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PRICING OF QUANTO CHAINED OPTIONS

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Abstract. A chained option is a barrier option activated in the event that the underlying asset price crosses barrier or barriers prior to maturity in a specified order. In this paper, we study the pricing of chained options with the quanto property called the "*Quanto chained option*". A quanto chained option is a chained option starting at time when the foreign exchange rate has the multiple crossing of specified barriers. We provide closed-form formulas for valuing the quanto chained options based on probabilistic approach.

1. Introduction

A quanto option is a foreign currency option which has a payoff is converted into a domestic currency at maturity at a foreign exchange rate. Since the quanto option is widely used to avoid the exchange rate risk in the currency-related market, the quanto option is the one of the most popular exotic options in the over-the-counter (OTC) markets. Thus, many researchers have studied for the valuation of this option. Huang and Hung [3] studied the quanto option under Lévy model. Bo et al. [1] adopted the Markov-modulated jump-diffusion models to formulate time-varying sovereign ratings in the currency market. Park et al. [7], Lee and Lee [6] and Giese [2] also provided pricing formulas for the price of quanto option in the stochastic volatility model. A recent research by Teng et al. [8] provided the pricing and hedging strategy of quanto option with the dynamic correlation.

Among exotic options, a barrier option also is one of the most popular types of path-dependent derivatives. The barrier option is a contingent claim whose payoff depends on the relationship between the specified barriers and the path of the underlying asset. For example, a down-and-in call will be the European call option if the value of the underlying asset falls below the specified

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level. Since barrier options are cheaper than the European options and provide flexibility, they are widely used in financial markets. So, the research for barrier options is an important issue in the theory of finance.

The barrier options of various types are developed by many researchers. Recently, Jun and Ku [4], [5] introduced a new-type barrier option named as a chained option. The chained option is an option when two or more barrier options are chained together. The explicit price formulas of the chained option when the barriers follow exponential functions also were provided by Jun and Ku [5]. However, these works did not deal with two underlying asset in the pricing of chained option. Thus, this paper considers chained options with two underlying asset. More concretely, we extend their works by combining the quanto options and the chained options. This paper is organized as follows. In Section 2 we introduce the quanto chained options and derive the closed-form formulas for the price of quanto chained options. In Section 3 we give a conclusion.

2. Quanto chained option

We assume that $(\Omega, P, \mathcal{F})$ is a filtered probability space, where P is the risk-neutral measure. Let $S(t)$ be the asset price of foreign company in foreign currency, and $V(t)$ be the foreign exchange rate in domestic currency per unit of the foreign currency at time t. Then the dynamics of processes under risk-neutral measure P are respectively given by

$$dV(t) = (r_d - r_f)V(t)dt + \sigma_v V(t)dZ_v(t),$$
$$dS(t) = r_f S(t)dt + \sigma_s S(t)dZ_s(t),$$

where r_d and r_f are the instantaneous domestic and foreign interest rates, σ_v and σ_s are the volatilities of the exchange rate and the foreign stock, and $Z_v(t)$ and $Z_s(t)$ are the two standard Brownian motions with correlation ρ under risk-neutral measure P, i.e., $dZ_v(t)dZ_s(t) = \rho dt$.

We consider the quanto chained option activated at time when the foreign exchange rate crosses fixed barriers. To evaluate price of this option, we rewrite respectively the above dynamics as

$$dV(t) = (r_d - r_f)V(t)dt + \sigma_v V(t)dW_v(t), \tag{1}$$

$$dS(t) = r_f S(t)dt + \sigma_s S(t)(\rho dW_v(t) + \sqrt{1-\rho^2}dW_s(t)), \tag{2}$$

where $W_v(t)$ and $W_s(t)$ are independent standard Brownian motions under the risk-neutral measure P. Applying the Ito's lemma, we can find the solutions of the dynamics (1) and (2) as

$$V(t) = Ve^{(r_d - r_f - \frac{1}{2}\sigma_v^2)t + \sigma_v W_v(t)}, \; S(t) = Se^{(r_f - \frac{1}{2}\sigma_s^2)t + \sigma_s \rho W_v(t) + \sigma_s \sqrt{1-\rho^2} W_s(t)},$$

respectively, where $V = V(0)$ and $S = S(0)$.

In order to prove our results, we introduce the following lemma.

