IEIE Transactions on Smart Processing and Computing

Review of the Application of Wavelet Theory to Image Processing

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Received November 16, 2016; Accepted December 6, 2016; Published December 30, 2016

** Regular Paper*

** Review Paper: This paper reviews the recent progress possibly including previous works in a particular research topic, and has been accepted by the editorial board through the regular reviewing process.*

***Abstract*:** This paper reviews recent published works dealing with the application of wavelets to image processing based on multiresolution analysis. After revisiting the basics of wavelet transform theory, various applications of wavelets and multiresolution analysis are reviewed, including image denoising, image enhancement, super-resolution, and image compression. In addition, we introduce the concept and theory of quaternion wavelets for the future advancement of wavelet transform and quaternion multiresolution applications.

***Keywords*:** Wavelets, Wavelet transform, Multiresolution analysis, Quaternion wavelets, Image processing

1. Introduction

The word wavelet was first introduced by Grossmann and Morlet [1] in the early 1980s. They used the French word *ondelette*, meaning small wave. Wavelets have recently become a popular topic of conversation in many scientific and engineering communities. Some view wavelets as a new basis for representing functions; some consider them a technique for time–frequency analysis, and others think of them as a new mathematics subject. Of course, all are justifiable, since the wavelet is a versatile tool with very rich mathematical content and great potential for applications.

Wavelets are already recognized as a powerful new mathematical tool in signal and image processing, time series analysis, geophysics, approximation theory, and many other areas. First of all, wavelets were introduced in seismology to provide a time dimension to seismic spectral analysis where Fourier analysis fails. Fourier analysis is ideal for studying stationary data (data whose statistical properties are invariant over time), but it is not well suited to analyzing data with transient events that cannot be statistically predicted from previous data. Since the wavelet theory was designed with such non-stationary data in mind, its generality and strong results have quickly become useful in a number of disciplines. The wavelet transform has been perhaps the most exciting development in the decade, bringing together researchers in several different fields, such as signal processing, quantum mechanics, image processing, communications, computer science, and mathematics [1-77].

At present, wavelet is not only the workspace in computer imaging and animation, but, for example, they are also used by the FBI to encode its database of a million fingerprints. In the future, scientists may use wavelet analysis to diagnose breast cancer, look for heart abnormalities, and predict the weather; for data compression, smoothing, and image compression; for fingerprint verification, DNA analysis, protein analysis, plus blood-pressure, heart-rate, and ECG analysis; as well as in finance, internet traffic descriptions, speech recognition, computer graphics, and many other fields. Wavelet analysis provides additional freedom, as compared to Fourier analysis, since the choice of atoms in the transform deduced from the analyzing wavelet is left to the user.

In this review, we summarize the concepts of wavelet theory in Section 2 as for image processing prospects.

Previous research in the area of wavelet applications in image processing is summarized in Section 3. In Section 4, a new developing area of wavelets, called quaternion wavelets, is discussed in detail. Finally, conclusions are drawn and future directions are suggested in Section 5.

2. Wavelet Theory

Wavelet theory involves representing general functions in terms of simpler building blocks at different scales and positions. The fundamental idea behind the wavelet transform is analyze according to scale. The wavelet transform was first introduced in the context of a mathematical transform by Grossman and Morlet in 1984 [1].

Definition 2.1. The continuous wavelet transform (CWT) of a function $f(x) \in L^2(R)$ with respect to $\psi(x) \in L^2(R)$ is

$$CWT(f)(a,b) = |a|^{j/2} \int_{-\infty}^{\infty} f(x)\psi(a(x-b))\,dx, \qquad (1)$$
$$a, b \in R, \quad a \neq 0, \quad j \in Z$$

Definition 2.2. The discrete wavelet transform (DWT) of a function $f(x) \in L^2(R)$ with respect to $\psi(x) \in L^2(R)$ is

$$DWT(f)(j,k) = |a_0|^{j/2} \int_{-\infty}^{\infty} f(x)\psi(a_0{}^j x - kb_0)\,dx, \qquad (2)$$
$$a_0, b_0 \in R, \quad a_0 \neq 0, \quad j,k \in Z$$

For the discrete wavelet transform, a_0 *and* b_0 are usually 2 and 1. In such case, the DWT becomes

$$DWT(f)(j,k) = 2^{j/2} \int_{-\infty}^{\infty} f(x)\psi(2^j x - k)\,dx. \qquad (3)$$

The notion of wavelets came into being because Fourier analysis, which depends on oscillating building blocks, is poorly suited to signals that change suddenly. A wavelet [2-6] is crudely a function which together with its dilates and their translates, determines all functions of our needs.

Definition 2.3. A function $\psi(x) \in L^2(R)$ is called an orthonormal wavelet if the system $\{\psi_{j,k}\}_{j,k\in Z}$ forms an orthonormal basis for $L^2(R)$, where $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}(x-k))$.

The equivalent conditions for wavelets are:

1) $\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty,$

2) $\int_{-\infty}^{\infty} |\psi(x)| dx = 0,$

3) $\int_{-\infty}^{\infty} \frac{|\psi(\xi)|^2}{|\xi|} d\xi < \infty,$

where $\hat{\psi}(\xi)$ is the Fourier transform of $\psi(x)$. Condition 3) is called the admissibility condition. The Haar wavelet (1910) is the oldest wavelet, which has limited application, as it is not continuous. To suit for approximating data with sharp discontinuities, wavelets that could automatically adapt to different components of a signal were targeted for investigation in early 1980s. Daubechies brought a big breakthrough in 1988 [3], and her work immediately stimulated a rapid development in the theory and application of wavelet analysis. It was 1989 [7, 8] when Mallat discovered the underlying concept for obtaining orthonormal wavelets, now popularly known as multiresolution analysis (MRA).

Definition 2.4. A Multiresolution analysis(MRA) is a sequence of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2(R)$ satisfying

a. $V_j \subset V_{j+1}$ for all $j \in Z$,
 or $V_j \subset V_{j-1}$ for all $j \in Z$,

b. $f \in V_j \Leftrightarrow f(2\bullet) \in V_{j+1}$ for all $j \in Z$,

c. $\cap_{j\in Z} V_j = \{0\}$,

d. $\cup_{j\in Z} V_j = L^2(R)$,

e. and there exist a function $\phi \in V_0$ (called a scaling function, or the father wavelet) such that $\{\phi(\bullet - k) : k \in Z\}$ is an orthonormal basis for V_0.

We note that an orthonormal basis for V_j, $j \in Z$, is given by translates of normalized dilations $\{2^{j/2}\phi(2^j(\bullet - k)) : k \in Z\}$ of ϕ. A multiresolution analysis gives

(1) an orthogonal direct sum decomposition of $L^2(R)$, and

(2) a wavelet called an MRA wavelet, or the mother wavelet.

Let W_0 be the orthogonal complement of V_0 in V_1; that is, $V_1 = V_0 \oplus W_0$. Then, if we dilate the elements of W_0 by 2^j, we obtain a closed subspace W_j of V_{j+1}, such that:

$$V_{j+1} = V_j \oplus W_j \text{ for each } j \in Z, \qquad (4)$$

Conditions (c) and (d) of the definition of an MRA provide

$$L^2(R) = \oplus_{j=-\infty}^{\infty} W_j. \qquad (5)$$

A wavelet arose through an MRA is known as an MRA wavelet. The Haar wavelet, the Shannon wavelet, and Daubechies wavelets are among the MRA wavelets. The

Journé wavelet is a non-MRA wavelet.

In the dyadic scale space, if we denote $A_j f$ as the approximation of a given $f(x)$ at the scale 2^{-j}, we express the difference between two successive approximations as

$$D_j f = A_{j-1} f - A_j f \tag{6}$$

where j stands for the number of a certain level. In general, the function $f(x)$ can be decomposed as

$$\begin{aligned} f(x) &= A_1 f + D_1 f = A_2 f + D_2 f + D_1 f \\ &= .... \\ &= A_n f + \sum_{j=1}^{n} D_j f. \end{aligned} \tag{7}$$

where
$$\begin{aligned} A_j f(x) &= \sum_{k \in Z} a_{j,k}(x) \phi_{j,k}(x), \\ D_j f(x) &= \sum_{k \in Z} b_{j,k}(x) \psi_{j,k}(x) \end{aligned} \tag{8}$$

and
$$\begin{aligned} a_{j,k}(x) &= \langle f(x), \phi_{j,k}(x) \rangle, \\ d_{j,k}(x) &= \langle f(x), \psi_{j,k}(x) \rangle. \end{aligned} \tag{9}$$

The multiresolution analysis of two-dimensional (2D) function $f(x,y) \in L^2(R^2)$ can be defined straightforwardly as an extension of the above equations:

$$\begin{aligned} f(x,y) &= A_1 f + \sum_{p=1}^{3} D_{1,p} f \\ &= A_2 f + \sum_{p=1}^{3} D_{2,p} f + \sum_{p=1}^{3} D_{1,p} f \\ &= .... \\ &= A_n f + \sum_{j=1}^{n} \left[D_{j,1} f + D_{j,2} f + D_{j,3} f \right]. \end{aligned} \tag{10}$$

