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NONHOMOGENEOUS DIRICHLET PROBLEM FOR ANISOTROPIC DEGENERATE PARABOLIC-HYPERBOLIC EQUATIONS WITH SPATIALLY DEPENDENT SECOND ORDER OPERATOR

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Abstract. There are fruitful results on degenerate parabolic-hyperbolic equations recently following the idea of Kružkov's doubling variables device. This paper is devoted to the well-posedness of nonhomogeneous boundary problem for degenerate parabolic-hyperbolic equations with spatially dependent second order operator, which has not caused much attention. The novelty is that we use the boundary flux triple instead of boundary layer to treat this problem.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. Denote $Q := (0,T)\times\Omega$ and $\Sigma := (0,T)\times\partial\Omega$ with $T>0$. We are interested in the following degenerate parabolic-hyperbolic equation

$$\partial_t u + \mathrm{div} f(t,x,u) = \nabla\cdot(K(t,x)\nabla A(u)) + g(t,x,u), \quad (t,x)\in Q, \tag{1.1}$$

with the initial-boundary conditions

$$\begin{cases} u(t,x) = u_b(t,x), & (t,x)\in\Sigma, \\ u(0,x) = u_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

Here $u(t,x)$ is the scalar unknown function that is sought. The vector-valued function $f(t,x,u) = (f_1(t,x,u),\ldots,f_d(t,x,u))$ is called convection flux. $K(t,x) = \mathrm{diag}\{k_i(t,x)\,|\,i=1,\ldots,d\}$. $A(u)$ and $g(t,x,u)$ are given scalar functions to be detailed in the next section. For the moment it suffices to say that $K(t,x)$ is a strictly positive matrix and $A(u)$ is nondecreasing with $A(0)=0$. Thus (1.1) is a strongly degenerate parabolic-hyperbolic equation meanwhile it takes the anisotropic form.

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Such an equation of quasilinear advection with degenerate diffusion occurs in several applications, such as diphasic flow in porous media and sedimentation-consolidation processes [6, 8]. In particular, the study of problem (1.1) includes the hyperbolic conservation law (where $A'(\cdot) = 0$). Since (1.1) is allowed to be degenerate, one has to define entropy solution to guarantee the uniqueness of weak solutions. The device of "doubling of variables" to prove uniqueness of entropy solutions for degenerate parabolic equations was first introduced in the fundamental work by Carrillo [7], which in turn followed the pioneering work of Kružkov [18] on conservation laws. We also refer the readers to [9, 10, 12, 13, 14] for various extensions of Carrillo's uniqueness result on the isotropic case for Cauchy problem.

The analysis of boundary problem for degenerate parabolic-hyperbolic equation is interesting and difficult. Even for the scalar hyperbolic conservation law, a correct formulation of the boundary condition has been done for a long time until Bardos etc. [4] put forward the well-known BLN condition. However the formulation of entropy solution given by Bardos etc. requires the solutions belong to BV space and thus have traces on the boundary. Later F. Otto [22] gave an integral formulation of entropy condition in L^∞ space. As for the degenerate parabolic problem, in [20] the authors first gave a definition of entropy solution with suitable entropy-entropy flux pairs for which uniqueness and consistency with the approximated strictly parabolic equation were proved. A. Michel and J. Vovelle [21] gave an original definition in integral form which is well suited to prove the convergence of approximated solutions with the FV method. They both used the doubling variables technique to prove the uniqueness. Later, K. Kobayasi [16] used the kinetic formulation to obtain a comparison property of entropy solution with respect to initial-boundary data.

The general anisotropic diffusion case is more delicate and was first treated by Chen and Perthame [11] in 2003. They introduced the fundamental chain-rule property and extended the notion of kinetic solution, which applied to more general situations than entropy solution. The initial-boundary value problem for the anisotropic case has been treated in recent years. B. Andreianov, M. Bendahame and Karlsen [2, 3, 5] considered doubly nonlinear degenerate parabolic equations with homogeneous Dirichlet boundary conditions. Lately, K. Ammar [1] obtained the existence and uniqueness of entropy solutions for the triple nonlinear degenerate parabolic-hyperbolic problem of the form

$$b(v)_t - \mathrm{div} a(v, \nabla g(v)) + \psi(v) = f$$

with nonhomogeneous boundary conditions. This work includes the isotropic degenerate parabolic equation, P-Laplace problem, triply degenerate problem with strong degeneracy and so on. In our knowledge there are little results on the boundary value problem for anisotropic case. In 1989, Wu and Zhao [25] proved the existence and uniqueness of generalized solutions in BV space with homogeneous boundary condition. Recently, Y. Li and Q. Wang [19] used doubling variables device and vanishing viscosity method to prove the well-posedness of bounded entropy solutions of the homogeneous boundary problem. K. Kobayasi and H. Ohwa [17] studied the well-posedness of the nonhomogeneous boundary problem on a bounded rectangle by using the kinetic formulation.

A typical feature of all the works on the boundary problem of degenerate parabolic equations is that the second order differential operator is independent of the spatial variable explicitly. For the Cauchy problem with explicit (t, x) dependent operator, there are several results [10, 15]. Another feature is that the methods dealing with homogeneous and nonhomogeneous boundary conditions are different: for the former, one can define boundary flux triples following the idea of Carrillo [7] while for the latter people mainly use boundary layer [20, 21]. In this paper we will take boundary flux triples to treat the nonhomogeneous boundary problems of degenerate parabolic equations with spatially dependent second order operator.

The rest of this paper is organized as follows. In Section 2, we state the notion of entropy solution as well as the main theorem. Section 3 and Section 4 are devoted to the proof of uniqueness and existence of entropy solutions, respectively.

2. Definitions and main results

Now we give some notations. Define

$$\mathrm{sgn}^+(z) = \begin{cases} 1, & z > 0, \\ 0, & z \le 0, \end{cases} \quad \mathrm{sgn}^-(z) = \begin{cases} 0, & z \ge 0, \\ -1, & z < 0, \end{cases}$$

and $r^+ = \max\{r, 0\} = r\mathrm{sgn}^+(r)$, $r^- = -\min\{r, 0\} = r\mathrm{sgn}^-(r)$.

For any $\epsilon > 0$, set

$$\mathrm{sgn}^+_\epsilon(z) = \begin{cases} 1, & z > \epsilon, \\ (\sin \frac{\pi z}{2\epsilon})^2, & 0 \le z \le \epsilon, \\ 0, & z < 0, \end{cases} \quad \mathrm{sgn}^-_\epsilon(z) = \begin{cases} 0, & z > 0, \\ -(\sin \frac{\pi z}{2\epsilon})^2, & -\epsilon \le z \le 0, \\ -1, & z < -\epsilon \end{cases}$$

which are C^1 approximations of $\mathrm{sgn}^\pm(z)$, respectively.

