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UNIT-DUO RINGS AND RELATED GRAPHS OF ZERO DIVISORS

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Abstract. Let R be a ring with identity, X be the set of all nonzero, nonunits of R and G be the group of all units of R. A ring R is called *unit-duo ring* if $[x]_\ell = [x]_r$ for all $x \in X$ where $[x]_\ell = \{ux \mid u \in G\}$ (resp. $[x]_r = \{xu \mid u \in G\}$) which are equivalence classes on X. It is shown that for a semisimple unit-duo ring R (for example, a strongly regular ring), there exist a finite number of equivalence classes on X if and only if R is artinian. By considering the zero divisor graph (denoted $\widetilde{\Gamma}(R)$) determined by equivalence classes of zero divisors of a unit-duo ring R, it is shown that for a unit-duo ring R such that $\widetilde{\Gamma}(R)$ is a finite graph, R is local if and only if $\mathrm{diam}(\widetilde{\Gamma}(R)) = 2$.

1. Introduction and basic definitions

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring. $X(R)$ (or simply, X) denotes the set of all nonzero nonunits in R, and $G(R)$ (or simply, G) denotes the group of all units in R. Let $J(R)$ (or simply, J) denote the Jacobson radical of R. $|S|$ denotes the cardinality of any set S. $GF(p^n)$ denotes the Galois field of order p^n.

Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$) and use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. Following the literature, we write $D_n(R) = \{(a_{ij}) \in U_n(R) \mid$ all diagonal entries are equal$\}$ and $V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{1k} = a_{2(k+1)} = \cdots = a_{hn}$ for $h = 1, 2, \ldots, n-1$ and $k = 2, \ldots, n\}$. Note $V_n(R) \cong \frac{R[x]}{x^n R[x]}$, so $V_n(R)$ is commutative if so is R.

For $x, y \in R$, we say that $x \sim_\ell y$ (resp. $x \sim_r y$) if and only if $y = ux$ (resp. $y = xu$) for some unit $u \in R$. Then both $\sim_\ell$ (resp. $\sim_r$) is an equivalence relation on R. Let $[x]_\ell$ (resp. $[x]_r$) be the equivalence class containing $x \in R$ under $\sim_\ell$ (resp. $\sim_r$). A ring R is called *unit-duo* if $[x]_\ell = [x]_r$ for all $x \in R$ (refer [8]). Any commutative ring and a finite direct product of division rings

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are unit-duo rings. Note that a ring R in which X is not an empty set is a unit-duo ring if and only if $[x]_\ell = [x]_r$ for all $x \in X$. Indeed, $[u]_\ell = [u]_r = G$ for all units $u \in R$, and $[0]_\ell = [0]_r = \{0\}$. For a unit-duo ring R, we denote $[x]_\ell = [x]_r$ by $[x]$ for each $x \in R$ and denote $\sim_\ell$ $(=\sim_r)$ by $\sim$.

In Section 2, we will show that (i) if R is a right (resp. left) unit-duo ring such that $X(R)$ is a finite union of equivalence classes under $\sim_\ell$ (resp. $\sim_r$), then every $x \in X(R)$ is a two-sided zero divisor; (ii) for a field F, there are just only two types of unit-duo subrings of $\mathrm{Mat}_n(F)$ $(n \geq 2)$ up to isomorphism, say, the ring of all diagonal matrices in $\mathrm{Mat}_n(R)$ and $V_n(F)$; (iii) for a strong regular ring R (which is unit-duo), the followings are equivalent:

(1) There exist a finite number of idempotents of R;
(2) There exist a finite number of equivalence classes under relation $\sim$;
(3) R is artinian;
(4) $R \simeq D_1 \times D_2 \times \cdots \times D_t$ for some positive integer t.

On the other hand, the zero divisor graph of a commutative ring has been studied extensively by Akbari, Mohammadian [1], Anderson, Livingston [2] since its concept had been introduced by Beck in [3]. The zero-divisor graph of a noncommutative ring (resp. a semigroup) has studied by Redmond and Wu (resp., F. DeMeyer and L. Demeyer) in [12, 13, 15] (resp., [5]). Zero divisor graph is very useful to find the algebraic structures and properties of rings. Recently, the graph of equivalence classes of zero divisors of a commutative Noetherian ring was studied by Spiroff and Wickham in [14]. Note that if R is a unit-duo ring, then for all $x, y \in R$, $[xy] = [x][y]$, i.e., multiplication is well defined on $\{[x] \mid x \in R\}$. For a ring R, let $Z(R)$ be the set of all left or right zero divisors of R and $Z(R)^* = Z(R) \setminus \{0\}$. In this article, loops (i.e., edges from some vertex to itself) can be considered edges in a zero-divisor graph and we study the graph of equivalence classes of zero-divisor of a unit-duo by considering the following definition:

Definition 1.1. Let R be a unit-duo ring. The graph of equivalence classes of elements in $Z(R)^*$, denoted $\widetilde{\Gamma}(R)$, is the graph associated to R whose vertices are the classes of elements in $Z(R)^*$, and edges $[x] \longrightarrow [y]$, which means that $[x][y] = [0]$ for each pair of vertices $[x], [y]$ (not necessarily distinct).

Example 1.2. Let D be a noncommutative division ring, and consider a ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in D \right\}.$$

Let e_{ij} denote the matrix in the R with 1 in the (i, j)-position and 0 elsewhere. Then R is a noncommutative local ring with $J^2 \neq 0 = J^3$ and there exist two equivalence classes $[x]_\ell = [x]_r = [x]$, $[y]_\ell = [y]_r = [y]$ in X such that $X = [x] \cup [y]$ where $x = e_{13}, y = e_{12} + e_{23}$. Thus R is a unit-duo ring and $\widetilde{\Gamma}(R)$ is a graph with two vertices in which there is just one loop.

Also note that even though $\Gamma(R)$ is infinite graph (a graph with infinite set of vertices), $\widetilde{\Gamma}(R)$ is a finite graph with two vertices.