We can characterize each approximation function $A_j f(x,y)$ and the difference components $D_{j,p} f(x,y)$, $p = 1,2,3$ by means of two-dimensional scaling function $\Phi(x,y)$ and its associated wavelet functions, $\Psi_p(x,y)$, as follows:

$$\begin{aligned} A_j f(x,y) &= \sum_{l \in Z} \sum_{k \in Z} a_{j,k,l} \Phi_{j,k,l}(x,y), \\ D_{j,p} f(x,y) &= \sum_{l \in Z} \sum_{k \in Z} d_{j,p,k,l} \Psi_{j,p,k,l}(x,y) \end{aligned} \tag{11}$$

where

$$\Phi_{j,k,l}(x,y) = 2^{-j} \Phi(2^{-j}(x-k),\ 2^{-j}(y-l)), \quad (j,\ k,\ l) \in Z^3, \tag{12}$$

$$\Psi_{j,p,k,l}(x,y) = 2^{-j} \Psi_p(2^{-j}(x-k),\ 2^{-j}(y-l)), \quad p = 1,2,3,\ \ (j,\ k,\ l) \in Z^3, \tag{13}$$

and

$$\begin{aligned} a_{j,k,l}(x,y) &= \langle f(x,y), \Phi_{j,k,l}(x,y) \rangle, \\ d_{j,p,k,l}(x,y) &= \langle f(x,y), \Psi_{j,p,k,l}(x,y) \rangle \end{aligned} \tag{14}$$

and in order to carry out a separable multiresolution analysis, we decompose scaling function $\Phi(x,y)$ and wavelet function $\Psi_p(x,y)$ as follows:

$$\begin{aligned} \Phi(x,y) &= \phi(x)\phi(y), \\ \Psi_1(x,y) &= \phi(x)\psi(y), \\ \Psi_2(x,y) &= \psi(x)\phi(y), \\ \text{and}\ \ \Psi_3(x,y) &= \psi(x)\psi(y), \end{aligned} \tag{15}$$

where function ϕ is a one-dimensional scale function, and function ψ is its associated wavelet function.

Most of the concepts in the theory of wavelets, such as orth $p = 1,2,3$ onormal wavelets, multiresolution analysis, MRA wavelets, scaling function, etc., can be introduced for $L^2(R^n)$, where n is a natural number greater than 1, by considering an $n \times n$ expansive matrix A with entries as integers for the dilation factor. Results as expected are obtained.

Often, signals we wish to process are in the time domain, but in order to process them more easily, other information (such as frequency) is required. Mathematical transforms convert the information of signals into different representations. For example, the Fourier transform converts a signal from the time domain to the frequency domain such that the frequencies of a signal can be seen. However, the Fourier transform cannot provide information on which frequencies occur at specific times in the signal, as time and frequency are viewed independently. To overcome this problem the short-term Fourier transform (STFT) introduced the idea of windows, through which different parts of a signal are viewed. For a given window in time, the frequencies can be viewed. However Heisenberg's Uncertainty Principle states that, as the resolution of the signal improves in the time domain, by zooming in on different sections, the frequency resolution gets worse. So, a method of multiresolution is needed that allows certain parts of the signal to be resolved well in time, and other parts to be resolved well in frequency. The power and magic of wavelets comes from the use of multiresolution. Rather than examine the entire signal through the same window, different parts of the signal are viewed through different-sized windows (or resolutions). High-frequency parts of the signal use a small window to give good time resolution; low-frequency parts use a big window to get good frequency information.

The discrete wavelet transform (DWT) provides sufficient information for the analysis and synthesis of a signal, but is advantageously much more efficient. Discrete wavelet analysis is computed using the concept of filter banks. Filters of different cut-off frequencies analyze the signal at different scales. Resolution is changed by the filtering, the scale is changed by upsampling and

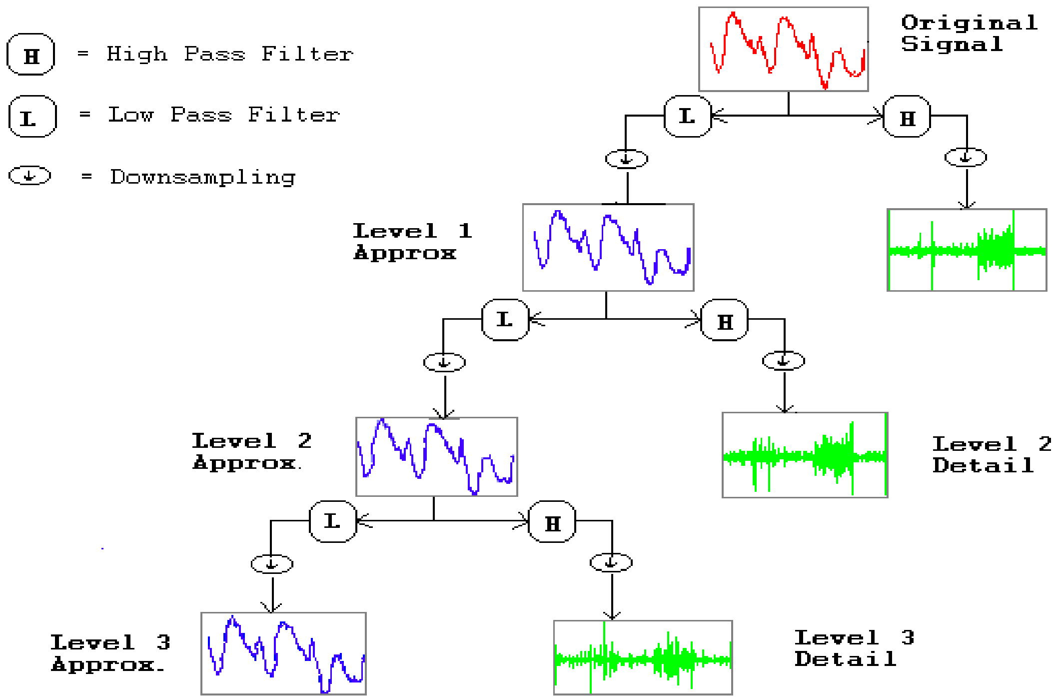

Fig. 1.

downsampling. If a signal is put through two filters (a high-pass filter and a low-pass filter), then the signal is effectively decomposed into two parts, a detail part (high frequency), and an approximation part (low frequency). A high-pass filter means high-frequency information is kept and low-frequency information is lost. Similarly, a low-pass filter means low-frequency information is kept, and high-frequency information is lost. The subsignal produced from the low filter will have a highest frequency equal to half that of the original signal. This change in frequency range means that only half of the original sample needs to be kept in order to perfectly reconstruct the signal. More specifically, this means that upsampling can be used to remove every second sample. The scale has now been doubled. The resolution has also been changed, and the filtering made the frequency resolution better, but reduced the time resolution. The approximation subsignal can then be put through a filter bank, and this is repeated until the required level of decomposition has been reached. The ideas are shown in Fig. 1.

Images contain large amounts of information that require a lot of storage space, large transmission bandwidths, and long transmission times. An image can be thought of as a matrix of pixel values. Images are treated as 2D signals. Two-dimensional wavelet analysis can be used to divide the information of an image into approximation and detail subsignals. The approximation subsignal shows the general trend of pixel values, and three detail subsignals show the vertical, horizontal, and diagonal details (or changes) in the image. If these details are very small, then they can be set to zero without changing the image. The value below which details are considered small enough to be set to zero is known as the threshold. Two-dimensional wavelet analysis uses the same mother wavelets, but requires an extra step at every level of decomposition.

One-dimensional wavelet analysis filters out the high-frequency information from the low-frequency information at every level of decomposition, so only two subsignals are produced at each level. In 2D wavelet analysis, the images are considered to be matrices with N rows and M columns. At every level of decomposition, the horizontal data are filtered, then the approximation and details produced from this are filtered on columns. The ideas are shown in Fig. 2.

After recalling the theory of the basics of wavelet transform, wavelets, and multiresolution analysis, in the next section, we review some of the applications in image processing, domain by domain, beginning with image denoising, image enhancement, super-resolution and lastly, image compression.

3. Application of Wavelet Theory in Image Processing

Image processing is a field that continues to grow, with new applications being developed at an ever increasing pace. It is an exciting and fascinating area to be involved in today, with application areas ranging from the entertainment industry to the space program. One of the most interesting aspects of this information revolution is the ability to send and receive complex data that transcend ordinary written text. Image information, transmitted in the form of digital images, has become a main method of communication for the 21st century. Image processing is one form of signal processing for which the input is an image; these photographs or frames of video and the output of image processing can be either an image or a set of characteristics or parameters related to image processing.

