\
&\quad - \sum_{i=1}^{m}|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}-1}\langle du(\nabla_{e_i}e_i), e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle \\
&= \sum_{i=1}^{m}|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}-1}\langle\widetilde{\nabla}_{e_i}du(e_i) - du(\nabla_{e_i}e_i), e^{\frac{|\tau(u)|^2}{2}}\tau(u)\rangle \\
&= |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}}|\tau(u)|,
\end{aligned}$$

where the fourth equality follows from that $|e^{\frac{|\tau(u)|^2}{2}}\tau(u)|$ is constant and $\widetilde{\nabla}_X[e^{\frac{|\tau(u)|^2}{2}}\tau(u)] = 0$, for $X \in \Gamma(TM)$. So we have

$$\begin{aligned}
\int_M [-\delta\alpha]dv_g &= \int_M |e^{\frac{|\tau(u)|^2}{2}}\tau(u)|^{\frac{p}{2}}|\tau(u)|dv_g \\
&\leq \left[\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g\right]^{\frac{1}{2}}\left[\int_M |\tau(u)|^2 dv_g\right]^{\frac{1}{2}}.
\end{aligned}$$

Since $\int_M e^{\frac{p|\tau(u)|^2}{2}}|\tau(u)|^p dv_g < \infty$ and $\int_M |\tau(u)|^2 dv_g < \infty$, the function $-\delta\alpha$ is also integrable over M.

From this and (17), we can apply Lemma 2.1 for the 1-form α. Therefore we have

$$0 = \int_M (-\delta\alpha)dv_g = \int_M e^{\frac{p|\tau(u)|^2}{4}}|\tau(u)|^{\frac{p}{2}+1}dv_g,$$

so we have $\tau(u) = 0$, that is, u is harmonic. □

4. Exponentially biharmonic submanifolds in nonpositive curvature forms

In this section, we obtain the following results:

Theorem 4.1. *Let $u : (M, g) \to (N, h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$.*

If

$$\int_M e^{\frac{pm^2|H|^2}{2}}|H|^q dv_g < \infty,$$

then u is minimal.

Proof. From the equation (7), we have

$$
\begin{aligned}
\triangle[e^{\frac{m^2|H|^2}{2}}|H|]^2 &= \triangle\langle e^{\frac{m^2|H|^2}{2}}H, e^{\frac{m^2|H|^2}{2}}H\rangle \\
&= 2\langle\triangle^{\perp}(e^{\frac{m^2|H|^2}{2}}H), e^{\frac{m^2|H|^2}{2}}H\rangle + 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 \\
&= 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2\sum_{i=1}^{m}\langle B(A_{e^{\frac{m^2|H|^2}{2}}H}e_i, e_i), e^{\frac{m^2|H|^2}{2}}H\rangle \\
&\quad - \sum_{i=1}^{m}\langle R^N(e^{\frac{m^2|H|^2}{2}}H, e_i)e_i, e^{\frac{m^2|H|^2}{2}}H\rangle \\
&\geq 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2\sum_{i=1}^{m}\langle B(A_{e^{\frac{m^2|H|^2}{2}}H}e_i, e_i), e^{\frac{m^2|H|^2}{2}}H\rangle, \qquad (18)
\end{aligned}
$$

where the inequality follows from the sectional curvature of (N, h) is non-positive. Now we state an inequality:

$$
\sum_{i=1}^{m}\langle B(A_{e^{\frac{m^2|H|^2}{2}}H}e_i, e_i), e^{\frac{m^2|H|^2}{2}}H\rangle \geq m[e^{\frac{m^2|H|^2}{2}}]^2|H|^4. \qquad (19)
$$