Example 1.3. Let D be a noncommutative division ring, and consider a ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in D \right\}.$$

Then R is a noncommutative local ring with $J \neq 0 = J^2$ and there exists the only one equivalence class $[e_{12} + e_{13}]_\ell = [e_{12} + e_{13}]_r = X$. Thus R is a unit-duo ring and $\widetilde{\Gamma}(R)$ is a graph with just one vertex and one edge which is a loop.

Example 1.4. Let $R = D_1 \oplus D_2$ be a direct product of two division rings D_1 and D_2. Then there exist two equivalences $[(1,0)]_\ell = [(1,0)]_r = [(1,0)]$, $[(0,1)]_\ell = [(0,1)]_r = [(0,1)]$ in X such that $X = [(1,0)] \cup [(0,1)]$, and so R is a unit-duo ring and $\widetilde{\Gamma}(R)$ is a graph with two vertices in which there is no loop.

The *indegree* of a vertex v in a graph, denoted *indegree*(v), is the number of edges arriving at v. Similarly, the *outdegree* of v, denoted *outdegree*(v), is the number of edges leaving at v. That is, *indegree*$(v) = |ann_\ell(v) \setminus \{0\}|$ and *outdegree*$(v) = |ann_r(v) \setminus \{0\}|$ where $ann_\ell(v)$ (resp. $ann_r(v)$) is a left (resp. right) annihilator of v. Of course, for a commutative ring case, *indegree*$(v) =$ *outdegree*(v) for any vertex of v in a graph, which is called the *degree* of v, denoted *degree*(v). A vertex of indegree 0 (resp. outdegree 0) is called a *source* (resp. *sink*). A *path of length* n from a vertex u to a vertex w is a sequence of distinct vertices v_i of the form $u = v_0 \longrightarrow v_1 \longrightarrow \cdots v_n = w$ such that $v_i \longrightarrow v_{i+1}$ is an edge for each $i = 0, \ldots, n-1$. The *distance* from a vertex u to a vertex w, denoted $d(u, w)$, is the length of the shortest path from u to w. When there is no path from a vertex u to a vertex w, we let $d(u, w) = \infty$.

Recall that a graph $\Gamma(R)$ over a ring R is *connected* if for all distinct vertices u, w there exists a path from u to w. The *diameter* of a graph (denoted by $\mathrm{diam}(\Gamma(R))$) is the supremum of $d(u, w)$ for all distinct vertices u and w. If $d(u, u) = k$ in a graph, then the path is called the *cycle* of length k. In particular, a cycle of length 2 in a graph is called a *loop* (i.e., an edge from some vertex to itself). In this paper, a loop can be considered an edge in a graph. If there is a cycle in a graph, then the *girth* of the graph (denoted by $g(\Gamma(R))$) (denoted by $\mathrm{diam}(\Gamma(R))$ is defined by the length of the shortest cycle in $\Gamma(R)$, otherwise, the girth of the graph is ∞. In [6, Proposition 1.3.2], if $\Gamma(R)$ contains a cycle, then $1 + 2\mathrm{diam}(\Gamma(R)) \geq g(\Gamma(R))$. We say that a graph is *complete* if there is an edge from u to w for any distinct vertices u, w of the graph.

In Section 3, we will show that (1) for a unit-duo ring R such that $X \neq \emptyset$, if $\widetilde{\Gamma}(R)$ is a finite graph (i.e., a graph with a finite number of vertices), then $\widetilde{\Gamma}(R)$ is connected and $\mathrm{diam}(\widetilde{\Gamma}(R))$ (resp. $\mathrm{g}(\widetilde{\Gamma}(R))$ is equal to or less than 3;

(2) for a semisimple unit-duo ring R, R is an artinian ring if and only if $\widetilde{\Gamma}(R)$ is a finite graph.

Let $\mathbb{Z}_n$ be the ring of integer of modulo n. In Section 3, we show that all vertices of $\widetilde{\Gamma}(\mathbb{Z}_n)$ are equivalences of all nonunit proper divisors of n. Let $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $n = q_1^{\beta_1} \cdots q_t^{\beta_t}$ be the prime factorization of m and n respectively. Then we have an equivalence relation $\simeq$ on the ring of integers defined by $m \simeq n$ if $s = t$ and each $\alpha_i = \beta_i$ by reordering $q_i's$. We also show that $m \simeq n$ if and only if $\widetilde{\Gamma}(\mathbb{Z}_m)$ is isomorphic to $\widetilde{\Gamma}(\mathbb{Z}_n)$.

2. Properties of unit-duo rings

In [8], it was shown that any right (left) unit-duo ring R is abelian (i.e., every idempotent in R is central). In [11], it was also shown that if a ring R has a finite number of equivalence classes under $\simeq_\ell$, then R is an artinian ring with $J(R)^{n+1} = 0$ where n is the number of classes under $\simeq_\ell$.

Proposition 2.1. *Let R be a right (resp. left) unit-duo ring such that $X(R)$ is a finite union of equivalence classes under $\sim_\ell$ (resp. $\sim_r$). Then every $x \in X(R)$ is a two-sided zero divisor.*

Proof. Let R be a right unit-duo ring and $x \in X(R)$ be arbitrary. If x is nilpotent, then clearly, x is a two-sided zero divisor. Suppose that x is not nilpotent and consider $x, x^2, \ldots$. Note that $x^k \in X(R)$ for all $k \geq 1$. Since $X(R)$ is a finite union of equivalence classes under $\sim_\ell$, $[x^r]_\ell = [x^s]_\ell$ for some positive integers r, s $(r > s)$. Then $x^r = ux^s$ for some unit $u \in R$, i.e., $(x^{r-s} - u)x^{s-1}x = 0$. If $(x^{r-s} - u)x^{s-1} \neq 0$, then x is a right zero divisor. If $(x^{r-s} - u)x^{s-1} = 0$, then by continuing in this way, we can deduce that x is a right zero divisor. Since R is right unit-duo, $x^r = ux^s = x^s v$ for some unit $v \in R$. By the similar argument, we have that x is a left zero divisor. Hence every $x \in X$ is a two-sided zero divisor. □