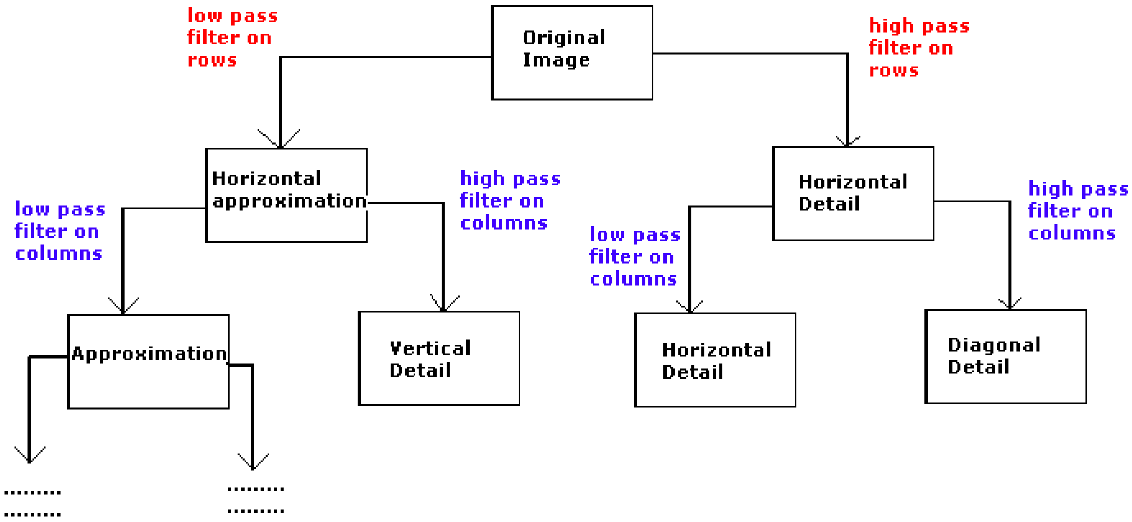

Fig. 2.

The majority of image processing techniques involve treating the image as a two-dimensional signal and applying standard signal-processing techniques to it. In this section, we are going to review image processing, domain by domain, by using wavelet analysis techniques.

3.1 Image Denoising

Image denoising (as expected an image corrupted by noise) is a classical problem in the field of signal or image processing systems. Additive random noise can easily be removed using simple threshold methods. Denoising of natural images using wavelet techniques is very effective because of its ability to capture the energy of a signal in a few energy transform values. The wavelet denoising technique thresholds the wavelet coefficients arising from the wavelet transform. A wavelet transform helps a large number of small coefficients and a small number of large coefficients. General denoising methods that employ the wavelet transform consist of the following steps.

- Calculate the wavelet transform of the given noisy signal.
- Modify noisy wavelet coefficients according to the rule.
- Compute the inverse wavelet transform by using the modified coefficients.

The subsequent literature review discusses denoising using wavelet transforms in a wide variety of scenarios, i.e. by using a number of thresholding methods for a broad variety of test images.

Mallat and Hwang [10] demonstrated that the wavelet transform is particularly suitable for applications of non-stationary signals that may spontaneously vary in time. Other wavelet denoising methods depend on the wavelet transform scale correlation between wavelet coefficients [11]. Donoho and Stone established thresholding by coining soft-threshold and hard-threshold wavelet denoising methods [12-14]. Wood and Johnson [15] applied denoising to synthetic, phantom, and volunteer cardiac images in either the magnitude or complex domains. The authors suggested denoising prior to rectification for superior edge resolution of real and imaginary images. Magnitude and complex denoising significantly improved signal-to-noise ratio (SNR). Rosenbaum et al. [16] used wavelet shrinkage denoising algorithms and Nowak's algorithm for denoising magnitude images. The wavelet shrinkage denoising methods were performed using both soft and hard thresholding, and it was suggested that changes in mean relative SNR are statistically associated with the type of threshold and type of wavelet. Data-adaptive wavelet filtering was found to provide the best overall performance, compared to direct wavelet shrinkage. Argenti and Torricelli [17] assumed Wiener-like filtering and performed it in a shift-invariant wavelet domain by means of adaptive rescaling of the coefficients of undecimated octave decomposition calculated from the parameters of the noise model and the wavelet filters. The proposed method found excellent background smoothing as well as preservation of edge sharpness and well details. Local linear minimum mean square error (LLMMSE) evaluation in an undecimated wavelet domain tested on both ultrasonic images and synthetically speckled images demonstrated efficient rejection of the distortion due to speckle. Xie et al. [18], using the minimum description length (MDL) principle, provided a denoising method based on a doubly stochastic process model of wavelet coefficients that gave a new spatially varying threshold. This method outperformed the traditional thresholding method in both mean square error (MSE) and compression gain. Wink and Roerdink [19] estimated two denoising methods for simulation of a functional magnetic resonance imaging (fMRI) series with a time signal in an active spot by the average temporal SNR inside the original activated spot, and by the shape of the spot detected by thresholding the temporal SNR maps. These methods were found to be better suited to low SNRs, but were not preferred for reasonable-quality images, as they introduced heavy decompositions. Therefore, wavelet-based denoising methods were used, since they preserved sharpness of the images from the original shapes of active regions, as well, and produced a smaller total number of errors than Gaussian noise.

But both Gaussian and wavelet-based smoothing methods introduced severe deformations, and blurred the edges of the active mark. For a low SNR, both techniques are found to be similar. For a high SNR, wavelet methods are better than the Gaussian method, giving a maximum output above 10 dB.

Choi and Baranuik [20] defined Besov Balls (a convex set of images whose Besov norms are bounded from above by their radii) in multiple wavelet domains, and projected them onto their intersection using the projection onto convex sets (POCS) algorithm. It resembled a type of wavelet shrinkage for image denoising. This algorithm provided significant improvement over the conventional wavelet shrinkage algorithm, based on a single wavelet domain, such as hard thresholding in a single wavelet domain. Yoon and Vaidyanathan [21] offered the custom thresholding scheme, and demonstrated that it outperformed the traditional soft and hard-thresholding schemes. Hsung et al. [22] improved the traditional wavelet method by applying multivariate shrinkage on multiwavelet transform coefficients. First, a simple second-order orthogonal pre-filter design method was used for applying multiwavelets of higher multiplicities (preserving an orthogonal pre-filter for any multiplicity). Then threshold selections were studied using Stein's unbiased risk estimator (SURE) for each resolution point, provided the noise constitution is known. Numerical experiments showed that a multivariate shrinkage of higher multiplicity usually gave better performance, and the proposed LSURE substantially outperformed the traditional SURE in multivariate shrinkage denoising, mainly at high multiplicity. Poornachandra [23] used wavelet-based denoising to recover a signal contaminated by white additive Gaussian noise, and investigated the noise-free reconstruction property of a universal threshold. Giaouris and Finch [24] presented a denoising scheme based on a wavelet transform (WT) that did not distort the signal and the noise component after the process was found to be small. There are several studies on thresholding wavelet coefficients [25, 26]. A methodology to denoise an image based on a least square approach using wavelet filters was presented by Veena et al. [27]. This work is the extension of the one-dimensional signal denoising approach based on least square (proposed by Selesnick) to two-dimensional image denoising.

3.2 Image Enhancement

Image enhancement is one of the measure issues in high-quality pictures from digital cameras and in high definition television (HDTV). Since clarity of the image is easily affected by weather, lighting, wrong camera exposures or aperture settings, a high dynamic range in the scene, etc., these conditions lead to an image that may suffer from loss of information. Many techniques have been developed to recover information in an image. In this section, we present a literature review of some of the image enhancement techniques for color image enhancement, such as contrast stretching, histogram equalization and its improved versions, homomorphic filtering, retinex, single or multiscale retinex, and the wavelet multiscale transform.

Since color images provide more and richer information for visual perception than gray images, color image enhancement plays an important role in digital image processing [28]. The main purpose of image enhancement is to obtain finer details of an image and to highlight useful information. The images appear darker or with low contrast under poor illumination conditions. Such low-contrast images need to be enhanced. Image enhancement is basically improving the interpretability or perception of information in images for human viewers, and providing better input for other automated image processing techniques [28]. In the literature, there exist many image enhancement techniques that can enhance a digital image without spoiling it. Image enhancement methods can broadly be divided into two categories:

1. spatial domain methods, and
2. frequency domain methods

In spatial domain methods, we directly deal with the image pixels. The pixel values are manipulated to achieve the desired enhancement. In frequency domain methods, the image is first transferred into the frequency domain. This means that the Fourier transform of the image is computed first. All the enhancement operations are performed on the Fourier transform of the image, and then, an inverse Fourier transform is performed to get the resultant image.

Because some features in an image are hardly detectable by eye, we often transform images before display. Histogram equalization is one of the most well-known methods for contrast enhancement. Such an approach is generally useful for images with poor intensity distribution. Since edges play a fundamental role in image understanding, one good way to enhance the contrast is to enhance the edges. Multiscale-edge enhancement [29] can be seen as a generalization of this approach, taking all resolution levels into account.