In fact, let $x \in M$, when $H(x) = 0$, we are done. If $H(x) \neq 0$, we have at x,

$$
\begin{aligned}
&\sum_{i=1}^{m}\langle B(A_{e^{\frac{m^2|H|^2}{2}}H}e_i, e_i), e^{\frac{m^2|H|^2}{2}}H\rangle \\
&= \sum_{i=1}^{m}[e^{\frac{m^2|H|^2}{2}}]^2|H|^2\langle B(A_{\frac{H}{|H|}}e_i, e_i), \frac{H}{|H|}\rangle \\
&= \sum_{i=1}^{m}[e^{\frac{m^2|H|^2}{2}}]^2|H|^2\langle A_{\frac{H}{|H|}}e_i, A_{\frac{H}{|H|}}e_i\rangle \\
&= \sum_{i,j=1}^{m}[e^{\frac{m^2|H|^2}{2}}]^2|H|^2|\langle B(e_i, e_j), \frac{H}{|H|}\rangle|^2 \\
&\geq m[e^{\frac{m^2|H|^2}{2}}]^2|H|^4.
\end{aligned}
$$

From (18) and (19), we have

$$
\triangle[e^{\frac{m^2|H|^2}{2}}|H|]^2 \geq 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2m[e^{\frac{m^2|H|^2}{2}}]^2|H|^4. \qquad (20)
$$

Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$
\begin{cases}
0 \leq \lambda(x) \leq 1, & x \in M, \\
\lambda(x) = 1, & x \in B_r(x_0), \\
\lambda(x) = 0, & x \in M - B_{2r}(x_0), \\
|\nabla\lambda| \leq \frac{C}{r}, & x \in M,
\end{cases} \qquad (21)
$$

where $B_r(x_0)=\{x\in M: d(x,x_0)<r\}$, C is a positive constant and d is the distance of (M,g). From (20), we have

$$\begin{aligned}
&-\int_M \nabla(\lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a)\nabla[e^{m^2|H|^2}|H|^2]dv_g\\
&=\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a\triangle[e^{m^2|H|^2}|H|^2]dv_g\\
&\geq 2\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&\quad+2m\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a e^{m^2|H|^2}|H|^4dv_g, \qquad (22)
\end{aligned}$$

where a is a positive constant to be determined later. On the other hand, we have

$$\begin{aligned}
&-\int_M \nabla(\lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a)\nabla[e^{m^2|H|^2}|H|^2]dv_g\\
&=-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|e^{\frac{m^2|H|^2}{2}}H|^a\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H],e^{\frac{m^2|H|^2}{2}}H\rangle dv_g\\
&\quad-2a\int_M \lambda^{a+4}|e^{\frac{m^2|H|^2}{2}}H|^{a-2}\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H],e^{\frac{m^2|H|^2}{2}}H\rangle^2dv_g\\
&\leq-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|e^{\frac{m^2|H|^2}{2}}H|^a\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H],e^{\frac{m^2|H|^2}{2}}H\rangle dv_g. \qquad (23)
\end{aligned}$$

From (22) and (23), we have

$$\begin{aligned}
&2\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&\quad+2m\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a e^{m^2|H|^2}|H|^4dv_g\\
&\leq-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|e^{\frac{m^2|H|^2}{2}}H|^a\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H],e^{\frac{m^2|H|^2}{2}}H\rangle dv_g \qquad (24)\\
&\leq\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&\quad+(a+4)^2\int_M \lambda^{a+2}e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}|\nabla\lambda|^2dv_g.
\end{aligned}$$

So we have

$$\begin{aligned}
&\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&\quad+2m\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{a+4}|H|^{a+4}dv_g\\
&\leq(a+4)^2\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{a+2}|H|^{a+2}|\nabla\lambda|^2dv_g. \qquad (25)
\end{aligned}$$