We will make the following assumptions throughout the paper:

(A1) $u_0 \in L^\infty(\Omega)$, $u_b \in C(\Sigma)$, $g(t, x, u) \in L^\infty(Q \times \mathbb{R})$.

(A2) For $i = 1, \dots, d$, $f_i(t, x, u) \in W^{1,\infty}(Q\times\mathbb{R})$, $k_i(t, x) \in W^{1,\infty}(Q)$. $A(u) \in W^{1,\infty}(\mathbb{R})$ is nondecreasing and $A(0) = 0$.

Let $M = \max\left\{||u_0||_{L^\infty(\Omega)}, ||u_b||_{C(\Sigma)}\right\}$, and

$$F^\pm(t, x, u, v) := \mathrm{sgn}^\pm(u - v)(f(t, x, u) - f(t, x, v)).$$

We present the definition of entropy solution as follows.

Definition 2.1 (**Entropy solution**). A measurable function $u(t, x)$ is called an entropy solution of problem (1.1)-(1.2) if

(D.1) $u \in L^\infty(Q) \cap C([0,T); L^1(\Omega))$, i.e., the initial data holds in the strong L^1 sense

$$ess \lim_{t\to 0+} \int_\Omega |u(t,x) - u_0(x)| dx = 0.$$

(D.2) $A(u) \in L^2(0,T; H^1(\Omega))$, and $A(u) = A(u_b)$ in the trace sense.

(D.3) The following entropy inequalities are satisfied:

$$\int_Q \Big\{ (u-k)^\pm \varphi_t + \Big(F^\pm(t,x,u,k) - K(t,x)\nabla(A(u) - A(k))^\pm \Big) \nabla\varphi + \operatorname{sgn}(u-k) g\varphi \Big\} dxdt + \int_\Sigma W^\pm((t,x),k,u_b)\varphi dr(x)dt$$
$$+ \int_\Omega (u_0 - k)^\pm \varphi(0,x) dx \geq 0 \tag{2.1}$$

hold for all $k \in [-M, M]$ and all $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^d)$ such that $\varphi \geq 0$ and $\operatorname{sgn}^\pm(A(u_b) - A(k))\varphi = 0$ a.e. on Σ, respectively.

We will use vanishing viscosity method to prove the existence of entropy solution. It's unfortunate that we can not get the strong convergence of the approximated solution $\{u^\epsilon\}_{\epsilon>0}$, so we will consider the measure-valued solutions or entropy process solutions. The idea is based on the following theorem.

Theorem 2.1 (Nonlinear convergence for weak-$*$ topology)**.** *Let Ω be a Borel subset of $\mathbb{R}^d$ and $\{u_n\}$ be a uniformly bounded sequence in $L^\infty(\Omega)$. Then there exists a subsequence, still denoted by $\{u_n\}$ and $\mu(t,x,\alpha) \in L^\infty(\Omega \times (0,1))$ such that, for any $h(\cdot) \in C(\mathbb{R})$,*

$$h(u_n) \to \int_0^1 h(\mu(\cdot,\cdot,\alpha)) d\alpha \quad \textit{in } L^\infty(\Omega) \textit{ weak-}* .$$

Now we put forward the definition of entropy process solution.

Definition 2.2 (**Entropy process solution**)**.** Let $u = u(t,x,\alpha)$ be in $L^\infty(Q \times (0,1))$. The function $u(t,x,\alpha)$ is an entropy process solution to problem (1.1)-(1.2) if

(D.1) $A(u) = A(u_b) \in L^2(0,T; H_0^1(\Omega))$.

(D.2) The following entropy inequalities hold:

$$\int_Q \int_0^1 \Big\{ (u(t,x,\alpha) - k)^\pm \varphi_t + \Big(F^+(t,x,u(t,x,\alpha),k) - K(t,x)\nabla(A(u(t,x)) - A(k))^\pm \Big) \nabla\varphi + \operatorname{sgn}^+(u(t,x,\alpha) - k) g\varphi \Big\} d\alpha dxdt + \int_\Omega (u_0 - k)^\pm \varphi(0,x) dx + \int_\Sigma W^\pm((t,x),k,u_b)\varphi dr(x)dt \geq 0 \tag{2.2}$$

for all $k \in [-M, M]$ and all nonnegative functions $\varphi \in \mathcal{D}([0,T) \times \mathbb{R}^d)$ with $\operatorname{sgn}^\pm(A(u_b) - A(k))\varphi = 0$ a.e. on Σ.

Let $E = \{r \,|\, A^{-1}(\cdot)\,\text{is discontinuous at}\, r\}$. We first state the following fundamental lemma.

Lemma 2.1. (1) *Under the assumptions* (A1)-(A2), *let* $u(t,x)$ *be a weak solution of problem* (1.1)-(1.2). *Then*

$$\begin{aligned}
&\int_Q \Big\{(u-k)^+\xi_t + \big(F^+(t,x,u,k) - K(t,x)\nabla(A(u)-A(k))^+\big)\nabla\xi \\
&\qquad + \operatorname{sgn}^+(u-k)g\xi\Big\}dtdx + \int_\Omega (u_0-k)^+\xi(0,x)dx \\
&= \lim_{\epsilon\to 0}\int_Q (\operatorname{sgn}^+_\epsilon)'(A(u)-A(k))\,|\sqrt{K(t,x)}\nabla A(u)|^2\xi dxdt \qquad (2.3)
\end{aligned}$$

holds for any $(k,\xi) \in \mathbb{R}\times\mathcal{D}([0,T)\times\Omega)$ *such that* $A(k)\notin E$ *and* $\xi\geq 0$, *and for any* $(k,\xi)\in[\operatorname{esssup} u_b,+\infty)\times\mathcal{D}([0,T)\times\mathbb{R}^d)$ *such that* $A(k)\notin E$ *and* $\xi\geq 0$. *Here* $\sqrt{K(t,x)} = \operatorname{diag}\{\sqrt{k_i(t,x)}\,|\, i=1,\dots,d\}$.