Theorem 2.2. *Let F be a field. Then there are just only two types of unit-duo subrings of* $\mathrm{Mat}_n(R)$ $(n \geq 2)$ *up to isomorphism as follows*:

(1) *The ring of all diagonal matrices in R;*

(2) *$V_n(R)$.*

Proof. Let S be a unit-duo subring of R. Note that if there is no nonzero nilpotent in S, then S is the ring of all diagonal matrices in R. Suppose that there is some nonzero nilpotent in S. First, we will show that every $g = (g_{ij}) \in G(S)$ is upper triangular by proceeding by induction on n. Take

$x_0 = e_{1n} \in X(S)$. Then

$$gx_0 = \begin{pmatrix} 0 & 0 & \cdots & g_{11} \\ 0 & 0 & \cdots & g_{21} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & g_{n1} \end{pmatrix}.$$

Since $[x_0]_\ell = [x_0]_r$, $gx_0 = x_0 h$ for some $h = (h_{ij}) \in G(S)$. Note that

$$x_0 h = \begin{pmatrix} h_{n1} & h_{n2} & \cdots & h_{nn} \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

yielding that $g_{21} = g_{31} = \cdots = g_{n1} = 0$. and so

$$g = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ 0 & g_{22} & \cdots & g_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & g_{n2} & \cdots & g_{nn} \end{pmatrix}.$$

Consider $x_1 = e_{1(n-1)} + e_{2n} \in X(S)$. Since $[x_1]_\ell = [x_1]_r$, $gx_1 = x_1 k$ for some $k = (k_{ij}) \in G(S)$, yielding that $g_{32} = g_{42} = \cdots = g_{n2} = 0$. Consider $x_t = e_{1(n-t)} + e_{2(n-t+1)} + \cdots + e_{tn} \in X(S)$ for each $t \geq 2$. By the similar argument, we have $g_{(t+1)t} = g_{(t+2)t} = \cdots = g_{nt} = 0$. Continuing in this way, we have that every $g \in G(S)$ is upper triangular.

Second, we will show that any $a = (a_{ij}) \in X(S)$ is upper triangular. Note that there exists $g \in G(S)$ such that ga is upper triangular. Since S is unit-duo, there exists $h = (h_{ij}) \in G(S)$ such that $ah = ga$. By the above argument, h is upper triangular, i.e., $h_{ij} = 0$ $(i > j)$. Let $ah = (b_{rs})$. Since $(b_{rs}) \in X(S)$ is upper triangular, $b_{rs} = 0$ for all $r, s = 1, \ldots, n$ $(r > s)$. For each $r = 2, \ldots, n$, we will show that $a_{rs} = 0$ for all $s = 1, \ldots, n$ with $r > s$ by using the mathematical induction on s. For $s = 1$, since $0 = b_{r1} = a_{r1}h_{11}$, $a_{ri} = 0$. Assume that $a_{r1} = a_{r2} = \cdots = a_{r(s-i)} = 0$. Then, for $r > s$, we have

$$0 = b_{rs} = a_{r1}h_{1s} + a_{r2}h_{2s} + \cdots + a_{r(s-1)}h_{(s-1)s} + a_{rs}h_{ss}.$$

By induction hypothesis, we have $a_{rs}h_{ss} = 0$, and so $a_{rs} = 0$ because $0 \neq h_{ss} \in F$. Therefore, any $a = (a_{ij}) \in X(S)$ is upper triangular.

Note that

$$a = diag_1(a_{11}, a_{22}, \ldots, a_{nn}) + diag_2(a_{12}, a_{23}, \ldots, a_{(n-1)n}) + \cdots + diag_n(a_{1n}),$$

where $diag_j(a_{1j}, a_{2(j+1)}, \ldots, a_{(n-j+1)n})$, for each j, is a matrix such that $a_{rs} = 0$ for all $(r,s) \neq (1,j), (2,j+1), \ldots, (n-j+1,n)$. Third, we will show that $diag_j(a_{1j}, a_{2(j+1)}, \ldots, a_{(n-j+1)n})$ is a matrix satisfying $a_{1j} = a_{2(j+1)} = \cdots = a_{(n-j+1)n}$ for each $j = 1, \ldots, n-1$.

Let $a^{(j)} = diag_j(a_{1j}, a_{2(j+1)}, \ldots, a_{(n-j+1)n}) \in X(S)$ for each $j = 1, \ldots, n-1$. Assume that there exists $a_{i(j+i-1)} \neq 0$, $a_{k(j+k-1)} = 0$ for some i, k $(i \neq k)$. Then $(i, j+k+1)$-entry of $ga^{(j)}$ is zero for all $g \in G(S)$, but $(i, j+k+1)$-entry of $a^{(j)}h$ is $a_{i(j+i-1)}h_{(j+i+1)(j+k+1)} \neq 0$ for some $h = (h_{ij}) \in G(S)$ with $h_{(i+1)(k+1)} \neq 0$, which implies that $[a^{(j)}]_\ell \neq [a^{(j)}]_r$, a contradiction to the assumption that S is unit-duo. Hence there exists no

$$diag_j(a_{1j}, a_{2(j+1)}, \ldots, a_{(n-j+1)n}) \in X(S)$$

such that $a_{i(j+i-1)} \neq 0$, $a_{k(j+k-1)} = 0$ for some i, k $(i \neq k)$. On the other hand, assume that there exists $a^{(j)} = diag_j(a_{1j}, a_{2(j+1)}, \ldots, a_{(n-j+1)n}) \in X(S)$ such that $a_{i(j+i-1)}, a_{k(j+k-1)} \neq 0$ and $a_{i(j+i-1)} \neq a_{k(j+k-1)}$ for some i, k $(i \neq k)$.