In color images, objects can exhibit variations in color saturation with little or no correspondence in luminance variation. Several methods have been proposed in the past for color image enhancement [30]. Many image enhancement techniques, such as gamma correction, contrast stretching, histogram equalization, and contrast-limited adaptive histogram equalization (CLAHE) have been discussed [31]. These are all old techniques that will not provide exact, enhanced images, and that give poor performance in terms of root mean square error (RMSE), peak signal-to-noise ratio (PSNR) and mean absolute error (MAE) [31]. Use of the old enhancement technique will not recover an exact true color in the image. Recently, retinex, single and multiscale retinex, and homomorphic and wavelet multiscale techniques have become popular for enhancing images. These methods are shown to perform much better than those listed earlier [32].

The retinex concept was introduced by Land [33] as a model for human color constancy. The single scale retinex (SSR) method [34] consists of applying the following transform to each band of the color image:

$$R_i(x,y) = \log(I_i(x,y)) - \log(F(x,y) * I_i(x,y))$$

where $R_i(x,y)$ is the retinex output, $I_i(x,y)$ is the image distribution in the i^{th} spectral band, F is a Gaussian function, and * is convolution. The retinex method is efficient for dynamic range compression, but does not provide good tonal rendition [35]. Multiscale retinex (MSR) combines several SSR outputs to produce a single output image that has both good dynamic range compression and color constancy (color constancy may be defined as the independence of the perceived color from the color of the light source [36, 37]), and good tonal rendition [38].

Multiscale retinex leads the concept of multiresolution for contrast enhancement. It implements dynamic range compression and can be used for different image processing goals. Barnard and Funt [39] presented improvements to the algorithm, leading to better color fidelity. MSR softens the strongest edges and leaves the faint edges almost untouched. Using the wavelet transform, the opposite approach was proposed by Velde [30] for enhancing the faintest edges and keeping the strongest untouched. The strategies are different, but both methods allow the user to see details that were hardly distinguishable in the original image, by reducing the ratio of strong features to faint features.

Wavelet analysis [40] has proven to be a powerful image processing tool in recent years. When images are to be viewed or processed at multiple resolutions, the wavelet transform is the mathematical tool of choice. In addition to being an efficient, highly intuitive framework for the representation and storage of multiresolution images, the WT provides powerful insight into an image's spatial and frequency characteristics. The image detail parts are stored in the high-frequency parts of the image transformed by the wavelet, and the imagery constant part is stored in the low-frequency part. Because the imagery constant part determines the dynamic range of the image, the low-frequency part determines the dynamic range of the image. We attenuate the low-frequency part in order to compress the dynamic range. But details must be lost when the low-frequency part is attenuated [41]. As some details are stored in the high-frequency parts very well, the image reconstructed by inverse wavelet transform has more detail.

In 2014, Provenzi and Caselles proposed a variational model of perceptually inspired color correction based on the wavelet representation of a digital image [42]. Motivation to consider wavelets arises from the results of Palma-Amestoy et al. (2009), where it was shown that such a color correction can be achieved by minimization of a class of functionals that induces a local contrast enhancement and a global dispersion control around the average image value (plus a conservative action that inhibits excessive departure from the original data). The wavelet framework was selected instead of the Fourier, because wavelets provide an intrinsically local frequency description of a signal directly related to local contrast, while the Fourier transform provides only global frequency information.

The review concludes that histogram equalization cannot preserve the brightness and color of the original image, and a homomorphic filtering technique has a problem with bleaching of the image. Modern technique retinex (SSR & MSR) performs much better than those listed above, because it is based on the color constancy theory, but it still suffers from color violation and the unnatural color rendition problem, as the wavelet transform is a very good technique for image enhancement and denoising, and input images always face noise during image processing. In future work, there is scope in applying a wavelet transform with retinex to improve the image enhancement results, such as PSNR, the color rendition problem, and minimum MSE.

3.3 Super-resolution

Super-resolution (SR) imaging is a method or technique that enhances the resolution of an imaging system. Super-resolution is the process of generating a raster image with a higher resolution than its source. The source can consist of one or more images or frames.

With the recent advancements in image and video imaging, there is a constant need to get a better resolution image. One approach is to use signal processing techniques on a low-resolution (LR) image to achieve a high-resolution (HR) image. This technique of generating an HR image from a single LR image or multi-LR images is referred to as super-resolution. This is categorized into two forms: one is with input from multi-source LR images with sub-pixel shift [43, 44], and the other is with a single LR image [45-47]. Work on the former has been around for a long time. Multi-frame SR suffers from the fact that it requires multiple LR input images with sub-pixel shift, leading to poor image registration.

Ford and Etter [48] proposed a method to interpolate missing values for a one-dimensional signal. Inspired by this work, Nguyen and Milanfar [49-51] proposed an extension for interlaced one-dimensional signals and two-dimensional images. Since the super-resolution image is computed giving multiple low-resolution images, forward transformation to go from a high-resolution image to a low-resolution image methods. Each low-resolution image executes a set of linear constraints on the unknown high-resolution image. If the number of low-resolution images is enough (where each image gives different information due to the subpixel shifts), the set of equations will be larger than the number of unknowns, and the system is underdetermined. This method [49-51] will lead to an underdetermined system, because the number of unknowns will be larger than the number of linear constraints given by the low-resolution image. Instead of obtaining a super-resolution image from several low-resolution images, the challenge was to generate a super-resolution image from only one low-resolution image. In order to get around these difficulties of an underdetermined system affected by a restricted number of low-resolution images, Glasner et al. [52] proposed a trick based on the property of patch redundancy inside a single image. The single-image super-resolution techniques are further categorized broadly into interpolation techniques, machine-learning techniques, and wavelet-based techniques.

The linear interpolation procedures (bicubic and bilinear) suffer from the fact that higher frequency details are lost when the magnification factor is increased, leading to deprivation in edge information. Li and Orchard [53] worked on this problem and proposed a solution based on edge-directed interpolation. The basic idea lies in working on geometric duality between the estimated covariance coefficients between the low-resolution image and the interpolated version of the image. In spite of appreciable performance, this method of covariance-based adaptation interpolation introduces complexity. Zhang and Wu [54] further worked on enhancing the edge information by proposing a new edge-guided nonlinear interpolation technique that uses directional filtering and data fusion.

However, owing to the complexity in estimation and computation, research on image enhancement now focuses on newer techniques. The second category is based on machine learning, which uses a learning step between HR images (faces, fingerprints, walls, etc.) and their LR counterparts. This learned knowledge is then incorporated in a priori terms for reconstruction. Extraordinary work done by Mallat and Yu [55] focused on adaptive estimators obtained by mixing a family of linear inverse estimators, derived from different priors on signal regularity. Path-breaking work done by Yang et al. [47] is based on a sparse-land local model, which assumes that each patch from the LR image can be represented using linear combinations from a dictionary. Generally, each patch is considered to be generated by multiplying a dictionary by sparse vector coefficients. However, for a higher magnification factor, the results are not satisfactory.

In parallel, many techniques using wavelet decomposition operators for the single-image scale-up problem (i.e. single-image super-resolution) have been around the corner in recent times. The third category is based on enhancement using wavelet decomposition [56-60]. In this method, generally, the input image is decomposed into structurally correlated sub-images, which allows exploiting the self-similarities between local neighboring regions. Demirel and Anbarjafari [59] first decomposed the input image into subbands. Then, the input image and the high-frequency subbands are both interpolated. The results of a stationary wavelet transform of high-frequency subbands are used to improve the interpolated subbands. The super-resolved HR output is generated by combining all of these subbands using an inverse discrete wavelet transform.

In 2016, Sowmya et al. [61] developed an effective image enhancement technique using wavelet transformations and the concept of sparse recovery. In this technique, the input image was decomposed into different subbands using DWT. Since using DWT decimates an image and loss in high-frequency components, they used a second-generation wavelet transformer, lifting wavelet transform (LWT), where the first derivative preserves the edges, and the second takes care of the curves present in the image. Then, the higher frequency components are added up. Parallel to that, they sparsely recover the input image with an interpolation factor of two. The three higher subbands and the sparse recovered image are applied to inverse discrete wavelet transform (IDWT), and finally, to a reconstruction-based algorithm to give a super resolved image. That method was applied to five well-known test images, and the experimental results showed prominence over traditional and other state-of-the-art techniques.

3.4 Image Compression

Image compression is very important for efficient transmission and storage of images. It has many applications in information theory, applied harmonic analysis, and many other fields. The objective of image compression is to minimize the size of an image by exploiting redundancy within the data without degrading the quality of the image. The reduction in file size allows more images to be stored in a given amount of disk or memory space. The common redundancies are spatial redundancy, temporal redundancy, inter-pixel redundancy, psycho-visual redundancy and statistical redundancy [62]. Image compression is divided into two main techniques: transforms (Discrete Cosine Transform [DCT], Joint Photographic Experts Group [JPEG], Fast Fourier Transform [FFT], and wavelets), and non-transforms (Pulse-code modulation [PCM], differential pulse code modulation [DPCM]). Image compression is the main application of the wavelet transform in image processing. Wavelet compression algorithms provide better compression and quality than the traditionally used JPEG algorithm. Many authors have contributed to the field [63-71, 73]. Even multispectral images (from satellite imagery, for instance) can be compressed with a wavelet-based method with multiwavelet bases, for instance [63]. Khashman and Dimililer [72] proposed a technique for compressing a digital image using neural networks and a Haar wavelet transform, with the aim being to develop an optimum image compression system.