Lemma 2.1. *We assume that $W_1(T)$ and $W_2(T)$ are independent standard Brownian motions under the measure P with respect to its natural filtration $\mathcal{F}$. Then, for any $T>0$ and $u>0$,*

$$\begin{aligned}&\mathbf{E}^P\big[e^{\theta_1 W_1(T)+\theta_2 W_2(T)}\mathbf{1}_{\{\nu_1 W_1(T)+\nu_2 W_2(T)>k,\sup_{t\in[0,T]}W_1(t)\geq u\}}\big]\\&= e^{\frac{\theta_1^2+\theta_2^2}{2}T+2\theta_1 u}\Phi_2\left(\frac{2u\nu_1+\nu_1\theta_1 T+\nu_2\theta_2 T-k}{\sqrt{(\nu_1^2+\nu_2^2)T}},-\frac{u+\theta_1 T}{\sqrt{T}},-\frac{\nu_1}{\sqrt{\nu_1^2+\nu_2^2}}\right)\\&\quad+e^{\frac{\theta_1^2+\theta_2^2}{2}T}\Phi_2\left(\frac{\nu_1\theta_1 T+\nu_2\theta_2 T-k}{\sqrt{(\nu_1^2+\nu_2^2)T}},\frac{-u+\theta_1 T}{\sqrt{T}},\frac{\nu_1}{\sqrt{\nu_1^2+\nu_2^2}}\right),\end{aligned}$$

where $\theta_1,\theta_2,\nu_1,\nu_2$ and k are real numbers, Φ_2 is the cumulative bivariate normal distribution function, $\mathbf{1}_{\{A\}}$ is the indicator function of an event A and $\mathbf{E}^A$ is the expectation under the measure A.

Proof. We set $\tau=\inf\{t\geq 0\,|\,W_1(t)=u\}$. And let us define a process $X(t)$ by setting

$$X(t)=W_1(t)\mathbf{1}_{\{t\leq\tau\}}+(2u-W_1(t))\mathbf{1}_{\{t>\tau\}}.$$

Then from the reflection principle for Brownian motion, $X(t)$ is a standard Brownian motion under the measure P. Moreover, we have

$$\begin{aligned}&\mathbf{E}^P\big[e^{\theta_1 W_1(T)+\theta_2 W_2(T)}\mathbf{1}_{\{\nu_1 W_1(T)+\nu_2 W_2(T)>k,\sup_{t\in[0,T]}W_1(t)\geq u\}}\big]\\&=\mathbf{E}^P\big[e^{\theta_1(2u-X(T))+\theta_2 W_2(T)}\mathbf{1}_{\{\nu_1(2u-X(T))+\nu_2 W_2(T)>k,2u-X(T)<u\}}\big]\\&\quad+\mathbf{E}^P\big[e^{\theta_1 W_1(T)+\theta_2 W_2(T)}\mathbf{1}_{\{\nu_1 W_1(T)+\nu_2 W_2(T)>k,W_1(T)\geq u\}}\big]\\&=I_1+I_2.\end{aligned}$$

In order to calculate I_1, if we define a new measure $\overline{P}$ equivalent to P by

$$\frac{d\overline{P}}{dP}=\exp\left\{-\frac{\theta_1^2+\theta_2^2}{2}T-\theta_1 X(T)+\theta_2 W_2(T)\right\},$$

then, by the Girsanov's theorem, $\overline{X}(t)=X(t)+\theta_1 t$ and $\overline{W}_2(t)=W_2(t)-\theta_2 t$ are independent standard Brownian motions under the measure $\overline{P}$ and

$$\begin{aligned}&I_1\\&=\mathbf{E}^{\overline{P}}\Big[\tfrac{dP}{d\overline{P}}e^{\theta_1(2u-X(T))+\theta_2 W_2(T)}\mathbf{1}_{\{-\nu_1\overline{X}(T)+\nu_2\overline{W}_2(T)>k-2\nu_1 u-\nu_1\theta_1 T-\nu_2\theta_2 T,\overline{X}(T)>u+\theta_1 T\}}\Big]\\&=e^{\frac{\theta_1^2+\theta_2^2}{2}T+2\theta_1 u}\overline{P}\left(\nu_1\overline{X}(T)-\nu_2\overline{W}_2(T)<2\nu_1 u+\nu_1\theta_1 T+\nu_2\theta_2 T-k,\overline{X}(T)>u+\theta_1 T\right)\\&=e^{\frac{\theta_1^2+\theta_2^2}{2}T+2\theta_1 u}\Phi_2\left(\frac{2\nu_1 u+\nu_1\theta_1 T+\nu_2\theta_2 T-k}{\sqrt{(\nu_1^2+\nu_2^2)T}},-\frac{u+\theta_1 T}{\sqrt{T}},-\frac{\nu_1}{\sqrt{\nu_1^2+\nu_2^2}}\right).\end{aligned}$$