(2) *On the other hand,*

$$\begin{aligned}
&\int_Q \Big\{(k-u)^+\xi_t + \big(F^+(t,x,k,u) - K(t,x)\nabla(A(k)-A(u))^+\big)\nabla\xi \\
&\qquad + \operatorname{sgn}^+(k-u)g\xi\Big\}dtdx + \int_\Omega (k-u_0)^+\xi(0,x)dx \\
&= \lim_{\epsilon\to 0}\int_Q (\operatorname{sgn}^+_\epsilon)'(A(k)-A(u))|\sqrt{K(t,x)}\nabla A(u)|^2\xi dtdx \qquad (2.4)
\end{aligned}$$

holds for any $(k,\xi)\in\mathbb{R}\times\mathcal{D}([0,T)\times\Omega)$ *such that* $A(k)\notin E$ *and* $\xi\geq 0$, *and for any* $(k,\xi)\in(-\infty,\operatorname{essinf} u_b]\times\mathcal{D}([0,T)\times\mathbb{R}^d)$ *such that* $A(k)\notin E$ *and* $\xi\geq 0$.

Proof. We observe that for k such that $A(k)\notin E$, we have

$$\operatorname{sgn}^+(u-k) = \operatorname{sgn}^+(A(u)-A(k)) \text{ a.e. in } Q.$$

Let (k,ξ) be as in the Lemma. Then $\operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi\in L^2(0,T;H^1_0(\Omega))$, and

$$\begin{aligned}
&-\int_0^T\langle\partial_t u, \operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi\rangle dt \\
&= \int_Q\Big(\int_k^u \operatorname{sgn}^+_\epsilon(A(z)-A(k))dz\Big)\partial_t\xi dtdx. \qquad (2.5)
\end{aligned}$$

Since u is a weak solution and taking $\operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi$ as the test function, we have

$$\begin{aligned}
&-\int_0^T\langle\partial_t u, \operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi\rangle dt \\
&+\int_Q\Big\{(f(t,x,u)-f(t,x,k)-K(t,x)\nabla A(u))\nabla(\operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi) \\
&+ g(t,x,u)\operatorname{sgn}^+_\epsilon(A(u)-A(k))\xi\Big\}dtdx = 0. \qquad (2.6)
\end{aligned}$$

For k such that $A(k) \notin E$, we can prove that

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_Q (\int_k^u \operatorname{sgn}_\epsilon^+(A(z)-A(k))dz)\partial_t\xi dtdx \\ &= \int_Q (u-k)^+\partial_t\xi dtdx + \int_\Omega (u_0-k)^+\xi(0,x)dx \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\lim_{\epsilon\to 0}\int_Q \Big\{(f(t,x,u)-f(t,x,k)-K(t,x)\nabla A(u))\nabla(\operatorname{sgn}_\epsilon^+(A(u)-A(k))\xi)dtdx \\ &= \int_Q \Big\{F^+(t,x,u,k)-K(t,x)\nabla(A(u)-A(k))^+\Big\}\nabla\xi dtdx \\ &\quad -\lim_{\epsilon\to 0}\int_Q (\operatorname{sgn}_\epsilon^+)'(A(u)-A(k))|\sqrt{K(t,x)}\nabla A(u)|^2\xi dtdx. \end{aligned} \tag{2.8}$$

In addition,

$$\lim_{\epsilon\to 0}\int_Q g(t,x,u)\operatorname{sgn}_\epsilon^+(A(u)-A(k))\xi dtdx = \int_Q \operatorname{sgn}^+(u-k)g(t,x,u)\xi dtdx.$$

Therefore from (2.5)-(2.8) we get (2.3). In the same way we can prove (2.4). $\square$

Remark 2.1. $n(t,x) := (sgn_\epsilon^\pm)'(A(u)-A(k))|\sqrt{K(t,x)}\nabla A(u)|^2$ is called the parabolic dissipation measure, which was first introduced in [7]. It plays a crucial role in the doubling variables device for the degenerate parabolic equations.

Basing on Lemma 2.1 we will prove the following comparison result, which plays a crucial role in the proof of uniqueness of entropy solution.

Theorem 2.2 (**Comparison principle**). *For $i = 1,2$, let $(u_{0i}, g_i) \in L^\infty(\Omega)\times L^\infty(Q\times\mathbb{R}), u_b \in C(\Sigma)$. Let $u_i \in L^\infty(Q)$ be the entropy solutions of problem (1.1)-(1.2) with data (u_{0i}, u_b, g_i), respectively. Then there exists $\kappa \in L^\infty(Q)$ with $\kappa \in \operatorname{sgn}^+(u_1-u_2)$ a.e. on Q such that, for any nonnegative $\xi \in ([0,T)\times\mathbb{R}^d)$,*

$$\begin{aligned} &\int_Q \Big\{(u_1-u_2)^+\xi_t + (F^+(t,x,u_1,u_2)-K(t,x)\nabla(A(u_1)-A(u_2))^+)\nabla\xi \\ &+\kappa(g_1-g_2)\xi\Big\}dtdx + \int_\Omega (u_{01}-u_{02})^+\xi(0,x)dx + C\int_Q (u_1-u_2)^+\xi dtdx \geq 0, \end{aligned} \tag{2.9}$$

where C is a positive constant. Furthermore, taking $\xi(t,x) = 1_{(0,t)}(s)$ in (2.9), we can obtain

$$\int_\Omega (u_1-u_2)^+(t)dx \leq \int_\Omega (u_{01}-u_{02})^+dx,$$

which implies the uniqueness of entropy solution.

Theorem 2.3 (**Existence**). *Under the assumptions* $(A1)$-$(A2)$*, there exists at least one entropy process solution of problem* (1.1)-(1.2)*. As stated in Theorem 2.2 we can prove entropy process solution is unique and just is the entropy solution.*

3. Uniqueness of entropy solutions

We will use the doubling variables device to prove the comparison result. We consider u_1 as a function of $(s,y) \in Q$ and u_2 as a function of $(t,x) \in Q$. For arbitrary $\alpha > 0$, let$(B_i^\alpha)_{i=0,\ldots,m_\alpha}$ be a covering of $\overline{\Omega}$ satisfying $B_0^\alpha \cap \partial\Omega = \emptyset$, such that, for each $i \geq 1$, B_i^α is a ball of diameter less than α, contained in some lager ball $\widetilde{B_i^\alpha}$ with $\widetilde{B_i^\alpha} \cap \partial\Omega$ is part of the graph of a Lipschitz function. Let $(\phi_i^\alpha)_{i=0,\ldots,m_\alpha}$ denote the partition of unity subordinate to the covering $(B_i^\alpha)_{i=0,\ldots,m_\alpha}$.