By using Young's inequalities, we have

$$\begin{aligned}
&(a+4)^2\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{a+2}|H|^{a+2}|\nabla\lambda|^2dv_g\\
&=(a+4)^2\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{s}|H|^{s}\lambda^{a+2-s}|H|^{a+2-s}|\nabla\lambda|^2dv_g\\
&\leq \int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{a+4}|H|^{a+4}dv_g\\
&\quad +C(a,s)\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|H|^{(a+2-s)\frac{a+4}{a+4-s}}|\nabla\lambda|^{2\frac{a+4}{a+4-s}}dv_g, \qquad (26)
\end{aligned}$$

where $s\in(0,a+2)$ and $C(a,s)$ is a constant depending on a,s. From (25) and (26), we have

$$\begin{aligned}
&\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&\quad+(2m-1)\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{a+4}|H|^{a+4}dv_g\\
&\leq C(a,s)\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|H|^{(a+2-s)\frac{a+4}{a+4-s}}|\nabla\lambda|^{2\frac{a+4}{a+4-s}}dv_g\\
&\leq C(a,s)(\frac{C}{r})^{2\frac{a+4}{a+4-s}}\int_M e^{\frac{m^2(a+2)|H|^2}{2}}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|H|^{(a+2-s)\frac{a+4}{a+4-s}}dv_g. \qquad (27)
\end{aligned}$$

Note that when s varies from 0 to $a+2$, we know that $(a+2-s)\frac{a+4}{a+4-s}$ varies from $a+2$ to 0. Set $q=(a+2-s)\frac{a+4}{a+4-s}$, we have $q\in(0,a+2)$. Set $p=a+2$. By the assumption $\int_M e^{\frac{m^2p|H|^2}{2}}|H|^q dv_g<\infty$ $(2\leq p<\infty,\ 0<q\leq p<\infty)$, letting $r\to\infty$ in (27), we have

$$\int_M e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g+(2m-1)\int_M e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+4}dv_g=0.$$

Thus, we have $H=0$. □

Theorem 4.2. *Let $u:(M,g)\to(N,h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N,h) with non-positive sectional curvature. If*

$$\int_{B_r(x_0)} e^{\frac{pm^2|H|^2}{2}}dv_g\leq C_0(1+r)^s \qquad (28)$$

for some positive integer s, C_0 independent of r and $p\geq 2$, then u is minimal.

Proof. From the equation (24), we have

$$\begin{aligned}
&2\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2dv_g\\
&+2m\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a e^{m^2|H|^2}|H|^4dv_g
\end{aligned}$$

$$\leq \; -2(a+4)\int_M \lambda^{a+3}\nabla\lambda|e^{\frac{m^2|H|^2}{2}}H|^a\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H], e^{\frac{m^2|H|^2}{2}}H\rangle dv_g. \tag{29}$$

By using Young's inequalities, we have

(30)

$$\begin{aligned}
&-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|e^{\frac{m^2|H|^2}{2}}H|^a\langle\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H], e^{\frac{m^2|H|^2}{2}}H\rangle dv_g\\
\leq &\int_M \lambda^{a+4}|e^{\frac{m^2|H|^2}{2}}H|^a|\nabla^\perp[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g + \int_M e^{\frac{(a+2)m^2|H|^2}{2}}\lambda^{a+4}|H|^{a+4}dv_g\\
&+C(a)\int_M e^{\frac{(a+2)m^2|H|^2}{2}}|\nabla\lambda|^{a+4}dv_g,
\end{aligned}$$

where $C(a)$ is a constant depending on a.

From (29) and (30), we have

$$\begin{aligned}
&\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^\perp[e^{\frac{m^2|H^2|}{2}}H]|^2 dv_g\\
&+(2m-1)\int_M \lambda^{a+4}e^{\frac{m^2a|H|^2}{2}}|H|^a e^{m^2|H|^2}|H|^4 dv_g\\
\leq\; & C(a)\int_M e^{\frac{(a+2)m^2|H|^2}{2}}|\nabla\lambda|^{a+4}dv_g\\
\leq\; & C(a)C^{a+4}\frac{1}{r^{a+4}}\int_{B_{2r}(x_0)} e^{\frac{(a+2)m^2|H|^2}{2}}dv_g\\
\leq\; & C(a)C^{a+4}C_0\frac{(1+2r)^s}{r^{a+4}}. \qquad (31)
\end{aligned}$$

We finish the proof by letting a be big enough and $r\to\infty$. □

Theorem 4.3. *Let $u:(M,g)\to(N,h)$ be an exponentially biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N,h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon>0$ and $\int_{B_r(x_0)} e^{\frac{pm^2|H|^2}{2}}|H|^p dv_g$ $(p\geq 2)$ is of at most polynomial growth of r. Then u is minimal.*

