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A CHARACTERIZATION OF L-FUNCTIONS IN THE EXTENDED SELBERG CLASS

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Abstract. In this article, we establish a characterization of meromorphic functions and L-functions in the extended Selberg class, which shows that how they are uniquely determined by their c-values.

1. Introduction

L-functions in the Selberg class, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively (see [10]). Value distribution of L-functions concerns distribution of the zeros of L-functions L and more generally, the c-values of L, that is, the roots of the equation $L(s) = c$, or the values in the pre-image

$$L^{-1}(c) = \{s \in \mathbb{C} : L(s) = c\},$$

where and in what follows, s and z denote the complex variables and c denotes a complex value. L-functions can be analytically continued as meromorphic functions. Two meromorphic functions f and g in the complex plane are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicities) if $f^{-1}(c) = g^{-1}(c)$ as two sets in $\mathbb{C}$. Moreover, f and g are said to share a value c CM (counting multiplicities) if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities.

This article concerns the question of how an L-function is uniquely determined in terms of the pre-images of complex values, or sharing values. We refer the reader to the monograph [10] for a detailed discussion on this topic and related works. L-functions in the Selberg class include the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichlet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of $s = \sigma + it$ with $a(1) = 1$, satisfying the following axioms:

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(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;

(ii) Analytic continuation: There is a non-negative integer m such that $(s-1)^m L(s)$ is an entire function of finite order;

(iii) Functional equation: L satisfies a functional equation of type

$$\Lambda_L(s) = \omega\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j, and complex numbers ν_j, ω with $\mathrm{Re}\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product: $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

At the same time, there are a whole host of interesting Dirichlet series not possessing a Euler product (see [4] and [10]). Throughout the paper, all L-functions are assumed to be functions from the extended Selberg class of those only satisfying the axioms (i)-(iii) (see [4]). Thus, the results obtained in the present paper particularly apply to L-functions in the Selberg class.

It is well-know that a non-constant meromorphic function in $\mathbb{C}$ is completely determined by five pre-images (see [3]). Since L-functions can be analytically continued as meromorphic functions, we trivially note that two L-functions share the value ∞ CM if and only if they both are entire or if they both have a pole at $s = 1$ of the same order, for example, the Riemann zeta function ζ and a Dedekind zeta function. The uniqueness of two L-functions were firstly studied by Steuding (see [10]), as seen from the following result:

Theorem 1.1. *If two L-functions L_1 and L_2 with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then $L_1 \equiv L_2$.*

In [13], the authors show that Theorem 1.1 is actually false when $c = 1$, an example given there is $L_1 = 1 + \frac{2}{4^s}$ and $L_2 = 1 + \frac{3}{9^s}$.

In order to study how an L-function is uniquely determined by pre-images of complex values in the extended plane, one should examine the situation involving an arbitrary L-function and an arbitrary meromorphic function. As the function ζ and ζe^g, where g is any entire function, share 0 CM, thus the above theorem no longer holds for an L-function and a meromorphic function. It is nature to consider whether two sharing values with counting multiplicities would force an L-function and a meromorphic function to be identically equal. This turns out not to be the case either. For instance, the function $f = \frac{2\zeta}{\zeta+1}$ and ζ share 0, 1 CM, but they are not identically equal. However, considering L-functions have only one possible pole at $s = 1$, one can consider the natural objects-those meromorphic functions with finitely many poles, the following theorem was then established by Li (see [6]):

Theorem 1.2. *Let a, $b \in \mathbb{C}$ be two distinct values and let f be a meromorphic function in $\mathbb{C}$ with finitely many poles. If f and a nonconstant L-function L share a CM and b IM, then $L \equiv f$.*

2. Main results

2.1. A characterization of L-functions and meromorphic functions

This article concerns the question of how meromorphic functions and L-functions are determined by their c-values, and we expect that L-functions share fewer values with a meromorphic function. More specifically, we have the following results, which are of their own interests:

Theorem 2.1. *Let f be an entire function of finite order with $\lim_{\Re(s)\to+\infty} f(s) = k$ (k cannot be ∞). Then f and an L-function L share value b ($b \neq 1$ and $b \neq k$) CM if and only if*

$$f = \frac{b-k}{b-1}(L-b)+b.$$

More generally, for meromorphic functions with finitely many poles, we have the following:

Theorem 2.2. *Let f be a meromorphic function of finite order with finitely many poles and $\lim_{\Re(s)\to+\infty} f(s) = k$ (k cannot be ∞). If f and an L-function L share value b ($b \neq 1$ and $b \neq k$) CM, then*

$$f = \frac{b-k}{b-1}(L-b)+b$$

or

$$f = \frac{b-k}{b-1}h(s)(L-b)+b,$$

where $h(s) = \frac{(s-1)^m}{\prod_{i=1}^{m}(s-b_i)}$, and b_i ($i = 1, 2, \ldots, m$) are poles of f (b_i may be equal to b_j for $i \neq j$), m is the non-negative integer in the axiom (ii)*.*

Remark 2.3. Theorem 2.1 shows that L and f are identically equal if $k = 1$. In Theorem 2.2, if f has only one possible pole at $s = 1$ (that is $h(s) \equiv 1$), then $f \equiv \frac{b-k}{b-1}(L-b)+b$. What is more, if one of the f and L is an entire function, it follows from the proof of Theorem 2.2, then the other one is also an entire function, and so $f \equiv \frac{b-k}{b-1}(L-b)+b$.

Remark 2.4. As an application of the theorems, the results above are valid for Riemann zeta function ζ and other related functions, for instance, $\sin z$, or equivalently, $\cos z$ ($= \sin(z+\frac{\pi}{2})$). That is, similar to the proof of Theorem 2.1, we have: f is an entire function of finite order with $f(0) = k$, then $f \equiv (1-\frac{k}{a})\sin z + k$ if f and $\sin z$ share a ($\neq 0$) CM for $a \neq k$.

Remark 2.5. The conclusion in Theorem 2.2 need not hold when f and L share value b ($\neq 1$) IM. In fact, ζ and ζ^2 have same zeros (ignoring multiplicities), but the conclusion does not hold. On the other hand, let $f = Le^{-s}$, then f and L satisfy all the conditions of the theorem except the assumption "$b \neq k$", but the conclusion does not hold. Thus the theorem is best possible in these senses.

3. Proof of theorems

Firstly, recall the following known results:

(i) The order of products of meromorphic functions (see [12]):

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}.$$

(ii) Hadamard's Factorization Theorem (see [1]): let f be an entire function of finite order p, then

$$f(z) = z^m e^{h(z)} \prod_{n \geq 1} (1 - \frac{z}{z_n}) e^{(z + \frac{z^2}{2} + \cdots + \frac{z^q}{q})},$$

where $h(z)$ is a polynomial of degree less than p, the $\{z_n\}_{n=1}^{\infty}$ from the family of zeros of f distinct from $z = 0$.

(iii) The order of L-functions is of not greater than 1 (see [10]).

3.1. Proof of theorems

Proof of Theorem 2.2. Set

$$L_1 = \frac{b-k}{b-1}(L - b) + b.$$

We first introduce the auxiliary function

$$G = \frac{Q(L_1 - b)}{f - b},$$

where Q is a rational function such that G has neither a pole nor a zero. We say that such a Q does exist since f has only finitely many poles and L has only one possible pole at $s = 1$, and thus L_1 also has only one possible pole at $s = 1$, and a possible zero or pole of $\frac{L_1 - b}{f - b}$ may only come from a pole of L_1 or f. It is easy to see that L and L_1 share value b CM for $k \neq b$, in view of the assumption that f and L share value b CM, which implies that f and L_1 share value b CM. Thus, G is an entire function without any zeros.