Due to the fact that both functions u_1, u_2 satisfy the classical semi-Kruzhkov entropy inequalities for any $k \in \mathbb{R}$ in $\mathcal{D}'([0,T) \times \Omega)$, we can prove the following local comparison principle:

Lemma 3.1. *There exist* $\kappa(t,x) \in L^\infty(\Omega)$ *with* $\kappa \in \operatorname{sgn}^+(u_1 - u_2)$ *a.e. in* Q *such that, for any* $0 \leq \xi \in \mathcal{D}([0,T) \times \Omega)$,

(3.1)
$$\int_Q (u_1 - u_2)^+ \xi_t + \big(F^+(t,x,u_1,u_2) - K(t,x)\nabla(A(u_1) - A(u_2))^+\big)\nabla\xi dtdx$$
$$+ \int_Q \kappa(g_1 - g_2)\xi dtdx + \int_\Omega (u_{01} - u_{02})^+ \xi(0,x)dx \geq 0.$$

Next we will take $0 \leq \xi \in \mathcal{D}([0,T) \times \mathbb{R}^d)$. It's obvious that (3.1) holds for $\xi\phi_0^\alpha$.

For $i \in \{1,\ldots,m_\alpha\}$, we choose a sequence of mollifiers $\{\rho_n\}$ in $\mathbb{R}^d$ such that $x \to \rho_n(x-y) \in \mathcal{D}(\Omega)$ for all $x \in B$. $\sigma_n(x) = \int_\Omega \rho_n(x-y)dy$ is an increasing sequence for all $x \in B$, and $\sigma_n(x) = 1$ for all $x \in B$ with $d(x, \mathbb{R}\backslash\Omega) > \frac{C}{n}$ for some $C = C(i,\alpha)$ depending on B_i^α. Let $\{\rho_m\}$ denote a sequence of mollifiers in $\mathbb{R}$ with $supp\rho_m \subset (-\frac{2}{m}, 0)$. Define

$$(\zeta_{m,n})_i^\alpha(t,x,s,y) = \xi(t,x)\phi_i^\alpha(x)\rho_n(x-y)\rho_m(t-s),$$

then for m,n sufficiently large,

$$(3.2)\qquad \begin{aligned}(t,x) &\longmapsto (\zeta_{m,n})_i^\alpha(t,x,s,y) \in \mathcal{D}([0,T) \times \Omega) \text{ for any } (s,y) \in Q_T,\\ (s,y) &\longmapsto (\zeta_{m,n})_i^\alpha(t,x,s,y) \in \mathcal{D}((0,T) \times \mathbb{R}^d) \text{ for any } (t,x) \in Q_T.\end{aligned}$$

Moreover, the function

$$(\zeta_n)_i^\alpha(t,x) = \int_Q (\zeta_{m,n})_i^\alpha(t,x,s,y)dsdy = \xi(t,x)\phi_i^\alpha(x)\sigma_n(x)$$

satisfies $\zeta_n(t,x) \in \mathcal{D}([0,T) \times \Omega), 0 \leq \zeta_n \leq \xi$ for any n. For convenience, we omit the indexes i, α later.

Let $k_i^\alpha = \max_{B_i^\alpha \cap \Sigma} u_b(s,y)$. Since $u_1(s,y)$ satisfies (2.3), choosing $k = u_2(t,x) \vee k_i^\alpha$ and test function $\zeta_{m,n}(t,x,s,y)$, we get

$$\begin{aligned}
(3.3)\quad &\int_Q \Big\{(u_1 - u_2 \vee k_i^\alpha)^+ \partial_s \zeta_{m,n} + \Big(F^+(s,y,u_1,u_2 \vee k_i^\alpha) - K(s,y)\nabla(A(u_1) \\
&\quad - A(u_2 \vee k_i^\alpha))^+\Big)\nabla_y \zeta_{m,n} + \operatorname{sgn}^+(u_1 - u_2 \vee k_i^\alpha) g_1(s,y,u_1)\zeta_{m,n}\Big\} dsdy \\
&= \lim_{\epsilon \to 0} \int_Q (\operatorname{sgn}_\epsilon^+)'(A(u_1) - A(u_2 \vee k_i^\alpha))|\sqrt{K(s,y)}\nabla A(u_1)|^2 \xi dsdy.
\end{aligned}$$

Since $\operatorname{sgn}^+(u_1 - k_i^\alpha) = 0$ if $u_1 < k_i^\alpha$, (3.3) can be changed into

(3.4)

$$\begin{aligned}
&\int_Q \Big\{(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)^+ \partial_s \zeta_{m,n} + \Big(F^+(s,y,u_1 \vee k_i^\alpha, u_2 \vee k_i^\alpha) \\
&\quad - K(s,y)\nabla(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))^+\Big)\nabla_y \zeta_{m,n} \\
&\quad + sgn^+(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha) g_1(s,y,u_1)\zeta_{m,n}\Big\} dsdy \\
&= \lim_{\epsilon \to 0} \int_Q (sgn_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))|\sqrt{K(s,y)}\nabla A(u_1 \vee k_i^\alpha)|^2 \xi dsdy.
\end{aligned}$$

As $u_2(t,x)$ is an entropy solution of $P(u_{02}, u_b, g_2)$, choosing $k = u_1(s,y) \vee k_i^\alpha$ and $\zeta_{m,n}$ in (2.4) we obtain

(3.5)

$$\begin{aligned}
&\int_Q \Big\{(u_1 \vee k_i^\alpha - u_2)^+ \partial_t \zeta_{m,n} + \Big(F^+(t,x,u_1 \vee k_i^\alpha, u_2) - K(t,x)\nabla(A(u_1 \vee k_i^\alpha) \\
&\quad - A(u_2))^+\Big)\nabla_x \zeta_{m,n} + sgn^+(u_1 \vee k_i^\alpha - u_2) g_2(t,x,u_2)\zeta_{m,n}\Big\} dtdx \\
&= \lim_{\epsilon \to 0} \int_Q (sgn_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2))|\sqrt{K(t,x)}\nabla A(u_2)|^2 \xi dtdx.
\end{aligned}$$

According to the fact that

$$\operatorname{sgn}^+(u_1 \vee k_i^\alpha - u_2) = \operatorname{sgn}^+(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)\operatorname{sgn}^+(u_2 - k_i^\alpha) + \operatorname{sgn}^+(k_i^\alpha - u_2),$$

and

$$\begin{aligned}
&(\operatorname{sgn}_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2))|\sqrt{K(t,x)}\nabla A(u_2)|^2 \\
&\geq (\operatorname{sgn}_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))|\sqrt{K(t,x)}\nabla A(u_2 \vee k_i^\alpha)|^2,
\end{aligned}$$