The discrete wavelet transform can be efficiently used in image-coding applications because of their data reduction capabilities. The basis of DWT can be composed of any function (wavelet) that satisfies the requirements of multiresolution analysis [3]. Elamaran and Praveen [74] described the basic idea of compression and attempted to reduce the average number of bits per pixel to adequately represent an image. Fourier-based transforms (e.g., DCT and DFT) are efficient at exploiting the low-frequency nature of an image. The high-frequency coefficients are coarsely quantized, and hence, a reconstructed image has poor quality at the edges. Gupta and Garg [75] developed some simple functions to compute DCT and to compress images. Image compression was studied using 2D discrete cosine transform. The original image is transformed in eight-by-eight blocks, then via inverse transform in eight-by-eight blocks to reconstruct the image and the error image (the difference between the original and reconstructed image). Chowdhury and Khatun [76] described a new image compression scheme with a pruning proposal based on discrete wavelet transform. It provides sufficient high-compression ratios with no appreciable degradation of image quality. Singh et al. [77] studied the behavior of different types of wavelet functions with different types of image, and suggested an appropriate wavelet function that can perform optimum compression for any given type of image. The effects of different wavelet functions and compression ratios were assessed. This investigation was carried out by calculating the compression ratio (CR), mean square error, bits per pixel (BPP) and PSNR for different wavelets.

4. Quaternion Wavelet

As a mathematical tool, wavelet transform is a major breakthrough from the Fourier transform. It has good time–frequency features and multiple resolution, and wavelet analysis theory has become one of the most useful tools in signal analysis, image processing, pattern recognition, and other fields. In image processing, the basic idea of the wavelet transform is to decompose image multiresolution; the original image is decomposed into different space and different frequency sub-images, and then coefficients of sub-images are processed. Commonly used wavelet transforms are real discrete wavelet transforms, complex wavelet transforms, and so on. The real discrete wavelet and the complex wavelet transforms have two general shortcomings; first, the real discrete wavelet transform signal's small shift will produce the energy of a wavelet coefficient distribution change; second, dual-tree complex wavelets, although they overcome the first problem, can generate signal phase ambiguity when representing two-dimensional image features. While the quaternion wavelet transform is a new multiscale analysis image processing tool, it is based on the Hilbert two-dimensional transform theory, which has approximate shift invariance and can well overcome the above drawbacks [78].

Quaternion wavelet transform was established based on quaternion algebra, the quaternion Fourier transform, and the Hilbert transform. Using four real discrete wavelet transforms, the first real discrete wavelet corresponds to the quaternion wavelet real part, and the other real discrete wavelet is obtained by the first real discrete wavelet transform's Hilbert transform, corresponding to the quaternion wavelet's three imaginary parts and the four real wavelets composed of the quaternion analytic signal. It can be understood as an improved real wavelet and a complex wavelet's promotion, which have approximate shift invariance, abundant phase information, limited redundancy, and so forth, while still retaining the traditional wavelet time–frequency localization ability and filter design using a Hilbert transform pair of a dual tree structure, which is easy to realize. In this section, we summarize the signal or image processing applications' oriented theory of quaternion wavelets [78-80].

The quaternion algebra over R, denoted by H, is associative, non-commutative, four-dimensional algebra. Every element of H is a linear combination of a scalar and three imaginary units, i, j, and k, with real coefficients:

$$H = \{q : q = q_0 + iq_1 + jq_2 + kq_3,\ q_0, q_1, q_2, q_3 \in R\}, \quad (16)$$

which obey Hamilton's multiplication rules:

$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j, \\ i^2 = j^2 = k^2 = ijk = -1. \quad (17)$$

The quaternion conjugate of a quaternion q is given by

$$\bar{q} = q_0 - iq_1 - jq_2 - kq_3, \qquad q_0, q_1, q_2, q_3 \in R. \quad (18)$$

The quaternion conjugation of q is a linear anti-involution

$$\bar{\bar{q}} = q,\ \ \overline{p+q} = \bar{p} + \bar{q},\ \ \overline{pq} = \bar{q}\,\bar{p}\ \ for\ all\ p, q \in H. \quad (19)$$

The multiplication of a quaternion q and its conjugate can be expressed as

$$q\bar{q} = q_0^{\ 2} + q_1^{\ 2} + q_2^{\ 3} + q_3^{\ 2} \quad (20)$$

and the modulus of quaternion q is defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^{\ 2} + q_1^{\ 2} + q_2^{\ 3} + q_3^{\ 2}}. \quad (21)$$

Finally, we can define the inverse of $q \in H - \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2}, \quad (22)$$

which shows that H is normed division algebra.

Furthermore, we get $|q^{-1}| = |q|^{-1}$. In addition, quaternion q can also be expressed as $q = |q| e^{i\varphi} e^{i\theta} e^{i\xi}$, where $|q|$ is the modulus of q and (φ, θ, ξ) are the three phase angles, which are uniquely defined within the range

$$(\varphi, \theta, \xi) \in [-\pi, \pi] \times [-\pi/2, \pi/2] \times [-\pi/4, \pi/4].$$

The quaternion module $L^2(R^2, H)$ is defined as

$$L^2(R^2, H) = \{f :\ f : R^2 \to H, \|f\|_{L^2(R^2,H)} < \infty\}. \quad (23)$$

For all quaternion functions $f, g : R^2 \to H$, the inner product is defined as follows:

$$\langle f, g \rangle_{L^2(R^2,H)} = \int_{R^2} f(x)\overline{g(x)}\, d^2x, \qquad x \in R^2. \quad (24)$$

If $f = g$ $a.e.$, we obtain its associated norm:

$$\|f\|_{L^2(R^2,H)} = \langle f, g \rangle^{1/2} = \left(\int_{R^2} |f(x)|^2 d^2x \right)^{1/2}, \quad x \in R^2. \quad (25)$$

With the usual addition and scalar multiplication of functions, together with the inner product, $L^2(R^2, H)$ becomes a Hilbert space.

Definition 4.1. [80] If *f(x, y)* is a real two-dimensional signal, then a quaternion analytic signal can be defined as

$$f^q(x,y) = f(x,y) + if_{H_x}(x,y) + jf_{H_y}(x,y) + kf_{H_{xy}}(x,y), \tag{26}$$

where $f_{H_x}(x,y),\ f_{H_y}(x,y),\ f_{H_{xy}}(x,y)$ are the Hilbert transform of $f(x,y)$ along the x-axis, y-axis, and xy-axis directions, respectively.

Lemma 4.2. [80] If $\{\phi(x-k): k\in Z\}$ is an orthogonal basis for space $V\subset L^2(R)$, set $\phi_g(x) = H(\phi(x))$, where $\phi_g(x)$ is the Hilbert transform of $\phi(x)$, then $\{\phi_g(x-k): k\in Z\}$ is the standard orthogonal basis for space $\tilde{V} = HV \subset L^2(R)$.

Theorem 4.3. [80] If $\{V_j\}_{j\in Z}$ is one-dimensional orthogonal multiresolution analysis, and $\phi_h(x)$ and $\psi_h(x)$ are the corresponding scale function and wavelet function, respectively, then $\{\phi_g(x-l)\phi_h(y-k): l,k\in Z\}$ or $\{\phi_h(x-l)\phi_g(y-k): l,k\in Z\}$ represent the standard orthogonal space for $V_0\otimes\tilde{V}_0$ or $\tilde{V}_0\otimes V_0$ where $\tilde{V}_0 = HV_0$, and

$$\begin{aligned} &H_x(\phi_h(x)\phi_h(y)) = \phi_g(x)\phi_h(y),\\ &H_y(\phi_h(x)\phi_h(y)) = \phi_h(x)\phi_g(y) \text{ and}\\ &H_{xy}(\phi_h(x)\phi_h(y)) = \phi_g(x)\phi_g(y)\end{aligned} \tag{27}$$

are the Hilbert transform of $(\phi_h(x)\phi_h(y))$ along the x-axis, y-axis and xy-axis directions, respectively.