Similarly, if we define a new measure $\widetilde{P}$ equivalent to P by

$$\frac{d\widetilde{P}}{dP}=\exp\left\{-\frac{\theta_1^2+\theta_2^2}{2}T+\theta_1 W_1(T)+\theta_2 W_2(T)\right\},$$

then $\widetilde{W_1}(t) = W_1(t) - \theta_1 t$ and $\widetilde{W_2}(t) = W_2(t) - \theta_2 t$ are independent standard Brownian motions under the measure $\widetilde{P}$. Therefore we have

$$I_2 = \mathbf{E}^{\widetilde{P}}\left[\frac{dP}{d\widetilde{P}}e^{\theta_1 W_1(T)+\theta_2 W_2(T)}\mathbf{1}_{\{\nu_1\widetilde{W}_1(T)+\nu_2\widetilde{W}_2(T)>k-\nu_1\theta_1 T-\nu_2\theta_2 T,\widetilde{W}_1(T)>u-\theta_1 T\}}\right]$$

$$= e^{\frac{\theta_1^2+\theta_2^2}{2}T}\widetilde{P}(-\nu_1\widetilde{W_1}(T)-\nu_2\widetilde{W_2}(T) < \nu_1\theta_1 T+\nu_2\theta_2 T-k, \widetilde{W_1}(T) > u-\theta_1 T)$$

$$= e^{\frac{\theta_1^2+\theta_2^2}{2}T}\Phi_2\left(\frac{\nu_1\theta_1 T+\nu_2\theta_2 T-k}{\sqrt{(\nu_1^2+\nu_2^2)T}}, \frac{-u+\theta_1 T}{\sqrt{T}}, \frac{\nu_1}{\sqrt{\nu_1^2+\nu_2^2}}\right). \qquad \square$$

Let us consider a quanto option with maturity T and strike K. Then a close-form valuation formula for a down-and-in quanto chained call option ($DIQCC_u$) with the upper barrier U and the lower barrier D is provided by the following theorem.

Theorem 2.2. *The price at time* 0 *of quanto chained option* ($DIQCC_u$) *which is activated at time* $\tau = \min\{t > 0|V(t) = U, V < U\}$ *is*

$$\begin{aligned} DIQCC_u = {} & VSe^{\rho\sigma_v\sigma_s T}\left(\frac{D}{U}\right)^{\frac{2\mu_1}{\sigma_v^2}}\Phi_2(a_1, a_2, \rho) \\ & + VSe^{\rho\sigma_v\sigma_s T}\left(\frac{U}{V}\right)^{\frac{2\mu_1}{\sigma_v^2}}\Phi_2(b_1, b_2, -\rho) \\ & - KVe^{-r_f T}\left(\frac{D}{U}\right)^{\frac{2\mu_2}{\sigma_v^2}}\Phi_2(c_1, c_2, \rho) \\ & - KVe^{-r_f T}\left(\frac{U}{V}\right)^{\frac{2\mu_2}{\sigma_v^2}}\Phi_2(d_1, d_2, -\rho), \end{aligned}$$

where

$$a_1 = \left(\frac{\mu_1\rho}{\sigma_v} - \sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right) - \frac{g_1}{\sigma_s\sqrt{T}},$$

$$a_2 = \left(\frac{\mu_1}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right),$$

$$b_1 = \left(\frac{\mu_1\rho}{\sigma_v} - \sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{g_1}{\sigma_s\sqrt{T}},$$

$$b_2 = -\left(\frac{\mu_1}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right),$$

$$c_1 = \left(\frac{\mu_2\rho}{\sigma_v} - 2\sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right) - \frac{g_2}{\sigma_s\sqrt{T}},$$

$$c_2 = \left(\frac{\mu_2}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right),$$

$$d_1 = \left(\frac{\mu_2\rho}{\sigma_v} - 2\sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{g_2}{\sigma_s\sqrt{T}},$$