(3.5) can be changed into

(3.6)

$$\begin{aligned}
&\int_Q (u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)^+ \partial_t \zeta_{m,n} + \Big(F^+(t,x,u_1 \vee k_i^\alpha, u_2 \vee k_i^\alpha) \\
&\quad - K(t,x)\nabla(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))^+\Big)\nabla_x \zeta_{m,n}
\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{sgn}^+(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)\operatorname{sgn}^+(u_2 - k_i^\alpha)g_2(t,x,u_2)\zeta_{m,n}dtdx \\
&+ \int_\Omega (u_1 \vee k_i^\alpha - u_{02} \vee k_i^\alpha)^+ \zeta_{m,n}(0,x,s,y)dx + I \\
\geq{}& \lim_{\epsilon\to 0}\int_Q (\operatorname{sgn}_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))|\sqrt{K(t,x)}\nabla A(u_2 \vee k_i^\alpha)|^2 \xi dtdx,
\end{aligned}
$$

where

$$
\begin{aligned}
I = \int_Q &\Big\{(k_i^\alpha - u_2)^+(\zeta_{m,n})_t + \Big(F^+(t,x,k_i^\alpha,u_2) \\
&- K(t,x)\nabla(A(k_i^\alpha) - A(u_2))^+\Big)\nabla_x\zeta_{m,n} \\
&+ \operatorname{sgn}^+(k_i^\alpha - u_2)g_2(t,x,u_2)\zeta_{m,n}\Big\}dtdx \\
&+ \int_\Omega (k_i^\alpha - u_{02})^+\zeta_{m,n}(0,x,s,y)dx.
\end{aligned}
$$

Integrating (3.4) in (t,x) and (3.6) in (s,y) over Q, respectively, and summing them up, we can obtain

(3.7)

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_{Q\times Q}(\operatorname{sgn}_\epsilon^+)'(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))\Big(|\sqrt{K(s,y)}\nabla_y A(u_1 \vee k_i^\alpha)|^2 \\
&+ |\sqrt{K(t,x)}\nabla_x A(u_2 \vee k_i^\alpha)|^2\Big)\zeta_{m,n}dtdxdsdy \\
\leq{}& \int_{Q\times Q}(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)^+(\partial_s + \partial_t)\zeta_{m,n} + \Big(F^+(s,y,u_1 \vee k_i^\alpha, u_2 \vee k_i^\alpha)\nabla_y\zeta_{m,n} \\
&+ F^+(t,x,u_1 \vee k_i^\alpha, u_2 \vee k_i^\alpha)\nabla_x\zeta_{m,n}\Big) - \Big(K(s,y)\nabla_y(A(u_1 \vee k_i^\alpha) \\
&- A(u_2 \vee k_i^\alpha))^+\nabla_y\zeta_{m,n} + K(t,x)\nabla_x(A(u_1 \vee k_i^\alpha) - A(u_2 \vee k_i^\alpha))^+\nabla_x\zeta_{m,n}\Big) \\
&+ \operatorname{sgn}^+(u_1 \vee k_i^\alpha - u_2 \vee k_i^\alpha)\operatorname{sgn}^+(u_1 - k_i^\alpha)(g_1 - \operatorname{sgn}^+(u_2 - k_i^\alpha)g_2)\zeta_{m,n}dtdxdsdy \\
&+ \int_{Q\times\Omega}(u_1 \vee k_i^\alpha - u_{02} \vee k_i^\alpha)^+\zeta_{m,n}(0,x,s,y)dxdsdy + \int_Q Idsdy =: \sum_{i=1}^6 E_i.
\end{aligned}
$$

Notice the facts that

$$
\begin{aligned}
(\partial_t + \partial_s)\zeta_{m,n} &= \partial_t\xi(t,x)\phi(x)\rho_n(x-y)\rho_m(t-s), \\
(\nabla_x + \nabla_y)\zeta_{m,n} &= \nabla_x(\xi(t,x)\phi(x))\rho_n(x-y)\rho_m(t-s), \\
(\nabla_{xx}^2 + \nabla_{xy}^2 + \nabla_{yy}^2)\zeta_{m,n} &= \nabla_{xx}^2(\xi(t,x)\phi(x))\rho_n(x-y)\rho_m(t-s).
\end{aligned}
$$

We have

$$
\begin{aligned}
E_2 = \int_{Q\times Q} &F^+(t,x,u_1 \vee k_i^\alpha, u_2 \vee k_i^\alpha)\nabla_x(\xi(t,x)\phi(x))\rho_n(x-y) \\
&\rho_m(t-s)dtdxdsdy
\end{aligned}
$$

$$+\int_{Q\times Q}(F^+(s,y,u_1\vee k_i^\alpha,u_2\vee k_i^\alpha)-F^+(t,x,u_1\vee k_i^\alpha,u_2\vee k_i^\alpha))$$
$$\xi\phi\nabla\rho_n\rho_m dtdxdsdy.$$

Moreover,

$$\begin{aligned}E_3 = &\int_{Q\times Q}\Big(K(s,y)\nabla_y(A(u_1\vee k_i^\alpha)-A(u_2\vee k_i^\alpha))^+ + K(t,x)\nabla_x(A(u_1\vee k_i^\alpha)\\ &-A(u_2\vee k_i^\alpha))^+\Big)\nabla^2_{xx}(\xi(t,x)\phi(x))\rho_n(x-y)\rho_m(t-s)dtdxdsdy\\ &-\int_{Q\times Q}(\mathrm{sgn}_\epsilon^+)'(A(u_1\vee k_i^\alpha)-A(u_2\vee k_i^\alpha))\nabla_y A(u_1\vee k_i^\alpha)\\ &\quad(K(s,y)+K(t,x))\nabla_x A(u_2\vee k_i^\alpha)\zeta_{m,n}dtdxdsdy\\ =: &\, E_{31}-E_{32}.\end{aligned}$$

Denote the left-side term in (3.7) as E_0, it's easy to prove

$$\lim_{m,n\to\infty}(E_0+E_{32})\geq 0.$$

Combining the preceding estimates and passing to the limit as m,n to $+\infty$, we can obtain