Consider $b^{(j)} = diag_j(b_{1j}, b_{2(j+1)}, \ldots, b_{(n-j+1)n}) \in X(S)$ such that $b_{i(i+j-1)} = a_{k(k+j-1)} - a_{i(i+j-1)}$, $b_{s(s+j-1)} = a_{s(s+j-1)}$ for all $s \neq i$. Then

$$b^{(j)} - a^{(j)} = diag_j(c_{1j}, c_{2(j+1)}, \ldots, c_{(n-j+1)n}) \in X(S),$$

having $c_{i(i+j-1)} = a_{k(k+j-1)} \neq 0$, $c_{s(s+j-1)} = 0$ for all $s \neq i$, a contradiction by the above argument. Therefore, $a^{(j)}$ is a matrix satisfying $a_{1j} = a_{2(j+1)} = \cdots = a_{(n-j+1)n}$ for each j.

Finally, it remains to show that for any $g = (g_{ij}) \in G(S)$, $g_{11} = g_{22} = \cdots = g_{nn}$. Let $p = g \cdot diag_2(1,1,\ldots,1), q = diag_2(1,1,\ldots,1) \cdot g \in X(S)$. Since $n - 1 \geq 2$, we have that

$$p^{(2)} = diag_2(g_{11}, g_{22}, \ldots, g_{(n-1)(n-1)}) \in X(S),$$

which yields $g_{11} = g_{22} = \cdots = g_{(n-1)(n-1)}$, by the above argument. Similarly,

$$q^{(2)} = diag_2(g_{22}, g_{33}, \ldots, g_{nn}) \in X(S)$$

also yields $g_{22} = g_{33} = \cdots = g_{nn}$. Hence we have that $g_{11} = g_{22} = \cdots = g_{nn}$.

Hence S is equal to $V_n(R)$ as desired. □

Remark 2.3. For a ring $V_n(R)$ as given Theorem 2.2, we note that $V_n(R)$ is a local ring with $J(R)^{n-1} \neq 0 = J(R)^n$ and there exist $(n-1)$ equivalence classes such as $[x], [x^2], \ldots, [x^{n-1}]$ where $x = e_{12} + e_{23} + \cdots + e_{(n-1)n}$, $x^k = \sum_{i=1}^{n-k} e_{i(i+k)}$ $(1 \leq k \leq n-1)$.

In Theorem 2.2, we can easily check that two types of unit-duo subrings of R are commutative. Also we can note that if R is a finite semisimple unit-duo ring, then R is a finite product of finite fields by help of Theorem 2.2, entailing that R is commutative. Hence we can raise a question:

Question 1. Is a finite unit-duo ring commutative?

The answer to Question 1 is negative by the following example. We use $GF(4)$ to denote the Galois field of order 2^2.

Example 2.4. We use the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a^2 \end{pmatrix} | a, b \in GF(4) \right\},$$

constructed by Xue [16, Example 2].

Then R is a noncommutative ring of order 16 with

$$J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} | b \in GF(4) \right\} \neq 0.$$

Note that $X(R) = J(R) \setminus \{0\}$. Let $GF(4) = \{0, 1, a, a^2\}$ with $a^3 = 1$. For every $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in X(R)$, we have

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This yields

$$[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]_\ell = [\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}]_\ell = X(R).$$

On the other hand, we also obtain

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ 0 & a^4 \end{pmatrix},$$
$$\begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^2 \end{pmatrix},$$

which implies that

$$[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]_r = [\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}]_r = X(R)$$

for all $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in X(R)$. Therefore,

$$[\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}]_\ell = [\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}]_r = X(R)$$

for all $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in X(R)$, and hence R is unit-duo.

Lemma 2.5 ([11, Lemma 3.10])**.** *Let R be a ring such that $X(R) \neq \emptyset$. If there exist finitely many equivalence classes under the relation $\sim_\ell$, then R is a left artinian ring with $J(R)^{n+1} = 0$, where n is the number of equivalence classes under the relation $\sim_\ell$.*

But the converse of Lemma 2.5 does not hold true, as the following example shows.

Example 2.6. Let R be a full matrix ring of 2 by 2 matrices over the quaternions $\mathbb{H}$. Then clearly R is a left (right) artinian ring. Let

$$x_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix},\ y_\alpha = \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix}$$

for all $\alpha \in \mathbb{H}$. Then we note that $[x_\alpha]_\ell \neq [x_\beta]_\ell$, (resp. $[x_\alpha]_r \neq [x_\beta]_r$) for all $\alpha, \beta \in \mathbb{H}$ ($\alpha \neq \beta$), and so there exist an uncountable equivalence classes under the relation $\sim_\ell$ (resp. $\sim_r$). Also note that $[x_\alpha]_\ell \neq [x_\alpha]_r$, i.e., R is not unit-duo.

Remark 2.7. Note that the class of unit-duo rings is closed under direct product and homomorphic images. Note also that if R is a unit-duo ring and J is the Jacobson radical of R, then R/J is a unit-duo ring. But the converse may not be true. For example, consider R, the ring of 2 by 2 upper triangular matrices over $\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then $[e_{11}]_\ell = \{e_{11}\} \neq \{e_{11}, e_{11} + e_{12}\} = [e_{11}]_r$, and so R is not unit-duo. On the other hand, R/J is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and so R/J is unit-duo.

Proposition 2.8. *Let R be a semisimple unit-duo ring. Then there exist a finite number of equivalence classes under $\sim$ if and only if R is an artinian ring.*

Proof. ($\Rightarrow$) It follows from Lemma 2.5.

($\Leftarrow$) Suppose that R is an artinian ring. Since $J = 0$, by the Wedderburn-Artin Theorem there exist division rings $D_1, \ldots, D_t$ and positive integers n_1, $\ldots, n_t$ such that R is isomorphic to the ring $M_{n_1}(D_1) \times M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$ where each $M_{n_i}(D_i)$ is the full matrix ring of all n_i by n_i matrices over a division ring D_i. We note that since R is unit-duo, $n_i = 1$ for each $i = 1, \ldots, t$. Indeed, assume that $n_i \geq 2$ for some i. For some nonunit $x = e_{11} \in M_{n_i}(D_i)$ and some unit $g = (g_{ij}) \in M_{n_i}(D_i)$ with $g_{21} \neq 0$ in D_i, there exists no unit $h \in M_{n_i}(D_i)$ so that $gx = xh$, which is a contradiction to that R is unit-duo. Hence R is isomorphic to $D_1 \times \cdots \times D_t$, and then clearly, there exist a finite number of equivalence classes under $\sim$. □

Recall that a ring R is called von Neumann regular (simply, regular) (resp. unit-regular) if for every $x \in R$ there exists $y \in R$ (resp. $u \in G$) such that $xyx = x$ (resp. $xux = x$). A ring R is called strongly regular if for every $x \in R$ there exists $y \in R$ such that $x^2y = x$. It is well-known that R is strongly regular if and only if R is abelian regular (a regular ring whose all idempotents are central).