$$d_2 = -\left(\frac{\mu_2}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right),$$

with

$$\mu_1 = r_d - r_f + \frac{\sigma_v^2}{2} + \rho\sigma_v\sigma_s, \quad \mu_2 = \mu_1 - \rho\sigma_v\sigma_s,$$

$$g_1 = \ln\left(\frac{K}{S}\right) - \left(r_f + \frac{3}{2}\sigma_s^2 + \sigma_s\rho(\sigma_v - \sigma_s\rho) - \frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T - \frac{2\sigma_s\rho}{\sigma_v}\ln\left(\frac{U}{V}\right),$$

$$g_2 = \ln\left(\frac{K}{S}\right) - \left(r_f + \frac{3}{2}\sigma_s^2 + \sigma_s\rho(\sigma_v - 2\sigma_s\rho) - \frac{\mu_2\sigma_s\rho}{\sigma_v}\right)T - \frac{2\sigma_s\rho}{\sigma_v}\ln\left(\frac{U}{V}\right).$$

Proof. The price at time 0 of the down-and-in quanto chained call option is given by

$$\begin{aligned} DIQCC_u &= e^{-r_dT}\mathbf{E}^P\left[V(T)(S(T)-K)^+\mathbf{1}_{\{V(\tau)=U,\tau<T,\inf_{t\in[\tau,T]}V(t)\le D\}}\right] \\ &= e^{-r_dT}\mathbf{E}^P\left[V(T)S(T)\mathbf{1}_{\{V(\tau)=U,\tau<T,\inf_{t\in[\tau,T]}V(t)\le D,S(T)>K\}}\right] \\ &\quad - Ke^{-r_dT}\mathbf{E}^P\left[V(T)\mathbf{1}_{\{V(\tau)=U,\tau<T,\inf_{t\in[\tau,T]}V(t)\le D,S(T)>K\}}\right]. \end{aligned} \tag{3}$$

Let us consider the first term of Eq. (3). Then, under the measure P, the dynamic $V(T)S(T)$ is as follows.

$$V(T)S(T) = VSe^{(r_d - \frac{\sigma_v^2+\sigma_s^2}{2})T + (\sigma_s\rho+\sigma_v)W_v(T) + \sigma_s\sqrt{1-\rho^2}W_s(T)}.$$

We define a new measure $\overline{P}$ equivalent to measure P by

$$\frac{d\overline{P}}{dP} = \exp\left\{-\frac{\sigma_v^2 + 2\rho\sigma_v\sigma_s + \sigma_s^2}{2}T + (\rho\sigma_s + \sigma_v)W_v(T) + \sigma_s\sqrt{1-\rho^2}W_s(T)\right\}.$$

By the Girsanov's theorem, under the measure $\overline{P}$ the standard Brownian motions are respectively rewritten as $\overline{W}_v(t) = W_v(t) - (\rho\sigma_s + \sigma_v)t$, $\overline{W}_s(t) = W_s(t) - \sigma_s\sqrt{1-\rho^2}t$. Then, under the measure $\overline{P}$, $V(T) = Ve^{\mu_1 T + \sigma_v\overline{W}_v(T)}$ and

$$\begin{aligned} &e^{-r_dT}\mathbf{E}^P\left[V(T)S(T)\mathbf{1}_{\{V(\tau)=U,\tau<T,\inf_{t\in[\tau,T]}V(t)\le D,S(T)>K\}}\right] \\ &= VSe^{\rho\sigma_v\sigma_sT}\mathbf{E}^{\overline{P}}\left[\mathbf{1}_{\{V(\tau)=U,\tau<T\inf_{t\in[\tau,T]}V(t)\le D,S(T)>K\}}\right]. \end{aligned}$$

We define a new measure $\widetilde{P}$ again by

$$\frac{d\widetilde{P}}{d\overline{P}} = \exp\left\{-\frac{(\mu_1/\sigma_v)^2 + \sigma_s^2(1-\rho^2)}{2}T - \left(\frac{\mu_1}{\sigma_v}\right)\overline{W}_v(T) + \sigma_s\sqrt{1-\rho^2}\overline{W}_s(T)\right\}.$$