Next, we will deal with the rational function Q. In view of the discussion about Q above, which implies that Q must be of the form:

$$Q(s) = \frac{A(s-1)^m}{\prod_{i=1}^{n}(s - b_i)}, \tag{1}$$

where A ($\neq 0$) is a finite complex number, m is the non-negative integer in the axiom (ii) of the definition of L-functions; b_1, b_2, …, b_n are poles of f (b_i may be equal to b_j for $i \neq j$), thus $\rho(Q) < 1$. Since L and f is of finite order, the function G is also of finite order. By the Hadamard factorization theorem we have that

$$G = \frac{Q(L_1 - b)}{f - b} = e^{p(s)}, \tag{2}$$

where $p(s)$ is a polynomial. We may write

$$\Re p(\sigma + it) = a_n(t)\sigma^n + a_{n-1}(t)\sigma^{n-1} + \cdots + a_0(t)$$

a polynomial in σ with $a_n(t), \ldots, a_0(t)$ being polynomials in t. We claim that $a_n(t) \equiv 0$ when $n \geq 1$. To this end, (2) means that

$$\frac{L_1 - b}{f - b} = e^{p(s)} Q^{-1}. \tag{3}$$

Since $L(s) \to 1$ as $\sigma \to +\infty$, thus $L_1(s) \to k$ as $\sigma \to +\infty$. Noting that $\lim_{\sigma \to +\infty} f(s) = k$ (k cannot be ∞) and $k \neq b$. Hence the left-hand side of (3) is equal to 1 as $\sigma \to +\infty$, that is,

$$\lim_{\sigma \to +\infty} \frac{L_1 - b}{f - b} = 1. \tag{4}$$

On the other hand, if $a_n(t) \not\equiv 0$, without loss of generality, say $a_n(t_0) > 0$ for some value t_0. Noting that

$$\mid \frac{L_1 - b}{f - b} \mid = \mid Q^{-1} \mid e^{\Re p(\sigma + it)},$$

thus $1 = \infty$ when $\sigma \to +\infty$ with $t = t_0$, this is a contradiction. Hence, $a_n(t) \equiv 0$ ($n \geq 1$). It follows from (3) and (4),

$$\lim_{\sigma \to +\infty} Q = e^{a_0(t)}. \tag{5}$$

Noting that the limit of Q as $\sigma \to +\infty$ is a non-vanishing finite constant for some value t and the characterization of rational function Q in (1), we thus obtain that $\lim_{\sigma \to +\infty} Q = A$, thus $m = n$. When $m = n = 0$, that is $Q \equiv A$, this and (5) imply that $e^{a_0(t)} = A$. Hence, it follows from (3), we get

$$\frac{L_1 - b}{f - b} = \frac{1}{A} \cdot A = 1,$$

which implies that $f \equiv L_1$. That is,

$$f = \frac{b - k}{b - 1}(L - b) + b.$$

When $m = n \neq 0$, $Q(s) = \frac{A(s-1)^m}{\prod_{i=1}^{m}(s - b_i)}$, thus (5) implies that $e^{a_0(t)} = A$, it follows from (3) again, we have

$$\frac{L_1 - b}{f - b} = A \cdot \frac{\prod_{i=1}^{m}(s - b_i)}{A(s - 1)^m},$$

then it is easy to check that

$$f = \frac{b - k}{b - 1} h(s)(L - b) + b,$$

where $h(s) = \frac{(s-1)^m}{\prod_{i=1}^{m}(s-b_i)}$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. The sufficiency is clear. For the necessity, it is similar to the proof of Theorem 2.2, but the rational function Q here is $Q(s) = A(s-1)^m$ in (1) for $A \neq 0$. By similar proof, we can also get that $\lim_{\sigma\to+\infty} Q = A$, that is $m = 0$, and thus the equation

$$f = \frac{b-k}{b-1}(L-b) + b$$

also holds. This completes the proof. □

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