Proposition 2.9. *Let R be a unit-regular ring. Then R is unit-duo if and only if R is strongly regular.*

Proof. If R is unit-duo, then clearly R is strongly regular. Suppose that R is strongly regular. Then every idempotent of R is central. Let $x \in X$ be arbitrary. Since R is unit-regular, there exists a unit $u \in R$ such that $xux = x$. Since R is strongly regular, $ux \in X$ is a central idempotent. Let $e = ux$.

Clearly, $[x]_\ell = [e]_\ell = [e]_r$. Since $e = ux$, $x = u^{-1}e = eu^{-1}$, and then $x \sim_r e$, i.e., $[x]_r = [e]_r$. Thus $[x]_\ell = [x]_r$, and so R is unit-duo. □

Lemma 2.10. *Let R be a unit-duo ring, and let $e \in R$ be a nonzero nonunit idempotent. If $f \in [e]$ for any idempotent f of R, then $e = f$.*

Proof. Since $f \in [e]$ and R is unit-duo, $f = ue = ev$ for some units $u, v \in R$. Then we have $fe = ef = f$. Since $e = u^{-1}f = fv^{-1}$, we also have $ef = fe = e$. Thus $e = f$. □

Proposition 2.11. *Let R be a strongly regular ring. Then we have the following equivalent conditions*:

(1) *There exist a finite number of idempotents in R*;
(2) *There exist a finite number of equivalence classes under relation $\sim$*;
(3) *R is artinian*;
(4) *$R \simeq D_1 \times D_2 \times \cdots \times D_t$ for some positive integer t.*

Proof. (1) ⇒ (2). Suppose that there exist a finite number of idempotents in R. Since R is strongly regular, R is unit-duo and every idempotent of R is central. Let $x \in X$ be arbitrary. Then $x = ue$ for some idempotent $e \in R$ and some unit $u \in R$, and so $[x] = [e]$. Since the number of idempotents of R is finite, There exist a finite number of equivalence classes under relation $\sim$ by Lemma 2.10.

(2) ⇒ (3). It follows from Lemma 2.5.

(3) ⇔ (4). Since R is unit-duo and $J = 0$, it follows from the proof of Proposition 2.8.

(4) ⇒ (1). Clear. □

3. Zero divisor graphs of unit-duo rings

For a ring R, let $\Gamma(R)$ be the zero divisor graph of R consisting of all vertices in $Z(R)$ and edges $x \to y$, which means that $xy = 0$ for all $x, y \in Z(R)^*$.

Proposition 3.1. *Let R be a unit-duo ring. Then*

(1) *$\Gamma(R)$ has no sources and no sinks if and only if $\widetilde{\Gamma}(R)$ has no sources and no sinks*;
(2) *$\Gamma(R)$ is connected if and only if $\widetilde{\Gamma}(R)$ is connected*;
(3) *$diam(\Gamma(R)) = diam(\widetilde{\Gamma}(R))$ and $g(\Gamma(R)) = g(\widetilde{\Gamma}(R))$*;
(4) *$\Gamma(R)$ is complete if and only if $\widetilde{\Gamma}(R)$ is complete.*

Proof. It follows from that for $x, y \in Z(R)^*$ $xy = 0$ if and only if $[x][y] = [0]$. □

In fact, we note that for a unit-duo ring R, there seems no distinction between $\Gamma(R)$ and $\widetilde{\Gamma}(R)$ except the number of vertices. Hence it is more efficient to consider $\widetilde{\Gamma}(R)$ than to consider $\Gamma(R)$ for a unit-duo ring R.

Proposition 3.2. *Let R be a unit-duo ring. If $\widetilde{\Gamma}(R)$ has no sources and no sinks, then $\widetilde{\Gamma}(R)$ is connected and diam$(\widetilde{\Gamma}(R))$ (resp. $g(\widetilde{\Gamma}(R))$ is equal to or less than 3.*

Proof. Let $[x], [y]$ ($[x] \neq [y]$) be arbitrary vertices of $\widetilde{\Gamma}(R)$. Since $\widetilde{\Gamma}(R)$ has no sources and sinks, there exists a vertex $[a]$ (resp. $[b]$) such that $[x][a] = [xa] = [0]$ (resp. $[b][y] = [by] = [0]$). If $[a][b] = [ab] = [0]$, then $[x] \longrightarrow [a] \longrightarrow [b] \longrightarrow [y]$ is a path of length 3. If $[a][b] = [ab] \neq [0]$, then $[x] \longrightarrow [ab] \longrightarrow [y]$ is a path of length 2. Hence diam$(\widetilde{\Gamma}(R))$ (resp. g$(\widetilde{\Gamma}(R))$ is equal to or less than 3. In particular, if we let $[x] = [y]$, then $g(\widetilde{\Gamma}(R)$ is equal to or less than 3. ☐

Corollary 3.3. *Let R be a commutative ring such that $Z(R)^* \neq \emptyset$. Then $\Gamma(R)$ is connected and diam$(\Gamma(R))$ (resp. $g(\Gamma(R))$ is equal to or less than 3.*

Proof. Since R is a commutative ring, R is unit-duo and $\Gamma(R)$ has no sources and no sinks. Hence it follows from Proposition 3.1 and Proposition 3.2. ☐

Theorem 3.4. *Let R be a unit-duo ring such that $X \neq \emptyset$. If $\widetilde{\Gamma}(R)$ is a finite graph (i.e., a graph with a finite number of vertices), then $\widetilde{\Gamma}(R)$ is connected and diam$(\widetilde{\Gamma}(R))$ (resp. $g(\widetilde{\Gamma}(R))$ is equal to or less than 3.*