Let $Y(t) = \overline{W}_v(t) + \left(\frac{\mu_1}{\sigma_v}\right)t = \frac{1}{\sigma_v}\ln\left(\frac{V_t}{V}\right)$ and $\widetilde{W}_s(t) = \overline{W}_s(t) - \sigma_s\sqrt{1-\rho^2}t$. Then $Y(t)$ and $\widetilde{W}_s(t)$ are respectively independent standard Brownian motions under the measure $\widetilde{P}$ and

$$\overline{P}\left(V(\tau) = U, \tau < T \inf_{t\in[\tau,T]} V(t) \le D, S(T) > K\right)$$

$$= \overline{P}\left(Y(\tau)=u, \tau<T, \inf_{t\in[\tau,T]} Y(t)\le d, S(T)>K\right), \tag{4}$$

where $u = \frac{1}{\sigma_v}\ln\left(\frac{U}{V}\right)$ and $d = \frac{1}{\sigma_v}\ln\left(\frac{D}{V}\right)$. Let us define a process $\widetilde{Y}_t$ by setting

$$\widetilde{Y}(t) = Y(t)\mathbf{1}_{\{\tau\le t\}} + (2u - Y(t))\mathbf{1}_{\{\tau<t\}}.$$

Then, by relying on the reflection principle, $\widetilde{Y}_t$ also follows a standard Brownian motions under the measure $\widetilde{P}$ and

$$\begin{aligned}
&\mathbf{E}^{\overline{P}}\left[\mathbf{1}_{\{V(\tau)=U,\tau<T \inf_{t\in[\tau,T]} V(t)\le D, S(T)>K\}}\right] \\
&= e^{-\frac{(\mu_1/\sigma_v)^2+\sigma_s^2(1-\rho^2)}{2}T} \\
&\quad\times \mathbf{E}^{\widetilde{P}}\left[e^{\left(\frac{\mu_1}{\sigma_v}\right)Y(T)-\sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)}\mathbf{1}_{\{Y(\tau)=u,\tau<T,\inf_{t\in[\tau,T]} Y(t)\le d, S(T)>K\}}\right] \\
&= e^{-\frac{(\mu_1/\sigma_v)^2+\sigma_s^2(1-\rho^2)}{2}T} \\
&\quad\times \mathbf{E}^{\widetilde{P}}\left[e^{\left(\frac{\mu_1}{\sigma_v}\right)(2u-\widetilde{Y}(T))-\sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)}\mathbf{1}_{\{\sup_{t\in[0,T]} \widetilde{Y}(t)\ge 2u-d, S(T)>K\}}\right] \\
&= e^{-\frac{(\mu_1/\sigma_v)^2+\sigma_s^2(1-\rho^2)}{2}T+\frac{2u\mu_1}{\sigma_v}}\mathbf{E}^{\widetilde{P}}\left[e^{-\left(\frac{\mu_1}{\sigma_v}\right)\widetilde{Y}(T)-\sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)}\times\right. \\
&\left.\mathbf{1}_{\{\sup_{t\in[0,T]}\widetilde{Y}(t)\ge 2u-d,\, \sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)-\sigma_s\rho\widetilde{Y}(T)>\ln\left(\frac{K}{S}\right)-\left(r_f+\frac{3}{2}\sigma_s^2+\rho\sigma_s(\sigma_v-\sigma_s\rho)-\frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T-2u\sigma_s\rho\}}\right].
\end{aligned}$$

Since the two Brownian motions $\widetilde{Y}(t)$ and $\widetilde{W}_s(t)$ are independent, we can apply Lemma 2.1. Then we have

$$\begin{aligned}
&\mathbf{E}^{\overline{P}}\left[\mathbf{1}_{\{V(\tau)=U,\tau<T \inf_{t\in[\tau,T]} V(t)\le D, S(T)>K\}}\right] \\
&= e^{-2\left(\frac{\mu_1}{\sigma_v}\right)(u-d)}\times \\
&\Phi_2\left(\frac{\left(\frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T-\sigma_s^2(1-\rho^2)T-2\sigma_s\rho(2u-d)-g_1}{\sigma_s\sqrt{T}}, \frac{-(2u-d)+\left(\frac{\mu_1}{\sigma_v}\right)T}{\sqrt{T}}, \rho\right) \\
&+ e^{2\left(\frac{\mu_1}{\sigma_v}\right)u}\times \\
&\Phi_2\left(\frac{\left(\frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T-\sigma_s^2(1-\rho^2)T-g_1}{\sigma_s\sqrt{T}}, \frac{-(2u-d)-\left(\frac{\mu_1}{\sigma_v}\right)T}{\sqrt{T}}, -\rho\right).
\end{aligned}$$

The second term of Eq. (3) also can be calculated in a similar way. Therefore the proof of Theorem 2.2 is completed. □

The following theorem gives a closed-form valuation formula for an up-and-in quanto call option ($UIQCC_{ud}$) activated at time when the foreign exchange rate touches the lower barrier D after crossing the upper barrier U.