Proof. From the equation (7), we have

$$\begin{aligned}
\triangle[e^{\frac{m^2|H|^2}{2}}|H|]^2 &= \triangle\langle e^{\frac{m^2|H|^2}{2}}H, e^{\frac{m^2|H|^2}{2}}H\rangle\\
&= 2\langle\triangle^\perp(e^{\frac{m^2|H|^2}{2}}H), e^{\frac{m^2|H|^2}{2}}H\rangle + 2|\nabla^\perp(e^{\frac{m^2|H|^2}{2}}H)|^2\\
&= 2|\nabla^\perp(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2\sum_{i=1}^m\langle B(A_{e^{\frac{m^2|H|^2}{2}}H}e_i,e_i), e^{\frac{m^2|H|^2}{2}}H\rangle\\
&\quad -\sum_{i=1}^m\langle R^N(e^{\frac{m^2|H|^2}{2}}H,e_i)e_i, e^{\frac{m^2|H|^2}{2}}H\rangle\\
&\geq 2|\nabla^\perp(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2me^{m^2|H|^2}|H|^4 + 2m\varepsilon e^{m^2|H|^2}|H|^2
\end{aligned}$$

$$\geq 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2m\varepsilon e^{m^2|H|^2}|H|^2,$$

that is,

$$\triangle[e^{\frac{m^2|H|^2}{2}}|H|]^2 \geq 2|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 + 2m\varepsilon e^{m^2|H|^2}|H|^2. \tag{32}$$

From (32), we have

$$\begin{aligned}
&-\int_M \nabla[\lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a]\nabla[e^{m^2|H|^2}|H|^2]dv_g \\
&= \int_M [\lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a]\triangle[e^{m^2|H|^2}|H|^2]dv_g \\
&\geq 2\int_M [\lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a]|\nabla^{\perp}(e^{\frac{m^2|H|^2}{2}}H)|^2 dv_g \\
&\quad + 2m\varepsilon \int_M \lambda^2 e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g,
\end{aligned} \tag{33}$$

where a is a nonnegative constant and λ is given by (21). On the other hand, we have

$$\begin{aligned}
&-\int_M \nabla[\lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a]\nabla[e^{m^2|H|^2}|H|^2]dv_g \\
=& -4\int_M \lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g \\
&-2a\int_M \lambda^2 e^{\frac{m^2(a-2)|H|^2}{2}}|H|^{a-2}\langle\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle^2 dv_g \\
\leq& -4\int_M \lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g \\
\leq& 2\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g \\
&+2\int_M e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}|\nabla\lambda|^2 dv_g \\
\leq& 2\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g \\
&+\frac{2C^2}{r^2}\int_{B_{2r}(x_0)-B_r(x_0)} e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g \\
\leq& 2\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla^{\perp}[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g \\
&+\frac{2C^2}{r^2}\int_{B_{2r}(x_0)} e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g.
\end{aligned} \tag{34}$$

From (33) and (34), we have

$$2m\varepsilon\int_{B_r(x_0)} e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g \leq \frac{2C^2}{r^2}\int_{B_{2r}(x_0)} e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g.$$

Set $f(r)=\int_{B_r(x_0)} e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g$, we have

$$f(r)\leq \frac{C_1}{r^2}f(2r),$$

where $C_1=\frac{C^2}{m\varepsilon}$. This implies that $f(r)\leq \frac{C_2}{r^{2n}}f(2^nr)$, where C_2 is a constant independent of r. By assumption, we have $f(r)\leq C_2(1+2^{ns}r^s)$ for some positive integer s, as r is big enough, hence $f(r)\leq \frac{C_2^2(1+2^{ns}r^s)}{\rho^{2n}}$. Let $2n>s$, we have $\lim_{r\to\infty} f(r)=0$. Therefore $H=0$. □

Theorem 4.4. *Let $u:(M,g)\to(N,h)$ be a complete ε-super exponentially biharmonic submanifold in N for $\varepsilon>0$. If*

$$\int_M e^{\frac{pm^2|H|^2}{2}}|H|^p dv_g<\infty, \tag{35}$$

then u is minimal, where $p\geq 2$.