Theorem 4.4. [80] If $\{V_j\}_{j\in Z}$ is one-dimensional orthogonal MRA, and $\phi_h(x)$ and $\psi_h(x)$ are the corresponding scale function and wavelet function, respectively, then $\{\Phi^q{}_{j,k,m}(x,y)\}_{m,k\in Z}$ is the orthogonal basis of the quaternion wavelet scale space in $L^2(R^2,H)$, where

$$\Phi^q(x,y) = \phi_h(x)\phi_h(y) + i\phi_g(x)\phi_h(y) + j\phi_h(x)\phi_g(y) + k\phi_g(x)\phi_g(y), \tag{28}$$

$$\Phi^q{}_{j,k,m}(x,y) = \phi_{h,j,k}(x)\phi_{h,j,m}(y) + i\phi_{g,j,k}(x)\phi_{h,j,m}(y) + j\phi_{h,j,k}(x)\phi_{g,j,m}(y) + k\phi_{g,j,k}(x)\phi_{g,j,m}(y), \tag{29}$$

and $\phi_{h,j,k}(x) = 2^{-j/2}\phi_h(2^{-j}x-k),\ j,k,m\in Z.$

$\Phi^q(x,y)$, defined above, is called a quaternion wavelet scale function in $L^2(R^2,H)$, and $\{\Phi^q{}_{j,k,m}(x,y)\}_{m,k\in Z}$ is called a discrete quaternion wavelet scale function in $L^2(R^2,H)$.

Theorem 4.5. [80] Let $\{V_j\}_{j\in Z}$ be one-dimensional orthogonal MRA, and let $\phi_h(x)$ and $\psi_h(x)$ be the corresponding scale function and wavelet function, respectively. Then $\{\Psi^{q,1}(x,y), \Psi^{q,2}(x,y), \Psi^{q,3}(x,y)\}$ are the quaternion wavelet basis functions in $L^2(R^2,H)$, and $\{\Psi^{q,1}_{j,k,m}(x,y), \Psi^{q,2}_{j,k,m}(x,y), \Psi^{q,3}_{j,k,m}(x,y)\}_{j,k,m\in Z}$ are the discrete quaternion wavelet basis functions in $L^2(R^2,H)$, where

$$\begin{aligned} \Psi^{q,1}(x,y) &= \phi_h(x)\psi_h(y) + i\phi_g(x)\psi_h(y) + j\phi_h(x)\psi_g(y) + k\phi_g(x)\psi_g(y),\\ \Psi^{q,2}(x,y) &= \psi_h(x)\phi_h(y) + i\psi_g(x)\phi_h(y) + j\psi_h(x)\phi_g(y) + k\psi_g(x)\phi_g(y),\\ \Psi^{q,3}(x,y) &= \psi_h(x)\psi_h(y) + i\psi_g(x)\psi_h(y) + j\psi_h(x)\psi_g(y) + k\psi_g(x)\psi_g(y), \end{aligned} \tag{30}$$

The shift and expand forms of $\{\Psi^{q,1}(x,y), \Psi^{q,2}(x,y), \Psi^{q,3}(x,y)\}$ are

$$\begin{aligned} \Psi^{q,1}_{j,k,m}(x,y) &= \phi_{h,j,k}(x)\psi_{h,j,m}(y) + i\phi_{g,j,k}(x)\psi_{h,j,m}(y) + j\phi_{h,j,k}(x)\psi_{g,j,m}(y) + k\phi_{g,j,k}(x)\psi_{g,j,m}(y),\\ \Psi^{q,2}_{j,k,m}(x,y) &= \psi_{h,j,k}(x)\phi_{h,j,m}(y) + i\psi_{g,j,k}(x)\phi_{h,j,m}(y) + j\psi_{h,j,k}(x)\phi_{g,j,m}(y) + k\psi_{g,j,k}(x)\phi_{g,j,m}(y),\\ \Psi^{q,3}_{j,k,m}(x,y) &= \psi_{h,j,k}(x)\psi_{h,j,m}(y) + i\psi_{g,j,k}(x)\psi_{h,j,m}(y) + j\psi_{h,j,k}(x)\psi_{g,j,m}(y) + k\psi_{g,j,k}(x)\psi_{g,j,m}(y), \end{aligned} \tag{31}$$

and $\psi_{h,j,k}(x) = 2^{-j/2}\psi_h(2^{-j}x-k),\ j,k,m\in Z.$

Definition 4.6. [80] For all $f(x,y)\in L^2(R^2,H)$, define

$$\begin{aligned} a^q_{j,k,m} &= (f(x,y),\ \Phi^q_{j,k,m}(x,y)),\ \&\\ d^{q,i}_{j,k,m} &= (f(x,y),\ \Psi^{q,i}_{j,k,m}(x,y)),\ i=1,2,3,\ j,k,m\in Z. \end{aligned} \tag{32}$$

$d^i_{j,k,m}$, $(i=1,2,3)$, therefore, is called the discrete quaternion wavelet transform of *f(x, y).*

Quaternion Multiresolution Analysis [78] For the two-dimensional image function *f(x, y);* a quaternion wavelet multiresolution analysis will be

$$\begin{aligned} f(x,y) &= A^q_1 f + \sum_{i=1}^{3} D^{q,i}_1 f\\ &= A^q_2 f + \sum_{i=1}^{3} D^{q,i}_2 f + \sum_{i=1}^{3} D^{q,i}_1 f\\ &= \ldots\ldots\ldots\ldots\\ &= A^q_n f + \sum_{j=1}^{n}\left[D^{q,1}_j f + D^{q,2}_j f + D^{q,3}_j f\right] \end{aligned} \tag{33}$$

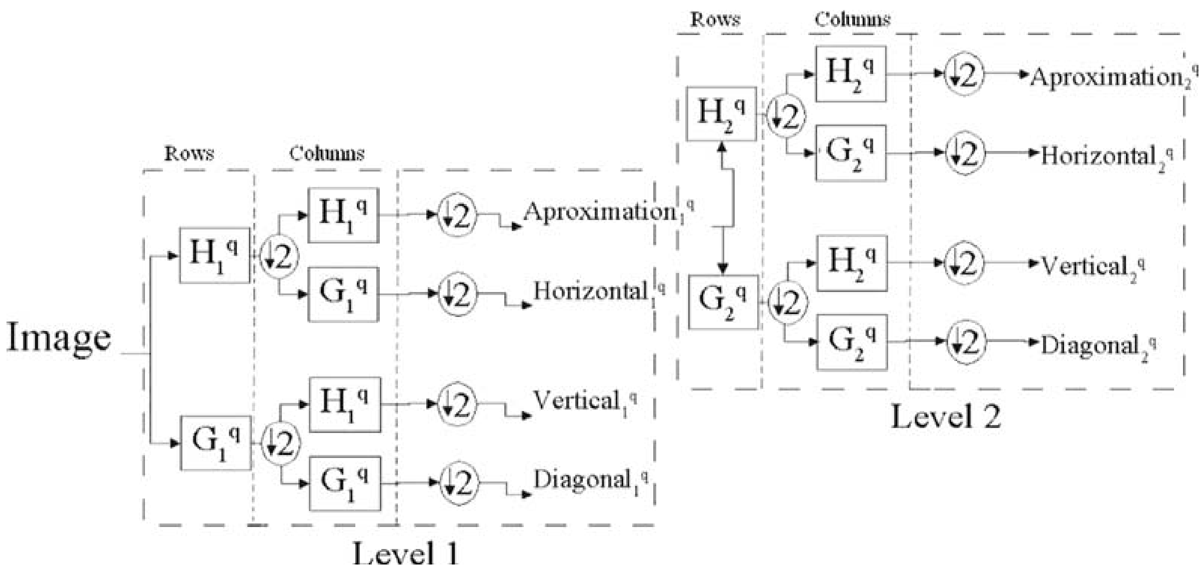

Fig. 3.

We can characterize each approximation function $A_j^q f(x,y)$ and the difference components $D_j^{q,i} f(x,y)$, $i=1,2,3$ by means of two-dimensional scaling function $\Phi^q(x,y)$ and its associated wavelet function, $\Psi^{q,i}(x,y)$, $i=1,2,3$, as follows:

$$A_j^q f(x,y) = \sum_{l\in Z}\sum_{k\in Z} a_{j,k,l}^q \Phi_{j,k,l}^q(x,y)\,, \tag{34}$$

$$D_j^{q,i} f(x,y) = \sum_{l\in Z}\sum_{k\in Z} d_{j,k,l}^{q,i}\ \Psi_{j,k,l}^{q,i}(x,y) \tag{35}$$

where

$$a_{j,k,l}^q(x,y) = \left\langle f(x,y), \Phi_{j,k,l}^q(x,y)\right\rangle \tag{36}$$

$$d_{j,k,l}^{q,i}(x,y) = \left\langle f(x,y), \Psi_{j,k,l}^{q,i}(x,y)\right\rangle, \quad i=1,2,3,\ \ (j,\,k,\,l)\in Z^3. \tag{37}$$

The quaternion wavelet analysis from level *j-1* to level *j* corresponds to the transformation of one quaternion approximation to a new quaternion approximation and three quaternion differences, i.e.

$$\left\{A_{j-1}^q\right\} \to \left\{A_j^q,\, D_j^{q,i},\, i=1,2,3\right\}$$

As a result, the quaternion wavelet tree is the compact and economic processing structure to be used for the case of n-dimensional multiresolution analysis.