Theorem 2.3. *The price at time* 0 *of quanto chained option* ($UIQCC_{ud}$) *which is activated at time* $\tau_2 = \min\{t > \tau_1 | V(t) = D, \tau_1 = \min\{t > 0 \,|\, V(t) = U, V < U\}\}$ *is*

$$\begin{aligned} UIQCC_{ud} &= VSe^{\rho\sigma_v\sigma_s T}\left(\frac{U^2}{VD}\right)^{\frac{2\mu_1}{\sigma_v^2}} \Phi_2\left(a_1, a_2, -\rho\right) \\ &+ VSe^{\rho\sigma_v\sigma_s T}\left(\frac{D}{U}\right)^{\frac{2\mu_1}{\sigma_v^2}} \Phi_2\left(b_1, b_2, \rho\right) \\ &- KVe^{-r_f T}\left(\frac{U^2}{VD}\right)^{\frac{2\mu_2}{\sigma_v^2}} \Phi_2\left(c_1, c_2, -\rho\right) \\ &+ KVe^{-r_f T}\left(\frac{D}{U}\right)^{\frac{2\mu_2}{\sigma_v^2}} \Phi_2\left(d_1, d_2, \rho\right), \end{aligned}$$

where

$$\begin{aligned} a_1 &= \left(\frac{\mu_1\rho}{\sigma_v} - \sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{D}{U}\right) - \frac{g_1}{\sigma_s\sqrt{T}}, \\ a_2 &= -\left(\frac{\mu_1}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^3}{VD^2}\right), \\ b_1 &= \left(\frac{\mu_1\rho}{\sigma_v} - \sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right) - \frac{g_1}{\sigma_s\sqrt{T}}, \\ b_2 &= \left(\frac{\mu_1}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^3}{VD^2}\right), \\ c_1 &= \left(\frac{\mu_2\rho}{\sigma_v} - 2\sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{D}{U}\right) - \frac{g_2}{\sigma_s\sqrt{T}}, \\ c_2 &= -\left(\frac{\mu_2}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^3}{VD^2}\right), \\ d_1 &= \left(\frac{\mu_2\rho}{\sigma_v} - 2\sigma_s(1-\rho^2)\right)\sqrt{T} - \frac{2\rho}{\sigma_v\sqrt{T}}\ln\left(\frac{U^2}{VD}\right) - \frac{g_2}{\sigma_s\sqrt{T}}, \\ d_2 &= \left(\frac{\mu_2}{\sigma_v}\right)\sqrt{T} - \frac{1}{\sigma_v\sqrt{T}}\ln\left(\frac{U^3}{VD^2}\right), \end{aligned}$$

with μ_1, μ_2, g_1 *and* g_2 *defined in Theorem 2.2.*

Proof. The price at time 0 of the up-in quanto chained option is given by

$$\begin{aligned} & UIQCC_{ud} \\ &= e^{-r_d T}\mathbf{E}^P\left[V(T)(S(T)-K)^+\mathbf{1}_{\{V(\tau_1)=U, V(\tau_2)=D, \tau_1<\tau_2<T, \sup_{t\in[\tau_2,T]} V(t)\geq U\}}\right] \\ &= e^{-r_d T}\mathbf{E}^P\left[V(T)S(T)\mathbf{1}_{\{V(\tau_1)=U, V(\tau_2)=D, \tau_1<\tau_2<T, \sup_{t\in[\tau_2,T]} V(t)\geq U, S(T)>K\}}\right] \\ &\quad - Ke^{-r_d T}\mathbf{E}^P\left[V(T)\mathbf{1}_{\{V(\tau_1)=U, V(\tau_2)=D, \tau_1<\tau_2<T, \sup_{t\in[\tau_2,T]} V(t)\geq U, S(T)>K\}}\right] \end{aligned}$$