Proof. From (3), we have

$$\begin{aligned}
&(\varepsilon-1)\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2dv_g\\
\leq&\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\triangle[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g\\
=&-\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2dv_g\\
&-\int_M 2\lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g\\
&-a\int_M \lambda^2 e^{\frac{m^2(a-2)|H|^2}{2}}|H|^{a-2}\langle\nabla[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle^2dv_g\\
\leq&-\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2dv_g\\
&-\int_M 2\lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g,
\end{aligned}$$

where λ is given by (21) and $a\geq 0$. So we have

$$\begin{aligned}
&\varepsilon\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2dv_g\\
\leq&-\int_M 2\lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g.
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
&\varepsilon\int_M \lambda^2 e^{\frac{m^2a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2dv_g\\
\leq&-\int_M 2\lambda\nabla\lambda e^{\frac{m^2a|H|^2}{2}}|H|^a\langle\nabla[e^{\frac{m^2|H|^2}{2}}H],[e^{\frac{m^2|H|^2}{2}}H]\rangle dv_g
\end{aligned}$$

$$\leq \frac{\varepsilon}{2}\int_M \lambda^2 e^{\frac{m^2 a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g$$
$$+\frac{2}{\varepsilon}\int_M e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}|\nabla\lambda|^2 dv_g.$$

So we have

$$\int_M \lambda^2 e^{\frac{m^2 a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g$$
$$\leq \frac{4}{\varepsilon^2}\frac{C^2}{r^2}\int_M e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g. \tag{36}$$

Since $\int_M e^{\frac{m^2(a+2)|H|^2}{2}}|H|^{a+2}dv_g$ is finite, let $r\to\infty$ in (36), we have

$$\int_M e^{\frac{m^2 a|H|^2}{2}}|H|^a|\nabla[e^{\frac{m^2|H|^2}{2}}H]|^2 dv_g \leq 0 \tag{37}$$

and hence $H=0$ or $\nabla[e^{\frac{m^2|H|^2}{2}}H]=0$.

In the following, we will show that $\nabla[e^{\frac{m^2|H|^2}{2}}H]=0$ implies $H=0$.

Now let $x\in M$ such that $\nabla[e^{\frac{m^2|H|^2}{2}}H]=0$. We choose an orthonormal basis $\{e_i\}_{i=1}^m$ of T_xM and an orthonormal basis $\{\upsilon_\alpha\}_{\alpha=1}^t$ of $(T_xM)^\perp$. We have

$$0=\langle\nabla_{e_i}[e^{\frac{m^2|H|^2}{2}}H],e_j\rangle=-\langle[e^{\frac{m^2|H|^2}{2}}H],B(e_i,e_j)\rangle. \tag{38}$$

From (38), we have

$$0=\sum_{i=1}^m\langle[e^{\frac{m^2|H|^2}{2}}H],B(e_i,e_i)\rangle=me^{\frac{m^2|H|^2}{2}}|H|^2,$$

so we have $H=0$. □

Theorem 4.5. *Let $u:(M^m,g)\to(N^{m+1}(c),\langle,\rangle)$ be a weakly convex exponentially biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c\leq 0$. Then u is minimal.*

Proof. Assume that $H=h\nu$, where ν is the unit normal vector field on M. Since M is weakly convex, we have $h\geq 0$. Set $C=\{q\in M: h(q)>0\}$. We will prove that C is an empty set.

If C is not empty, we see that C is an open subset of M. We assume that C_1 is a nonempty connect component of C. We will prove that $h\equiv 0$ in C_1, thus a contradiction.

Firstly, we prove that h is a constant in C_1.

Let $q\in C_1$ be a point. Choose a local orthonormal frame $\{e_i, i=1,\dots,m\}$ around q such that $\langle B,\nu\rangle$ is a diagonal matrix and $\nabla_{e_i}e_j|_q=0$.