Proof. Since $\widetilde{\Gamma}(R)$ is a finite graph, there exist a finite number of equivalence classes under $\sim$, any $x \in X$ is a two-sided zero divisor, i.e., $X = Z(R)^*$ by Proposition 2.1. Since for $x, y \in Z(R)^*$ $xy = 0$ if and only if $[x][y] = [0]$, there is no origin and no sink in $\widetilde{\Gamma}(R)$. Hence we have the desired result by Proposition 3.2. ☐

It was shown in [4, Lemma 2.2] that if there exists the only one equivalence $[x]_\ell$ in a ring R such that $X(R) \neq \emptyset$ under $\sim_\ell$, then R is local. In this case, $[x]_\ell = [x]_r = X(R)$ by [10, Theorem 2.9], and so R is a unit-duo ring.

Proposition 3.5. *If R and S are unit-duo rings, then $\widetilde{\Gamma}(R \times S)$ is isomorphic to $\widetilde{\Gamma}(R) \times \widetilde{\Gamma}(S)$ as graphs.*

Proof. Clearly, $R \times S$ is also a unit-duo ring. Define $\phi : \widetilde{\Gamma}(R \times S) \to \widetilde{\Gamma}(R) \times \widetilde{\Gamma}(S)$ by $\phi([(x, y)] = ([x], [y])$ for all $[(x, y)] \in \widetilde{\Gamma}(R \times S)$. It is straightforward to show that ϕ is a graph isomorphism. ☐

On other hand, it was also shown in [9, Proposition 3.3] that if R is an abelian ring with a finite number of equivalence classes under $\sim_\ell$, then R is a finite product of local rings. Hence it is enough to consider $\widetilde{\Gamma}(R)$ for a local unit-duo ring R.

For a given unit-duo ring R, we denote a loop in $\widetilde{\Gamma}(R)$ from a vertex $[x]$ to itself by $\widehat{[x]}$.

Lemma 3.6 ([7, Lemma 2.9])**.** *Let R be a ring such that $X(R)$ is a union of n equivalence classes under $\sim_\ell$. Then the following are equivalent:*

(i) *There exists* $x \in J(R)$ *such that* $x^n \neq 0$;
(ii) R *is a local ring,* $J^n \neq 0 = J^{n+1}$;
(iii) $J > J^2 > \cdots > J^{n-1} > J^n \neq 0$.

Theorem 3.7. *Let* R *be a ring such that* $X(R)$ *is a finite union of equivalence classes under* $\sim$*. Then* R *is local if and only if there exists an element* $b \in X(R)$ *such that* $ann(b) = X \cup \{0\}$.

Proof. Suppose that R is local. Since $J^{n+1} = 0$ by Lemma 2.5 where n is the number of equivalence classes under $\sim$. Let r be the least positive integer so that $J^r \neq 0 = J^{r+1}$. Let $\bar{R} = R/J^r$ and $\bar{X} = X(\bar{R})$. Note that $\bar{R}$ is local. First, we will show that $\bar{X}$ is a union of $(r-1)$ classes under the regular action. Take $a_i \in J^i \setminus J^{i+1}$ for each $i = 1, 2, \ldots, r$. Observe that $[\bar{a_i}]$ are all distinct. Indeed, assume that $[\bar{a_j}] = [\bar{a_k}]$ for some j, k $(1 \leq j < k \leq r)$. Then $\bar{a_j} = \bar{u}\bar{a_k}$ for some unit $\bar{u}$ of $\bar{R}$, and so $a_j - ua_k \in J^r$. Since $a_k \in J^k \subseteq J^j$, $a_j \in J^k \subseteq J^{j+1}$, which is a contradiction. Hence there are at least $(r-1)$ classes in $\bar{X}$. Let s be the number of classes in $\bar{X}$. To show $s = r-1$, assume that $s \geq r$. Since $\bar{R}$ is local, by Lemma 3.6, there exists $\bar{x} \in J(\bar{R}) = J/J^r$ such that $\bar{x}^s \neq \bar{0} = J^r$, i.e., $x^s \notin J^r$. But $x^s \in J^s \subseteq J^r$, a contradiction. Thus if $J^r \neq 0 = J^{r+1}$, $\bar{X}$ is a union of $(r-1)$ classes under the regular action, and so by Lemma 3.6, there exists $\bar{x} \in J(\bar{R}) = J/J^r$ such that $\bar{x}^{r-1} \neq \bar{0} = J^r$, i.e., $x^{r-1} \notin J^r$, yielding that $0 \neq x^{r-1} \in J(R)$. Since $J^r \neq 0$, we can have the following two cases:

Case 1. There exists $a \in J$ such that $ax^{r-1} \neq 0$ or $x^{r-1}a \neq 0$, say, $ax^{r-1} \neq 0$.

Let $x_0 = ax^{r-1} \in J^r$. Since $J^{r+1} = 0$, $yx_0 = x_0y = 0$ for all $y \in X(R)$, which yields that $ann(x_0) = X \cup \{0\}$.

Case 2. $yx^{r-1} = x^{r-1}y = 0$ for all $y \in J$.

Hence we have that $ann(x^{r-1}) = X \cup \{0\}$ in this case.

The converse is clear. □

Corollary 3.8. *Let* R *be a finite ring. Then* R *is local if and only if there exists an element* $b \in X(R)$ *such that* $ann(b) = X \cup \{0\}$.

Proof. It follows from Theorem 3.7. □

Corollary 3.9. *Let* R *be a unit-duo ring such that* $\widetilde{\Gamma}(R))$ *is a finite graph. Then* R *is local if and only if* $diam(\widetilde{\Gamma}(R)) = 2$.