(3.8)
$$\begin{aligned}&\int_Q(u_1\vee k_i^\alpha-u_2\vee k_i^\alpha)^+\xi_t\phi_i+\Big(F^+(t,x,u_1\vee k_i^\alpha,u_2\vee k_i^\alpha)-K(t,x)\\ &\nabla_x(A(u_1\vee k_i^\alpha)-A(u_2\vee k_i^\alpha))^+\Big)\nabla_x(\xi\phi_i)+C\int_Q\xi\phi_i(u_1\vee k_i^\alpha-u_2\vee k_i^\alpha)^+dtdx\\ &+\int_Q\kappa_1\mathrm{sgn}^+(u_1-k_i^\alpha)(g_1-\mathrm{sgn}^+(u_2-k_i^\alpha)g_2)\xi\phi_i\\ &+\int_\Omega(u_{01}\vee k_i^\alpha-u_{02}\vee k_i^\alpha)^+\xi(0,x)\phi_i(x)dx+\lim_{n\to\infty}\Upsilon(\zeta_n)\geq 0,\end{aligned}$$

where

$$\begin{aligned}\Upsilon(\zeta_n):=&\int_Q\Big\{(k_i^\alpha-u_2)^+(\zeta_n)_t+\Big(F^+(t,x,k_i^\alpha,u_2)\\ &-K(t,x)\nabla(A(k_i^\alpha)-A(u_2))^+\Big)\nabla_x\zeta_n+\mathrm{sgn}^+(k_i^\alpha-u_2)g_2\zeta_n\Big\}dtdx\\ &+\int_\Omega(k_i^\alpha-u_{02})^+\zeta_n(0,x)dx.\end{aligned}$$

Next we will prove the other half estimate for $(u_1\wedge\widetilde{k_i^\alpha}-u_2\wedge\widetilde{k_i^\alpha})^+$ with $\widetilde{k_i^\alpha}=\min_{B_i^\alpha\cap\Sigma}u_b$. We would like to choose the test function as in (3.2). Since u_1

satisfies (2.1), choosing $k = u_2(t,x) \wedge \widetilde{k_i^\alpha}$, we can obtain

(3.9)
$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_Q (\mathrm{sgn}_\epsilon^+)'(A(u_1) - A(u_2\wedge \widetilde{k_i^\alpha}))|\sqrt{K(s,y)}\nabla_y A(u_1)|^2\zeta_{m,n}dsdy\\
&= \int_Q (u_1(s,y) - u_2\wedge \widetilde{k_i^\alpha})^+\partial_s\zeta_{m,n} + \Big(F^+(s,y,u_1,u_2\wedge \widetilde{k_i^\alpha})\\
&\quad - K(s,y)\nabla_y(A(u_1) - A(u_2\wedge \widetilde{k_i^\alpha}))^+\Big)\nabla_y\zeta_{m,n}\\
&\quad + \mathrm{sgn}^+(u_1 - u_2\wedge \widetilde{k_i^\alpha})g_1\zeta_{m,n}dsdy + \int_\Omega (u_{01} - u_2\wedge \widetilde{k_i^\alpha})^+\zeta_{m,n}(t,x,0,y)dy\\
&= \int_Q (u_1(s,y)\wedge \widetilde{k_i^\alpha} - u_2\wedge \widetilde{k_i^\alpha})^+\partial_s\zeta_{m,n} + \Big(F^+(s,y,u_1\wedge \widetilde{k_i^\alpha},u_2\wedge \widetilde{k_i^\alpha})\\
&\quad - K(s,y)\nabla_y(A(u_1\wedge \widetilde{k_i^\alpha}) - A(u_2\wedge \widetilde{k_i^\alpha}))^+\Big)\nabla_y\zeta_{m,n}\\
&\quad + \mathrm{sgn}^+(u_1\wedge \widetilde{k_i^\alpha} - u_2\wedge \widetilde{k_i^\alpha})\mathrm{sgn}^+(k_i^\alpha - u_1)g_1(s,y,u_1)\zeta_{m,n}dsdy\\
&\quad + \int_\Omega (u_{01}\wedge \widetilde{k_i^\alpha} - u_2\wedge \widetilde{k_i^\alpha})^+\zeta_{m,n}(t,x,0,y)dy + II,
\end{aligned}
$$

where

$$
\begin{aligned}
II := &\int_Q (u_1 - \widetilde{k_i^\alpha})^+\partial_s\zeta_{m,n} + \Big(F^+(s,y,u_1,\widetilde{k_i^\alpha})\\
&- K(s,y)\nabla_y(A(u_1) - A(\widetilde{k_i^\alpha}))^+\Big)\nabla_y\zeta_{m,n}\\
&+ \mathrm{sgn}^+(u_1 - \widetilde{k_i^\alpha})g_1(s,y,u_1)\zeta_{m,n}dsdy + \int_\Omega (u_{01} - \widetilde{k_i^\alpha})^+\zeta_{m,n}(t,x,0,y)dy.
\end{aligned}
$$

As $u_2(t,x)$ is an entropy solution, choosing $k = u_1(s,y)\wedge \widetilde{k_i^\alpha}$ and test function $\zeta_{m,n}$ in (2.4) yields

(3.10)
$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int_Q (\mathrm{sgn}_\epsilon^+)'(A(u_1(s,y)\wedge \widetilde{k_i^\alpha}) - A(u_2))|\sqrt{K(t,x)}\nabla_x A(u_2)|^2\zeta_{m,n}dtdx\\
&= \int_Q (u_1(s,y)\wedge \widetilde{k_i^\alpha} - u_2)^+\partial_t\zeta_{m,n} + \Big(F^+(s,y,u_1(s,y)\wedge \widetilde{k_i^\alpha},u_2)\\
&\quad - K(t,x)\nabla_x(A(u_1\wedge \widetilde{k_i^\alpha}) - A(u_2))^+\Big)\nabla_x\zeta_{m,n}\\
&\quad + \mathrm{sgn}^+(u_1(s,y)\wedge \widetilde{k_i^\alpha} - u_2)g_2(t,x,u_2)\zeta_{m,n}dtdx\\
&= \int_Q (u_1(s,y)\wedge \widetilde{k_i^\alpha} - u_2\wedge \widetilde{k_i^\alpha})^+\partial_t\zeta_{m,n} + \Big(F^+(s,y,u_1(s,y)\wedge \widetilde{k_i^\alpha},u_2\wedge \widetilde{k_i^\alpha})\\
&\quad - K(t,x)\nabla_x(A(u_1(s,y)\wedge \widetilde{k_i^\alpha}) - A(u_2)\wedge \widetilde{k_i^\alpha})^+\nabla_x\zeta_{m,n}\\
&\quad + \mathrm{sgn}^+(u_1\wedge \widetilde{k_i^\alpha} - u_2\wedge \widetilde{k_i^\alpha})g_2(t,x,u_2)\zeta_{m,n}dtdx.
\end{aligned}
$$