The procedure for quaternion wavelet multiresolution analysis depicted partially in Fig. 3 can be found in detail elsewhere [78]. A dual tree using complex wavelets [81] uses an extra mirror tree structure; in contrast, the quaternion wavelet consists of only one tree, which offers three phases for analysis using the same amount of computational resources as the complex wavelet tree. The way of applying the quaternion phase concept is explained and illustrated with real experiments elsewhere [78].

5. Conclusion

In this paper, we reviewed wavelet theory, an important mathematical tool for signal and image processing, with an emphasis on its application to image processing. Representative work was identified in image compression, image denoising, super-resolution, and image enhancement, analysis, and classification. Compared to other tools, such as the Fourier transform, wavelet transforms often provide a better spatial domain localization property, critical to many image applications.

The application of wavelets in image processing is only a decade old. Wavelets have demonstrated their importance in almost all areas of signal processing and image processing. In many areas, techniques based on wavelet transforms represent the best of the available solutions. In the coming years, we expect to see more (and more successful) wavelet-based techniques in the field of image processing. We expect image restoration incorporating wavelets, wavelet transforms, and other statistical techniques to achieve greater success. Theoretical research inspired by wavelets has led to new techniques, such as quaternion wavelets and quaternion transforms [78-80] that are more promising in certain situations. Explorations of these new frontiers are likely to bring us more successful applications in image denoising, image enhancement, image restoration, etc.

Acknowledgement

This research was supported by the Commercializations Promotion Agency for R&D Outcomes(COMPA) funded by the Ministry of Science, ICT and Future Planning (MSIP) and by Ministry of Culture, Sports and Tourism (MCST) and Korea Creative Content Agency(KOCCA) in the Culture Technology(CT) Research & Development Program (R2014040014).

References

[1] Grossmann, A. and J. Morlet, "Decomposition of Hardy functions into square integrable wavelets of constant shape", *SIAM J. Anal*. 15, pp. 723-736, 1984. Article (CrossRefLink)

[2] C. K. Chui, "An Introduction to Wavelets", *Boston, Academic Press*, 1992.

[3] I. Daubechies, "Ten Lectures on Wavelets", *Capital City Press*, 1992. Article (CrossRefLink)

[4] Y. Meyer, "Wavelets Algorithms and Applications", *Philadelphia:SIAM*, 1993.

[5] Hernandez and Weiss, "A first course on Wavelets", *CRC Press*, 1996.

[6] G. Kaiser, "A Friendly Guide to Wavelets", *Boston, Birkhauser*, 1994.

[7] S. G. Mallat, "A theory for multiresolution signal decomposition: the wavelet representation", *IEEE Trans Pattern Analysis and Machine Intelligence*, 11(7), 674-93, 1989. Article (CrossRefLink)

[8] S. G. Mallat, "Multiresolution approximations and wavelet orthonormal bases of L^2(R)", *Trans. Amer. Math. Soc.*, 315(1), 69-87, 1989. Article (CrossRefLink), Article (CrossRefLink)

[9] S. G. Mallat, "A wavelet tour of signal processing", *Academic press, San Diego, California, USA*, 1998.

[10] S. Mallat and W. L. Hwang, "Singularity detection and processing with Wavelets", *IEEE Transactions on Information Theory*, 38 (2), 617-643, 1992. Article (CrossRefLink)

[11] Y. Xu, B. Weaver and D. M. Healy, "Wavelet transform domain filters: A spatially selective noise filtration technique", *IEEE Transactions on Image Processing*, 3(6), 217-237, 1994.

[12] D. L Donoho and I. M. John stone, "Ideal spatial adaptation via wavelet shrinkage", *Biometrika*, 81 (3), 425-455, 1994. Article (CrossRefLink)

[13] D. L Donoho, "De-Noising by Soft-Threshold", *IEEE Transactions on Information Theory*, 41(3), 613-627, 1995. Article (CrossRefLink)

[14] D. L. Donoho and I. M. John stone, "Adapting to Unknown Smoothness via Wavelet Shrinkage", *Journal of American Stat Assoc*, 12, 1200- 1224, 1995. Article (CrossRefLink)

[15] J. C. Wood and K. Johnson, "Wavelet-denoising of complex magnetic resonance images", *IEEE*, 1998. Article (CrossRefLink)

[16] W. L. Rosenbaum, M. S. Atkinsa and G. E. Sarty, "Classification and performance of denoising algorithms for low signal to noise ratio magnetic resonance images", *SPIE*, 3979 , 2000.

[17] F. Argenti and G. Torricelli, "Speckle suppression in ultrasonic images based on undecimated wavelets", *EURASIP Journal on Applied Signal Processing*, 2003. Article (CrossRefLink)

[18] J. Xie, D. Zhang, and W. Xu, "Spatially adaptive wavelet denoising using the minimum description length principle", *IEEE Transactions on Image Processing*, 13(2) ,2004. Article (CrossRefLink)

[19] A. M. Wink and Jos B. T. M. Roerdink, "Denoising functional MR Images – a comparison of wavelet denoising and gaussian smoothing", *IEEE Transactions on Medical Imaging*, 23(3), 2004. Article (CrossRefLink)

[20] H. Choi and R. G. Baranuik, "Multiple wavelet basis image denoising using besov ball projections", *IEEE Signal Processing Letters*, 11(9), 2004. Article (CrossRefLink)

[21] B. J. Yoon and P. P. Vaidyanathan,"Wavelet-based denoising by customized thresholding". *IEEE Int. Conf. on Acoustics, Speech, and signal processing (ICASSP)*, 2004.

[22] T. C. Hsung, D. P. Kong Lun and K. C. Ho, "Optimizing the multiwavelet shrinkage denoising", *IEEE Transactions on Signal Processing*, 53(1), 2005.

[23] S. Poornachandra, "Wavelet-based denoising using subband dependent threshold for ECG signals", *Digital Signal Processing*, 18, 49–55, 2008. Article (CrossRefLink)

[24] D. Giaouris and J. W. Finch, "Denoising using wavelets on elective drive applications", *Electric Power Systems Research,* Article (CrossRefLink).

[25] Sachin D Ruikar, Dharampal D Doye, "Wavelet based image denoising technique", *International Journal of Advanced Computer Science and Applications*, 2(3), 2011.

[26] Rakesh Kumar and B. S. Saini , "Improved Image Denoising Technique Using Neighboring Wavelet Coefficients of Optimal Wavelet with Adaptive Thresholding", *International Journal of Computer Theory and Engineering* , 4(3), 2012. Article (CrossRefLink)

[27] P.V. Leena, G. Remmiya Devi, V. Sowmya and K. P. Somam, "Least square based image denoising using wavelet filters", *Indian journal of science and technology*, 9(30), 2016.

[28] Rafael, C. Gonzalez and Richard E. Woods. "Digital Image Processing," *2nd edition, Prentice Hall*, 2002.

[29] A. Toet, "Multiscale color image enhancement", *in Proc. SPIE Int. Conf. Image Processing and Its Applications*, 583–585, 1992. Article (CrossRefLink)

[30] K. V. Velde, "Multi-scale color image enhancement", *Proc. Int. Conf. Image Processing*, 3, 584–587, 1999. Article (CrossRefLink)

[31] Dileep MD, and A. Sreenivasa Murthy, "A Comparison Between Different Color Image Contrast Enhancement Algorithms", *Proceedings of IEEE ICETECT*, 708-712, 2011.

[32] M. C. Hanumantharaju, M. Ravishankar, D. R. Rameshbabu, and S. Ramchandran, "Color Image Enhancement using Multiscale Retinex with Modified Color Restoration Technique", *Second International Conference on Emerging Applications of Information Tecnology*, 93-97, 2011. Article (CrossRefLink)

[33] E. Land, "Recent advances in retinex theory", *Vis. Res.*, 26(1), 7–21, 1986. Article (CrossRefLink)

[34] D. J. Jobson, Z. Rahman, and G. A. Woodell, "Properties and performance of a center/surround Retinex", *IEEE Trans. Image Processing*, 6, 451–462, 1997. Article (CrossRefLink)

[35] Z. Rahman, D. J. Jobson, and G. A. Woodell, "Multi-

scale retinex for color image enhancement", *IEEE Int. Conf. Image Processing,* 1996. Article (CrossRefLink)

[36] H. A. Mallot, "Computational Vision: Information Processing in Perception and Visual Behavior", *Cambridge, MA: MIT Press*, 2000.

[37] C. Munteanu and A. Rosa, "Color image enhancement using evolutionary principles and the retinex theory of color constancy", *Proc. Of IEEE Signal Processing Society Workshop on Neural Networks for Signal Processing XI*, 393–402, 2001. Article (CrossRefLink)

[38] D. J. Jobson, Z. Rahman, and G. A. Woodell, "A multi-scale retinex for bridging the gap between color images and the human observation of scenes", *IEEE Trans. Image Processing*, 6, 965–976, 1997. Article (CrossRefLink)

[39] K. Barnard and B. Funt, "Investigations into multi-scale Retinex", *Color Imaging: Vision and Technology, New York: Wiley*, 9–17, 1999.