Proof. Let $\widetilde{\Gamma}(R)$ be a finite graph with n vertices $[v_1], \ldots, [v_n]$. Note that $X(R)$ is a union of $[v_1], \ldots, [v_n]$. If R is local, there exists an element $b \in X(R)$ such that $ann(b) = X \cup \{0\}$ by Theorem 3.7, yielding that $[b][v_i] = [v][b] = [0]$ for all vertices $[v_i]$ of $\widetilde{\Gamma}(R)$, and so $\mathrm{diam}(\widetilde{\Gamma}(R)) = 2$. Conversely, suppose that $\mathrm{diam}(\widetilde{\Gamma}(R)) = 2$. Then there exists a vertex $[b]$ of $\widetilde{\Gamma}(R)$ such that $[b][v_i] = [v_i][b] = [0]$ for all vertices $[v_i]$ of $\widetilde{\Gamma}(R)$, i.e., $bv_i = v_ib = 0$. Let $y \in X(R)$ be arbitrary. Since $\widetilde{\Gamma}(R)$ is a finite graph, $[y] = [v_i]$ for some $[v_i]$. Since R is unit-duo, $y = uv_i = v_iw$ for some units u, w of R. Thus $yb = (uv_i)b = u(v_ib) = 0$ and $by = b(v_iw) = (bv_i)w = 0$, which implies that $ann(b) = X \cup \{0\}$, and then R is local as desired. □

4. Graph of equivalence classes of zero divisors of $\mathbb{Z}_n$

Throughout this section, n is considered as any positive non-prime integer otherwise stated. Let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ be the ring of integers modulo n. For all $a, b \in \mathbb{Z}_n$, ab means the product of a and b under the multiplication modulo n. We will denote the greatest common divisor of any two positive integers s and t by (s, t) and $s \mid t$ means that s is a divisor of t.

Lemma 4.1. *Let n be any positive integer and $x, y \in X(\mathbb{Z}_n)$ be distinct divisors of n such that $x < y$. Then $[x] \neq [y]$.*

Proof. Assume that $[x] = [y]$. Then $y = gx$ for some $g \in G(\mathbb{Z}_n)$. Since x, y are distinct divisors of n such that $x < y$, we can choose an element $a \in X(\mathbb{Z}_n)$ so that $ax \neq 0, ay = 0$. On the other hand, since $0 = ay = a(gx)$ and $g \in G(\mathbb{Z}_n)$, we have $ax = 0$, which is a contradiction. Hence $[x] \neq [y]$. □

Lemma 4.2. *Let n be any positive integer and $y \in X(\mathbb{Z}_n)$ be arbitrary. Then there exists $x \in X(\mathbb{Z}_n)$ such that $x \mid n$ and $(x, n) = (y, n)$.*

Proof. Let $x = (y, n)$. Then clearly, $x \mid n$ and $(x, n) = ((y, n), n) = (y, n)$. □

Lemma 4.3. *Let n be any positive integer and k be a divisor of n. If $\bar{g} \in G(\mathbb{Z}_k)$, then there exists $g \in G(\mathbb{Z}_n)$ such that $g \equiv \bar{g} \pmod{k}$.*

Proof. Note that since k is a divisor of n, $\mathbb{Z}_n/\langle k\rangle$ is isomorphic to $\mathbb{Z}_k$ where $\langle k\rangle$ is an ideal of $\mathbb{Z}_n$ generated by k. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ be the prime factorization of n where $p_1, p_2, \ldots, p_t$ are distinct primes for some positive integer t. Then $k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$ with $\alpha_i \geq \beta_i \geq 0$ for all $i = 1, \ldots, t$. Without loss of generality, we can assume that $\mathbb{Z}_n = \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$ (resp. $\mathbb{Z}_k = \mathbb{Z}_{p_1^{\beta_1}} \times \mathbb{Z}_{p_2^{\beta_2}} \times \cdots \times \mathbb{Z}_{p_t^{\beta_t}}$). Then we can consider a ring epimorphism $\pi : \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}} \to \mathbb{Z}_{p_1^{\beta_1}} \times \mathbb{Z}_{p_2^{\beta_2}} \times \cdots \times \mathbb{Z}_{p_t^{\beta_t}}$ given by $\pi(a_1, \ldots, a_t) = (\bar{a}_1, \ldots, \bar{a}_t)$ for all $(a_1, \ldots, a_t) \in \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$ where $\bar{a}_i$ is the remainder obtained from dividing a_i by $p_i^{\beta_i}$ for all i.

Case 1. Suppose that $\beta_i \geq 1$ for all $i = 1, \ldots, t$.

Let $\bar{g} = (\bar{g}_1, \ldots, \bar{g}_t) \in \mathbb{Z}_{p_1^{\beta_1}} \times \mathbb{Z}_{p_2^{\beta_2}} \times \cdots \times \mathbb{Z}_{p_t^{\beta_t}}$ be an arbitrary unit. Then there exists an element $g = (g_1, \ldots, g_t) \in \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$ such that $\pi(g) = \bar{g}$ i.e., $g_i \equiv \bar{g}_i \pmod{p_i^{\beta_i}}$ for all i. Since $\bar{g}$ is a unit in $\mathbb{Z}_{p_1^{\beta_1}} \times \mathbb{Z}_{p_2^{\beta_2}} \times \cdots \times \mathbb{Z}_{p_t^{\beta_t}}$, we have $(\bar{g}_i, p_i^{\beta_i}) = 1$ and so $(g_i, p_i^{\alpha_i}) = 1$ for all $i = 1, \ldots, t$, which implies that $g \in \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$ is a unit.

Case 2. Suppose that $\beta_i = 0$ for some i.