We integrate (3.9) in (t,x) and (3.10) in (s,y) over Q, respectively, and add them up. Taking the limit $m,n\to+\infty$, we can obtain

(3.11)

$$\begin{aligned}
&\int_Q \Big\{(u_1\wedge \widetilde{k_i^\alpha}-u_2\wedge \widetilde{k_i^\alpha})^+\partial_t\xi\phi_i+\Big(F^+(t,x,u_1\wedge \widetilde{k_i^\alpha},u_2\wedge \widetilde{k_i^\alpha})\\
&-K(t,x)\nabla_x(A(u_1\wedge \widetilde{k_i^\alpha})-A(u_2\wedge \widetilde{k_i^\alpha}))^+\Big)\nabla_x(\xi\phi_i)+\kappa(g_1-g_2)\xi\phi_i\Big\}dtdx\\
&+\int_\Omega (u_{01}\wedge \widetilde{k_i^\alpha}-u_{02}\wedge \widetilde{k_i^\alpha})^+\xi(0,x)\phi_i(x)dx\\
&+C\int_Q (u_1\wedge \widetilde{k_i^\alpha}-u_2\wedge \widetilde{k_i^\alpha})^+\xi\phi_i dtdx+\lim_{n\to\infty}\widetilde{\Upsilon}(\zeta_n)\ge 0,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\Upsilon}(\zeta_n):=&\int_Q (u_1-\widetilde{k_i^\alpha})^+\partial_t\zeta_n+\Big(F^+(t,x,u_1,\widetilde{k_i^\alpha})\\
&-K(t,x)\nabla_x(A(u_1)-A(\widetilde{k_i^\alpha}))^+\Big)\nabla_x\zeta_n\\
&+\mathrm{sgn}^+(u_1-\widetilde{k_i^\alpha})g_1(t,x,u_1)\zeta_n dsdy+\int_\Omega (u_{01}-\widetilde{k_i^\alpha})^+\zeta_n(0,x)dy.
\end{aligned}$$

Now for any $\varepsilon>0$, we can choose $\alpha>0$ such that for arbitrary $(t,x),(s,y)\in B_i^\alpha$ with $d((t,x),(s,y))<\alpha$, $|k_i^\alpha-\widetilde{k_i^\alpha}|\le\varepsilon$. Thus with such covering of $\overline{\Omega}$,

$$(u_1\vee k_i^\alpha-u_2\vee k_i^\alpha)^++(u_1\wedge \widetilde{k_i^\alpha}-u_2\wedge \widetilde{k_i^\alpha})^+=(u_1-u_2)^++O(\varepsilon).$$

Therefore, summation of (3.8) and (3.11) yields

$$\begin{aligned}
(3.12)\qquad &\int_Q \Big\{(u_1-u_2)^+\xi_t\phi_i+\Big(F^+(t,x,u_1,u_2)\\
&-K(t,x)\nabla_x(A(u_1)-A(u_2))^+\Big)\nabla_x(\xi\phi_i)+\kappa(g_1-g_2)\xi\phi_i\Big\}dtdx\\
&+\int_\Omega (u_{01}-u_{02})^+\xi(0,x)\phi_i(x)dx\\
&+C\int_Q (u_1-u_2)^+\xi\phi_i dtdx+\lim_{n\to\infty}\Upsilon(\zeta_n)+\lim_{n\to\infty}\widetilde{\Upsilon}(\zeta_n)\ge 0.
\end{aligned}$$

According to the superposition principle we have

(3.13)

$$\begin{aligned}
&\Lambda(u_1,u_2,\xi)\\
=&\int_Q \Big\{(u_1-u_2)^+\xi_t+\Big(F^+(t,x,u_1,u_2)-K(t,x)\nabla_x(A(u_1)-A(u_2))^+\Big)\nabla_x\xi\\
&+\kappa(g_1-g_2)\xi\Big\}dtdx+\int_\Omega (u_{01}-u_{02})^+\xi(0,x)dx+C\int_Q (u_1-u_2)^+\xi dtdx\\
\ge&-\lim_{n\to\infty}\Upsilon(\zeta_n)-\lim_{n\to\infty}\widetilde{\Upsilon}(\zeta_n)
\end{aligned}$$

for any $0 \le \xi \in \mathcal{D}([0,T)\times\mathbb{R}^d)$. Using the local comparison principle with test function $\xi_{m'}(t,x) = \xi(t,x)\sigma_{n'}(x)$, we have

$$\Lambda(u_1,u_2,\xi_{n'}) \ge 0 \quad \text{for each } n'.$$

Hence

$$\begin{aligned}\Lambda(u_1,u_2,\xi) &= \Lambda(u_1,u_2,\xi_{n'}) + \Lambda(u_1,u_2,\xi(1-\chi_{n'})) \ge \Lambda(u_1,u_2,\xi(1-\sigma_{n'}))\\ &\ge -\lim_{n\to\infty}\Upsilon(\xi\sigma_n(1-\sigma_{n'})) - \lim_{n\to\infty}\widetilde{\Upsilon}(\xi\sigma_n(1-\sigma_{n'})).\end{aligned}$$

Since

$$\sigma_n(1-\sigma_{n'}) = \sigma_n - \sigma_{n'} \quad \text{for } n \ge n',$$

then passing to the limit as $n, n' \to \infty$, we can get

$$\Lambda(u_1,u_2,\xi) \ge 0, \quad \forall \xi \in \mathcal{D}([0,T)\times\mathbb{R}^d), \quad \xi \ge 0,$$

which complete the proof of Theorem 2.2. In the same way we can prove the uniqueness of entropy process solutions.

4. Existence of entropy process solution

Consider the approximated equation

(4.1)
$$\begin{cases} \partial_t u^\epsilon + \mathrm{div} f(t,x,u^\epsilon) - \nabla\cdot(K(t,x)\nabla A^\epsilon(u^\epsilon)) = q(t,x,u^\epsilon), & (t,x)\in Q,\\ u(t,x) = u_b(t,x,) & (t,x)\in\Sigma,\\ u(0,\cdot) = u_0^\epsilon, & x\in\Omega,\end{cases}$$

where $A^\epsilon(u^\epsilon) = A(u^\epsilon) + \epsilon u^\epsilon$, and u_0^ϵ is the smooth approximated function of u_0 such that $\lim\limits_{\epsilon\to 0} u_0^\epsilon(x) = u_0(\mathrm{x})$ in $L^1(\Omega)$. We also assume $u_0^\epsilon(x)$ and $u_b(t,x)$ satisfy compatibility conditions on $\overline{\Sigma}\cap\overline{\Omega}$. For $\epsilon > 0$ fixed, the existence of a unique weak solution u^ϵ of problem (4.1) is a classical result according to the pioneering work of Leray-Lions. The main goal of this section is to show that if $u_0^\epsilon(x)$ approximates $u_0(x)$ in $L^1(\Omega)$, then the sequence of solution u^ϵ of problem (4.1) approximates a function $u(t,x,\alpha) \in L^\infty(Q\times(0,1))$, which is just the entropy process solution of problem (1.1). Furthermore, according to the uniqueness of entropy process solution we reduce that the entropy process solution is just the entropy solution of problem(1.1).