From equation (8), we have at q

$$0=\langle\sum_{i=1}^m(\nabla_{e_i}A_{(e^{\frac{m^2h^2}{2}}H)})(e_i),e_k\rangle+\langle\sum_{i=1}^m A_{\nabla^\perp_{e_i}(e^{\frac{m^2h^2}{2}}H)}(e_i),e_k\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} e_i \langle A_{(e^{\frac{m^2h^2}{2}}H)}(e_i), e_k\rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(e^{\frac{m^2h^2}{2}}H)\rangle \\
&= \sum_{i=1}^{m} e_i \langle e^{\frac{m^2h^2}{2}}H, B(e_i, e_k)\rangle + \sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(e^{\frac{m^2h^2}{2}}H)\rangle \\
&= \sum_{i=1}^{m} \langle e^{\frac{m^2h^2}{2}}H, \nabla^{\perp}_{e_i}B(e_i, e_k)\rangle + 2\sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(e^{\frac{m^2h^2}{2}}H)\rangle \\
&= \sum_{i=1}^{m} \langle e^{\frac{m^2h^2}{2}}H, \nabla^{\perp}_{e_k}B(e_i, e_i)\rangle + 2\sum_{i=1}^{m} \langle B(e_i, e_k), \nabla^{\perp}_{e_i}(e^{\frac{m^2h^2}{2}}H)\rangle \\
&= m\langle e^{\frac{m^2h^2}{2}}H, \nabla^{\perp}_{e_k}H\rangle + 2\langle \lambda_k \nu, \nabla^{\perp}_{e_i}(e^{\frac{m^2h^2}{2}}H)\rangle \\
&= me^{\frac{m^2h^2}{2}}he_k(h) + e^{\frac{m^2h^2}{2}}2(m^2h^2+1)\lambda_k e_k(h) \\
&= (mh + 2\lambda_k + 2m^2h^2\lambda_k)e^{\frac{m^2h^2}{2}}e_k(h),
\end{aligned}
$$

where λ_k is the kth principle curvature of M at q, which is nonnegative by the assumption that M is weakly convex. Since $(mh + 2\lambda_k + 2m^2h^2\lambda_k)e^{\frac{m^2h^2}{2}} > 0$ at q, we have $e_k(h) = 0$ at q, for $k = 1, \ldots, m$, which implies that $\nabla h = 0$ at q. Because q is an arbitrary point in C_1, we have $\nabla h = 0$ in C_1. Therefore we obtain that h is constant in C_1.

Secondly, we prove that h is zero in C_1.

From (20), we have

$$
\triangle[e^{\frac{m^2h^2}{2}}h]^2 \geq 2m[e^{\frac{m^2h^2}{2}}]^2h^4. \tag{39}
$$

From equation (39), we have in C_1

$$
0 = \triangle[e^{\frac{m^2h^2}{2}}h]^2 \geq 2m[e^{\frac{m^2h^2}{2}}]^2h^4.
$$

We know that $h \equiv 0$ in C_1. This is a contradiction. □

Acknowledgements. This work was supported by the National Natural Science Foundation of China (Grant No.11201400), Basic and Frontier Technology Research Project of Henan Province (Grant No.142300410433), Project for youth teacher of Xinyang Normal University (Grant No.2014-QN-061), Nanhu Scholars Program for Young Scholars of XYNU.

References

[1] M. Ara, *Geometry of F-harmonic maps*, Kodai Math. J. **22** (1999), no. 2, 243–263.

[2] ———, *Instability and nonexistence theorems for F-harmonic maps*, Illinois J. Math. **45** (2001), no. 2, 657–679.

[3] ———, *Stability of F-harmonic maps into pinched manifolds*, Hiroshima Math. J. **31** (2001), no. 1, 171–181.

[4] A. Balmus, S. Montaldo, and C. Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space form*, Math. Nachr. **283** (2010), no. 12, 1696–1705.

[5] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. **130** (2002), 109–123.