Let $I_1 = \{i \in \{1,\dots,t\} : \beta_i \geq 1\}$ and $I_2 = \{i \in \{1,\dots,t\} : \beta_i = 0\}$. Consider $R = R_1 \times R_2$ where $R_1 = \prod_{i\in I_1} \mathbb{Z}_{p_i^{\beta_i}}$ and $R_2 = \prod_{i\in I_2}\{1_i\}$ where 1_i is the unity of $\mathbb{Z}_{p_i^{\beta_i}}$. By changing the order of the $\mathbb{Z}_{p_i^{\beta_i}}$ if necessary we can assume that $R = \mathbb{Z}_k = \mathbb{Z}_{p_1^{\beta_1}} \times \mathbb{Z}_{p_2^{\beta_2}} \times \cdots \times \mathbb{Z}_{p_t^{\beta_t}}$. Let $G(R)$ be the group of all units in R. Let $\bar{g} = (\bar{g}_1,\dots,\bar{g}_{|I_1|}, 1_1,\dots,1_{|I_2|}) \in G(R)$ be arbitrary. Then by the similar argument given in Case 1, there exists a unit $g_i \in \mathbb{Z}_{p_1^{\alpha_1}}$ such that $g_i \equiv \bar{g}_i \pmod{p_i^{\beta_i}}$ for all $i = 1,\dots,|I_1|$. Let $g = (g_1,\dots,g_{|I_1|}, 1_1,\dots,1_{|I_2|}) \in \mathbb{Z}_{p_1^{\alpha_1}} \times\cdots\times \mathbb{Z}_{p_t^{\alpha_t}}$. Then g is a unit in $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$ such that $\pi(g) = \bar{g}$. □

Theorem 4.4. *Let n be any positive integer. Then for all $x, y \in X(\mathbb{Z}_n)$, $[x] = [y]$ if and only if $(x, n) = (y, n)$.*

Proof. ($\Rightarrow$) Suppose that for all $x, y \in X(\mathbb{Z}_n)$, $[x] = [y]$. Then $y = gx$ for some $g \in G(\mathbb{Z}_n)$. Since $(g, n) = 1$, we have $(y, n) = (gx, n) = (x, n)$.

($\Leftarrow$) Suppose that for all $x, y \in X(\mathbb{Z}_n)$, $(x, n) = (y, n)$. It is enough to consider $x \mid n$, i.e., $x = (x, n)$ by Lemma 4.1. Since $x \mid y$, $y = ax$ for some integer a. Since $x = (y, n)$, $x = by + cn$ for some integers b and c. Hence $x \equiv by \equiv bax \pmod{n}$, and then $1 \equiv ba \pmod{\frac{n}{x}}$. Let $\bar{a}$ be an element of $\mathbb{Z}_{\frac{n}{x}}$ so that $a \equiv \bar{a} \pmod{\frac{n}{x}}$. Then $1 \equiv b\bar{a} \pmod{\frac{n}{x}}$, which implies that $\bar{a} \in G(\mathbb{Z}_{\frac{n}{x}})$. By Lemma 4.2, there exists $a_0 \in G(\mathbb{Z}_n)$ such that $a_0 \equiv \bar{a} \pmod{\frac{n}{x}}$. Since $a_0 = \bar{a} + k(\frac{n}{x})$ for some integer k, we have $a_0 x \equiv (\bar{a} + k(\frac{n}{x}))x \equiv \bar{a}x \equiv ax \equiv y \pmod{n}$, which implies that $o(x) = o(y)$. □

Let $V_n = \{[x] \mid x \mid n, x \neq 1, n\}$. By Theorem 3.4, $\widetilde{\Gamma}(\mathbb{Z}_n)$ is the graph of equivalence with vertices in V_n and edges $[x] \longrightarrow [y]$, which means that $[x][y] = [0]$ (i.e., $xy = 0$) for each pair of vertices $[x], [y] \in V_n$ (not necessarily distinct). Let m, n be non-prime positive integers, and $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}, n = q_1^{\beta_1} \cdots q_t^{\beta_t}$ be the prime factorizations of m and n. Define $m \simeq n$ (m is similar to n) if $s = t$ and each $\alpha_i = \beta_i$ by reordering $q_i's$ if necessary. For example, $12 \simeq 18 \simeq 245$. Then $\simeq$ is clearly an equivalence relation on $\mathbb{Z}$, the ring of integers.

Theorem 4.5. *Let m, n be non-prime positive integers. Then $m \simeq n$ if and only if $\widetilde{\Gamma}(\mathbb{Z}_m)$ is isomorphic to $\widetilde{\Gamma}(\mathbb{Z}_n)$.*

Proof. ($\Rightarrow$) Suppose that $m \simeq n$. Then we can let $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}, n = q_1^{\alpha_1} \cdots q_1^{\alpha_s}$ be the prime factorization of m and n respectively. Then clearly $|V_m| = |V_n|$. Define $\theta : V_m \to V_n$ by $\theta([p_1^{\beta_1} \cdots p_s^{\beta_s}]) = [q_1^{\beta_1} \cdots q_s^{\beta_s}]$ for all $[p_1^{\beta_1} \cdots p_s^{\beta_s}] \in V_m$ where $1 \leq \beta_i \leq \alpha_i$ for each $i = 1,\dots,s$. We note that $[x][y] = 0$ for all $[x], [y] \in V_m$ if and only if $\theta([x])\theta([y]) = 0$, and so θ is isomorphism.

($\Leftarrow$) Assume that m is not similar to n. Let $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}, n = q_1^{\beta_1} \cdots q_t^{\beta_t}$ be the prime factorization of m and n respectively.

Case 1. $r = s$

Since m is not similar to n, $\alpha_i \neq \beta_i$ for some i. Let k be the smallest positive integer so that $\alpha_k \neq \beta_k$. Without loss of generality, we assume that $\alpha_k < \beta_k$.

We can also assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_s$. Note that the number of vertices in $\widetilde{\Gamma}(\mathbb{Z}_m)$ having degree β_k is less than $(s-k+1)$ which is equal to the number of vertices in $\widetilde{\Gamma}(\mathbb{Z}_n)$ having degree β_k. Hence $\widetilde{\Gamma}(\mathbb{Z}_m)$ is not isomorphic to $\widetilde{\Gamma}(\mathbb{Z}_n)$.

Case 2. $r \neq s$

Note that the number of vertices in $\widetilde{\Gamma}(\mathbb{Z}_m)$ (resp. $\widetilde{\Gamma}(\mathbb{Z}_n)$) having degree 1 is r (resp. s). Since $r \neq s$, $\widetilde{\Gamma}(\mathbb{Z}_m)$ is not isomorphic to $\widetilde{\Gamma}(\mathbb{Z}_n)$. □

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