First, let $M = \max\big\{\sup\limits_{\Sigma} u_b(t,x), \sup\limits_{\Omega} u_0^\epsilon(x)\big\}$,

$$\varphi_\delta(z) = \begin{cases} \Big((z-M)^2+\delta^2\Big)^{\frac{1}{2}} - \delta, & \text{for } z \ge M,\\ 0, & \text{for } z < M.\end{cases}$$

Then $\varphi_\delta(z) \to (z-M)^+$ as $\delta \to 0$. Multiplying (4.1) by $\varphi'_\delta(u^\epsilon)$, we can obtain

(4.2)
$$\int_\Omega \varphi_\delta(u^\epsilon(t,x))dx = \int_0^t\int_\Omega \varphi''_\delta(u^\epsilon)\Big(\nabla u^\epsilon f(t,x,u^\epsilon) - K(t,x)\nabla u^\epsilon\nabla A^\epsilon(u^\epsilon)\Big)dxdt$$

$$+\int_0^t\int_\Omega g(t,x,u^\epsilon)\varphi'_\delta(u^\epsilon)dxdt.$$

The first term of the right side can be estimated by $\frac{M^2\delta}{4\epsilon}$ with Young inequality. Let $\delta\to 0$ in (4.2), we have

$$\int_\Omega \max\big\{u^\epsilon - M, 0\big\}dx \le \int_0^t\int_\Omega ||g||_{L^\infty}dxd\tau,$$

which implies that

$$u^\epsilon(t,x)\le M+\int_0^T ||g(t,x,u^\epsilon)||_{L^\infty(\Omega)}dt. \tag{4.3}$$

In the same way we can prove that

$$u^\epsilon(t,x)\ge \min\big\{\inf_\Sigma u_b(t,x), \inf_\Omega u_b^\epsilon(x)\big\}-\int_0^T ||g(t,x,u^\epsilon)||_{L^\infty(\Omega)}dt. \tag{4.4}$$

Thus the sequence $\{u^\epsilon\}$ is bounded uniformly.

Next, taking any convex entropy $\eta(u^\epsilon)$ and multiplying (4.1) by $\eta'(u^\epsilon)$ we get

$$\begin{aligned}&\partial_t\eta(u^\epsilon)+\operatorname{div}\big(\eta'(u^\epsilon)f(t,x,u^\epsilon)\big)-\eta''(u^\epsilon)\nabla u^\epsilon f(t,x,u^\epsilon)\\&\quad-\operatorname{div}\big(\eta'(u^\epsilon)K(t,x)\nabla A^\epsilon(u^\epsilon)\big)+\eta'(u^\epsilon)g(t,x,u^\epsilon)\\&\le\ -\eta''(u^\epsilon)K(t,x)\nabla u^\epsilon\nabla A^\epsilon(u^\epsilon)\end{aligned}$$

holds in $\mathcal{D}([0,T)\times\mathbb{R}^d)$. Let $\eta(u^\epsilon)=\frac{(u^\epsilon)^2}{2}$ and take the test function $\varphi=1_Q$, we have

$$\int_Q K(t,x)\nabla u^\epsilon\nabla A(u^\epsilon)dtdx\le+\infty,$$

which implies $\sqrt{K(t,x)}\nabla A(u^\epsilon)\in L^2(Q)$, where $\sqrt{K(t,x)}=\operatorname{diag}\{\sqrt{k_i(t,x)}, i=1,\dots,d\}$. Since $k_i\ge C>0$ for any $(t,x)\in Q$, it follows that

$$A(u^\epsilon)\in L^2(0,T;H^1(\Omega)). \tag{4.5}$$

Let us introduce the function

$$u(t,x)=\int_0^1\mu(t,x,\alpha)d\alpha\quad \text{for a.e.}\,(t,x)\in Q.$$

Thanks to the convergence of $A(u^\epsilon)$, we can identify the limit of $A(u^\epsilon)$ with respect to $\epsilon\to 0$ with $A(u(t,x))$, which is actually independent of $\alpha\in(0,1)$.

For the entropy condition, one can easily prove that for all $k\in[-M,M]$, for all $\varphi\in\mathcal{D}([0,T)\times\mathbb{R}^d)$ such that $\varphi\ge 0$ and $\operatorname{sgn}^\pm(A(u_b)-A(k))\varphi=0$ a.e. on Σ,

$$\begin{aligned}&\int_Q\Big\{(u^\epsilon-k)^\pm\varphi_t+\Big(F_k^\pm(t,x,u^\epsilon)-K(t,x)\nabla(A(u^\epsilon)-A(k))^\pm\Big)\nabla\varphi\\&\quad+\operatorname{sgn}^\pm(u^\epsilon-k)g(t,x,u^\epsilon)\varphi\Big\}dtdx+\int_\Omega(u_0^\epsilon-k)^\pm\varphi(0,x)dx\end{aligned}\tag{4.6}$$

$$+ \int_\Sigma W^\pm((t,x),k,u_b)\varphi dr(x)dt \geq 0.$$

With the former estimations (4.3)-(4.5) we can pass the limit $\epsilon \to 0$ in the above entropy inequality to get

$$\int_Q \int_0^1 \Big\{(u(t,x,\alpha)-k)^\pm \varphi_t + \Big(F_k^+(t,x,u(t,x,\alpha))$$
$$- K(t,x)\nabla(A(u(t,x)) - A(k))^\pm\Big)\nabla\varphi + \operatorname{sgn}^+(u(t,x,\alpha)-k)g\varphi\Big\}d\alpha dt dx$$
$$+ \int_\Omega (u_0 - k)^\pm \varphi(0,x)dx + \int_\Sigma W^\pm((t,x),k,u_b)\varphi dr(x)dt \geq 0,$$

hold for all $k \in [-M, M]$ and all nonnegative functions $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^d)$ such that $\varphi \geq 0$ and $\operatorname{sgn}^\pm(A(u_b) - A(k)) = 0$ a.e. on Σ. The proof of Theorem 2.3 is completed.

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