[6] R. Caddeo, S. Montaldo, and P. Piu, *On biharmonic maps*, Contemp. Math. **288** (2001), 286–290.

[7] X. Z. Cai and Y. Luo, *On p-biharmonic submanifolds in nonpositively curved manifolds*, preprint.

[8] B. Y. Chen, *Some open problems and conjectures on submanifolds of finite type*, Michigan State University, 1988.

[9] L. F. Cheung and P. F. Leung, *Some results on stable p-harmonic maps*, Glasg. Math. J. **36** (1994), no. 1, 77–80.

[10] Y. J. Chiang, *Developments of Harmonic Maps, Wave Maps and Yang-Mills Fields into Biharmonic Maps, Biwave Maps and Bi-Yang-Mills Fields*, Frontiers in Mathematics, 2013.

[11] S. Dragomir and G. Tomassini, *Differential Geometry and Analysis on CR manifolds*, Progress in Mahematics, 246, Birkhäuser, 2006.

[12] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, 50. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1983.

[13] M. P. Gaffney, *A special Stoke's theorem for complete Riemannian manifold*, Ann. of Math. **60** (1954), 140–145.

[14] Y. B. Han, *Some results of p-biharmonic submanifolds in a Riemannian manifold of non-positive curvature*, J. Geometry **106** (2015), no. 3, 471–482.

[15] Y. B. Han and S. X. Feng, *Some results of F-biharmonic maps*, Acta Math. Univ. Comenian. **83** (2014), no. 1, 47–66.

[16] Y. B. Han and W. Zhang, *Some results of p-biharmonic maps into a non-positively curved manifold*, J. Korean Math. Soc. **52** (2015), no. 5, 1097–1108.

[17] P. Hornung and R. Moser, *Intrinsically p-biharmonic maps*, preprint (Opus: University of Bath online publication store).

[18] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Note Mat. **28** (2009), 209–232.

[19] J. C. Liu, *Nonexistence of stable exponentially harmonic maps from or into compact convex hypersurfaces in* $\mathbb{R}^{m+1}$. , Turk. J. Math. **32** (2008), no. 2, 117–126.

[20] E. Loubeau, S. Montaldo, and C. Oniciuc, *The stress-energy tensor for biharmonic maps*, Math. Z. **259** (2008), no. 3, 503–524.

[21] E. Loubeau and C. Oniciuc, *The index of biharmonic maps in spheres*, Compos. Math. **141** (2005), no. 3, 729–745.

[22] Y. Luo, *Weakly convex biharmonic hypersurfaces in nonpositive curvature space forms are minimal*, Results Math. **65** (2014), no. 1-2, 49–56.

[23] ______, *On biharmonic submanifolds in non-positively curved manifolds*, J. Geom. Phys. **88** (2015), 76–87.

[24] ______, *The maximal principle for properly immersed submanifolds and its applications*, Geom. Dedicata, DOI 10.1007/s10711-015-0114-4, (2015).

[25] S. Maeta, *Biharmonic maps from a complete Riemannian manifold into a non-positively curved manifold*, Ann. Global Anal. Geom. **46** (2014), no. 1, 75–85.

[26] N. Nakauchi, H. Urakawa, and S. Gudmundsson, *Biharmonic submanifolds in a Riemannian manifold with non-positive curvature*, Results Math. **63** (2013), no. 1-2, 467–474.

[27] C. Oniciuc, *Biharmonic maps between Riemannian manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 48 (2002), no. 2, 237–248.

[28] Y. L. Ou, *ff-harmonic morphisms between Riemannian manifolds*, Chin. Ann. Math. Ser. B **35** (2014), no. 2, 225–236.

[29] G. Wheeler, *Chen's conjecture and ε-superharmonic submanifolds of Riemannian manifolds*, Internat. J. Math. **24** (2013), no. 4, 1350028, 6 pp.

Yingbo Han
College of Mathematics and Information Science
Xinyang Normal University
Xinyang,464000, Henan, P. R. China
E-mail address: yingbohan@163.com