[40] S. Kim, W. Kang, E. Lee, and J. Paik, "Wavelet-Domain Color Image Enhancement using Filtered Directional Bases and Frequency- Adaptive Shrinkage", *IEEE Trans. on consumer Electronics*, 56(2), 2010. Article (CrossRefLink)

[41] F. Xiao, M. Zhou, and G. Geng, "Detail Enhancement and Noise Reduction with Color Image Detection Based on Wavelet Multi-scale", *Proceedings of the 2nd International Conference on Artificial Intelligence, Management Science and Electronic Commerce. Zhengzhou, China: IEEE Press,* 1061-1064, 2011. Article (CrossRefLink)

[42] E. Provenzi and Vicent Caselles, "A Wavelet Perceptive on Variational Perceptually-Inspired Color Enhancement", *Int. J. Comput. Vis.,* 106, 153-171, 2014. Article (CrossRefLink)

[43] T. S. Huang and R. Y. Tsay, "Multiple frame image restoration and registration", *Advances in Computer Vision and Image Processing*, 1, 317-339, 1984.

[44] S. Lertrattanapanich and N. Bose, "High resolution image formation from low resolution frames using delaunay triangulation", *IEEE Transactions on Image Processing,* 11(12), 1427-1441, 2002. Article (CrossRefLink)

[45] X. Gao, K. Zhang, D. Tao, and X. Li, "Joint learning for single-image super-resolution via a coupled constraint", *IEEE Transactions on Image Processing*, 21(2), 469-480, 2012. Article (CrossRefLink)

[46] S. Dai, M. Han, W. Xu, Y. Wu, Y. Gong, and A. Katsaggelos, "Softcuts: A soft edge smoothness prior for color image super-resolution", *IEEE Transactions on Image Processing*, 18(5), 969-981, 2009. Article (CrossRefLink)

[47] J. Yang, J. Wright, T. S. Huang, and Y. Ma, "Image super-resolution via Sparse Representation", *IEEE Transactions on Image Processing,* 19(11), 2861-2873, 2010. Article (CrossRefLink)

[48] C. Ford and D. M. Etter, "Wavelet basis reconstruction of nonuniformly sampled data", *IEEE Transactions on Circuits and Systems,* 45, 1165–1168, 1998. Article (CrossRefLink)

[49] N. Nguyen and P. Milanfar, "A wavelet-based interpolation-restoration, method for superresolution (wavelet superresolution)", *IEEE Transactions on Circuits and Systems*, 19, 321–338, 2000. Article (CrossRefLink)

[50] N. Nguyen, P. Milanfar, and G. H. Golub, "A computationally efficient superresolution image reconstruction algorithm", *IEEE Transactions on Image Processing*, 10(4), 573–583, 2001. Article (CrossRefLink)

[51] N. Nguyen and P. Milanfar, "An efficient wavelet-based algorithm for image Superresolution*", ICIP*, 2000. Article (CrossRefLink)

[52] D. Glasner, S. Bagon, and M. Irani, "Super-resolution from a single image*", ICCV*, 2009. Article (CrossRefLink)

[53] X. Li and M. T. Orchard, "New edge-directed interpolation", *IEEE Transactions on Image Processing*, 10, 1521–1527, 2001. Article (CrossRefLink)

[54] L. Zhang and X. Wu, "An edge-guided image interpolation algorithm via directional filtering and data fusion", *IEEE Transactions on Image Processing*, 15, 2226-2238, 2006. Article (CrossRefLink)

[55] S. Mallat and G. Yu, "Super-resolution with sparse mixing estimators*", IEEE Transactions Image Processing,* 19(11), 2889–2900, 2010. Article (CrossRefLink)

[56] Y. Piao, I. Shin, and H. W. Park, "Image resolution enhancement using inter-subband correlation in wavelet domain", *Proceedings International Conference on Image Processing*, 1, 445–448, 2007. Article (CrossRefLink)

[57] H. Demirel and G. Anbarjafari, "Satellite image resolution enhancement using complex wavelet transform", *IEEE Geoscience and Remote Sensing Letter*, 7(1), 123–126, 2010. Article (CrossRefLink)

[58] C. B. Atkins, C. A. Bouman, and J. P. Allebach, "Optimal image scaling using pixel classification*", Proc. International Conference on Image Processing*, 3, 864–867, 2001. Article (CrossRefLink)

[59] H. Demirel and G. Anbarjafari, "Image Resolution Enhancement by Using Discrete and Stationary Wavelet Decomposition", *IEEE Transactions on Image Processing*, 20(5), 1458-1460, 2011. Article (CrossRefLink)

[60] H. Chavez-Roman and V. Ponomaryov, "Super Resolution Image Generation Using Wavelet Domain Interpolation With Edge Extraction via a Sparse Representation", *IEEE Geoscience and remote sensing Letters*, 11(10), 1777-17781, 2014. Article (CrossRefLink)

[61] K. Sowmya, P. Surya Kumari, A. Ranga, "Single Image Super Resolution with Wavelet Domain Transformation and Sparse Representation", *International Journal of Innovative Research in Computer Science & Technology (IJIRCST),* ISSN: 2347-5552, 4(1), 2016.

[62] Z. Liu, "Context-Based and Perceptual-Based wavelet Image Compression with Application to JPEG2000", *Arizona State University*, 2003.

[63] J. C. Feauveau, "Analyse multirésolution avec un facteur de resolution", *J. de traitement du signal*, 7(2),

117-128, 1990.
[64] R.A. DeVore, B. Jawerth, B.J. Lucier, "Image compression through wavelet transform coding", *IEEE Trans. on Information Theory*, 38(2), 719-746, 1992. Article (CrossRefLink)
[65] Vetterli, M. and Kovacevic, "J. Wavelets and Subband Coding, Englewood Cliffs", *NJ, Prentice Hall*, 1995.
[66] A. Averbuch, D. Lazar, and M. Israeli, "Image Compression Using Wavelet Transform and Multiresolution Decomposition", *IEEE Transactions on Image Processing*, 5(1), 1996. Article (CrossRefLink)
[67] N. Farvardin, H. Jafarkhani, J. Johnson, R. Bhattacharya, "Real-time wavelet-based video compression system using multiple TMS320C40s", *Application Report of Texas Instrument*, 1997.
[68] M. N. Do and M. Vetterli, "The finite ridgelet transform for image representation", *IEEE Trans. on Image Processing,* 12(1), 16-28, 2003. Article (CrossRefLink)
[69] H. Farid and S. Lyu, "Higher-order wavelet statistics and their application to digital forensics", *Proc. of IEEE Workshop on Statistical Analysis in Computer Vision, Madison (Wisconsin),* 2003. Article (CrossRefLink)
[70] K. H. Talukder and K. Harada, "A Scheme of Wavelet Based Compression of 2D Image", *Proc. IMECS, Hong Kong*, 531-536, 2006.
[71] K. H. Talukder, and K. Harada, "Haar Wavelet Based Approach for Image Compression and Quality Assessment of Compressed Image," *IAENG International Journal of Applied Mathematics,* 36(1), 2007.
[72] A. Khashman, and K. Dimililer, "Image Compression using Neural Networks and Haar Wavelet", *WSEAS Transactions on Image Processing,* 4(5), 330-339, 2008.
[73] G Boopathi, Dr.S. Arockiasamy "Image Compression: An approach using Wavelet Transform and Modified FCM", *International Journal of Computer Applications,* 28(2), 2011.
[74] V. Elamaran, and A. Praveen, "Comparison of DCT and wavelets in image coding", *International Conference on Computer Communication and Informatics (ICCCI), IEEE,* 1-4 ,2012. Article (CrossRefLink)
[75] M. Gupta and A. K. Garg, "Analysis Of Image Compression Algorithm Using DCT", *International Journal of Engineering Research and Applications (IJERA),* 2(1), 515-521, 2012.
[76] M. M. H. Chowdhury, and A. Khatun, "Image Compression Using Discrete Wavelet Transform", *International Journal of Computer Science,* 9(4), 327-330, 2012.
[77] Tarlok Singh; Pooja and P. Manchanda, "Image Compression Using Wavelets", *International Journal of Computer Science*, 11(4), No 2, 2014, ISSN (Print): 1694-0814 | ISSN (Online): 1694-0784.
[78] E. B. Corrochano, "Multiresolution image analysis using the quaternion wavelet transform", *Numerical Algorithms*, 39 (2005), 35-55. Article (CrossRefLink)
[79] E. B. Corrochano, "The theory and the use of the quaternion wavelet transform", *Journal of Mathematical Imaging and Vision,* 24(1),19-35, (2006). Article (CrossRefLink)
[80] M. Yin, W. Liu, J. Shui and J. Wu, "Quaternion wavelet analysis and application in Image Denoising", *Mathematical problems in Engineering, Hindwai publishing Corporation*, (2012).
[81] N. Kingsbury, "Image processing with complex wavelets", *Phil. Trans. Roy. Soc. London Ser, A* 357,2543–2560, (1999).

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