$= I_1 - I_2.$

From the measures and notations defined in Theorem 2.2, we have

(5)

$$
\begin{aligned}
& I_1 \\
&= VSe^{\rho\sigma_v\sigma_s T}\mathbf{E}^{\overline{P}}\left[\mathbf{1}_{\{V(\tau_1)=U,V(\tau_2)=D,\tau_1<\tau_2<T,\sup_{t\in[\tau_2,T]}Y(t)\geq U,S(T)>K\}}\right] \\
&= VSe^{\rho\sigma_v\sigma_s T-\frac{(\mu_1/\sigma_v)^2+\sigma_s^2(1-\rho^2)}{2}T}\times \\
&\quad \mathbf{E}^{\widetilde{P}}\left[e^{\left(\frac{\mu_1}{\sigma_v}\right)(2u-\widetilde{Y}(T))-\sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)}\mathbf{1}_{\{\widetilde{Y}(\tau_2)=2u-d,\tau_2<T,\inf_{t\in[\tau_2,T]}\widetilde{Y}(t)\leq u,S(T)>K\}}\right] \\
&= VSe^{\rho\sigma_v\sigma_s T-\frac{(\mu_1/\sigma_v)^2+\sigma_s^2(1-\rho^2)}{2}T+(-2u+2d)\left(\frac{\mu_1}{\sigma_v}\right)}\times \\
&\quad \mathbf{E}^{\widetilde{P}}\left[e^{\left(\frac{\mu_1}{\sigma_v}\right)\widehat{Y}(T)-\sigma_s\sqrt{1-\rho^2}\widetilde{W}_s(T)}\mathbf{1}_{\{\sup_{t\in[\tau_2,T]}\widehat{Y}(t)\geq 3u-2d,S(T)>K\}}\right],
\end{aligned}
$$

where $\widetilde{Y}(t)$ and $\widehat{Y}(t)$ are respectively defined by

$$
\begin{aligned}
\widetilde{Y}(t) &= Y(t)\mathbf{1}_{\{t\leq\tau_1\}} + (2u - Y(t))\mathbf{1}_{\{\tau_1<t\}}, \\
\widehat{Y}(t) &= \widetilde{Y}(t)\mathbf{1}_{\{t\leq\tau_2\}} + (2(2u-d) - \widetilde{Y}(t))\mathbf{1}_{\{\tau_2<t\}}.
\end{aligned}
$$

Then from the reflection principle, $\widetilde{Y}(t)$ and $\widehat{Y}(t)$ are the standard Brownian motions under the measure $\widetilde{P}$. Here, we apply Lemma 2.1 to Eq. (5). Then

$$
\begin{aligned}
I_1 &= VSe^{\rho\sigma_v\sigma_s T+(4u-2d)\left(\frac{\mu_1}{\sigma_v}\right)}\times \\
&\quad \Phi_2\left(\frac{\left(\frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T-\sigma_s^2(1-\rho^2)T-g_1-2\sigma_s\rho(d-u)}{\sigma_s\sqrt{T}}, -\frac{3u-2d+\left(\frac{\mu_1}{\sigma_v}\right)T}{\sqrt{T}}, -\rho\right) \\
&\quad + VSe^{\rho\sigma_v\sigma_s T-(2u-2d)\left(\frac{\mu_1}{\sigma_v}\right)}\times \\
&\quad \Phi_2\left(\frac{\left(\frac{\mu_1\sigma_s\rho}{\sigma_v}\right)T-\sigma_s^2(1-\rho^2)T-g_1-2\sigma_s\rho(2u-d)}{\sigma_s\sqrt{T}}, \frac{-(3u-2d)+\left(\frac{\mu_1}{\sigma_v}\right)T}{\sqrt{T}}, \rho\right).
\end{aligned}
$$

In a similar way, I_2 also can be calculated. This completes the proof. □

Remark 2.4. The prices of quanto chained options which have more hitting times can be generalized by the methodology in Theorems 2.2 and 2.3. Our results also can be extended to quanto chained options with curved barriers

3. Conclusion

We provide closed-form formulas for valuing a new type of quanto option starting at time when the foreign exchange rate correlated with the asset of foreign company crosses two or more barriers. The Girsanov theorem and the reflection principle are used repeatedly to find the formulas. We present the valuation formulas of the quanto chained option by the cumulative bivariate normal distribution function.

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