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ON THE CONJUGACY OF MÖBIUS GROUPS IN INFINITE DIMENSION

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Abstract. In this paper, we establish some conjugacy criteria of Möbius groups in infinite dimension by using Clifford matrices. This extends the corresponding known results in finite dimensional setting.

1. Introduction

It's well-known that a Kleinian group G of $\mathbf{SL}(2,\mathbb{C})$ is *Fuchsian* if there exists a G-invariant disc $\mathbb{D}$ in Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If we regard $\mathbb{D}$ as the upper half plane $\mathbb{H}^2$, then G is a subgroup of $\mathbf{SL}(2,\mathbb{R})$. The following classical result is due to Maskit (see [11]).

Theorem 1.1. *Let $G < \mathbf{SL}(2,\mathbb{C})$ be a non-elementary Kleinian group in which $tr(f) \in \mathbb{R}$ for all $f \in G$. Then G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{R})$.*

This implies that if the traces of all elements in G of $\mathbf{SL}(2,\mathbb{C})$ are real, then G preserves a hyperbolic plane in $\mathbb{H}^3$. For higher dimensional case, Apanasov [2] proved that if G is a non-elementary subgroup of $\mathbf{M}(\overline{\mathbb{R}}^n)$ of which each non-trivial element is either hyperbolic, strictly parabolic or strictly elliptic, then G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{R})$.

Subsequently, in [12], Wang and Yang gave a generalization of Apanasov's result by using n-dimensional Clifford matrices of $\mathbf{SL}(2,\Gamma_n)$ as follows:

Theorem 1.2. *Let $G < \mathbf{SL}(2,\Gamma_n)$ be non-elementary. If each loxodromic element of G is hyperbolic, then G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{R})$.*

However, in general, the trace for matrices of $\mathbf{SL}(2,\Gamma_n)$ is not invariant under conjugations. By adding some assumptions, Chen [3] proved that:

Theorem 1.3. *Let G be a non-elementary subgroup of $\mathbf{SL}(2,\Gamma_n)$ containing hyperbolic elements. Then G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{C})$ if and only if G is conjugate in $\mathbf{SL}(2,\Gamma_n)$ to G' which satisfies the following properties*:

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(1) *there exist hyperbolic elements* $g_0, h \in G'$ *such that*

$$fix(g_0) = \{0, \infty\}, \ fix(g_0) \cap fix(h) = \emptyset \ \text{and} \ fix(h) \cap \mathbb{C} \neq \emptyset,$$

(2) $tr(g) \in \mathbb{C}$ *for each* $g \in G'$,
where for each $h \in G'$, $fix(h)$ *denotes its fixed points set.*

For the case of complex (resp. quaternionic) hyperbolic isometric groups, We refer to [5, 6]. Recently, with the help of Clifford matrices of $\mathbf{SL}(\Gamma)$ introduced by Frunză [4], Li studied the discreteness and Jørgensen's inequality for Möbius groups in infinite dimension (see [7, 8, 9, 10]). Motivated by the above mentioned results, in this article, we study the corresponding problems for the case of Möbius groups in infinite dimension. Our main results are the following:

Theorem 1.4. *Let* G *be a non-elementary subgroup of* $\mathbf{SL}(\Gamma)$ *containing a loxodromic element fixing* 0 *and* ∞. *If* $tr(f) \in \mathbb{R}$ *for each element* $f \in G$, *then* G *is conjugate to a subgroup of* $\mathbf{SL}(2, \mathbb{R})$.

Theorem 1.5. *Let* $G < \mathbf{SL}(\Gamma)$ *be non-elementary. Assume that*:
(1) G *contains two loxodromic elements* f, g *such that*

$$fix(f) = \{0, \infty\}, \ fix(f) \cap fix(g) = \emptyset \ \text{and} \ fix(g) \cap \mathbb{C} \neq \emptyset;$$

(2) $tr(w) \in \mathbb{C}$ *for each* $w \in G$.
Then G *is a subgroup of* $\mathbf{SL}(2, \mathbb{C})$.

For the general case, we have:

Theorem 1.6. *Let* G *be a non-elementary subgroup of* $\mathbf{SL}(\Gamma)$ *containing hyperbolic elements. Then* G *is conjugate to a subgroup of* $\mathbf{SL}(2, \Gamma_m)$ *if and only if* G *is conjugate in* $\mathbf{SL}(\Gamma)$ *to* G' *which satisfies the following properties*:
(1) G' *contains a loxodromic element* f *and a hyperbolic element* g *such that*

$$fix(f) = \{0, \infty\}, \ fix(f) \cap fix(g) = \emptyset \ \text{and} \ fix(g) \cap \mathbb{R}^m \neq \emptyset;$$

(2) $tr(h) \in \Gamma_m$ *for each* $h \in G'$.

2. Preliminaries

We need the following preliminary material (see [4, 7, 8, 9, 10] for the details).

The Clifford algebra ℓ is the associative algebra over the real field $\mathbb{R}$, generated by a countable family $\{i_k\}_{k=1}^{\infty}$ subject to the following relations:

$$i_h i_k = -i_k i_h \quad (h \neq k), \quad i_k^2 = -1, \quad \forall h, k \geq 1.$$

Each element of ℓ can be expressed of the following type

$$a = \sum a_I I,$$

where $I = i_{v_1} i_{v_2} \cdots i_{v_p}, 1 \leq v_1 < v_2 < \cdots < v_p, p \leq n$, n is a fixed natural number depending on a, $a_I \in \mathbb{R}$ are the coefficients and $\sum_I a_I^2 < \infty$. If $I = \emptyset$, then a_I is called the real part of a and denoted by $\mathrm{Re}(a)$; the remaining part is called the imaginary part of a and denoted by $\mathrm{Im}(a)$.

For $a \in \ell$ the Euclidean norm of a is defined by

$$|a| = \sqrt{\sum_I a_I^2} = \sqrt{|\mathrm{Re}(a)|^2 + |\mathrm{Im}(a)|^2}.$$

The algebra ℓ has three involutions:

(1) " $'$ ": replacing each i_k $(k \geq 1)$ of a by $-i_k$, we get a new number denoted by a'. The mapping " $'$ " is an isomorphism of ℓ satisfying:

$$(ab)' = a'b', \ (a+b)' = a' + b'$$

for $a, \ b \in \ell$;

(2) " $*$ ": replacing each $i_{v_1} i_{v_2} \cdots i_{v_p}$ of a by $i_{v_p} i_{v_{p-1}} \cdots i_{v_1}$. We know that "$*$" is an anti-isomorphism of ℓ and for $a, b \in \ell$:

$$(ab)^* = b^* a^*, \ (a+b)^* = a^* + b^*;$$

(3) " $-$ ": $\bar{a} = (a^*)' = (a')^*$. It is obvious that the mapping"$a \to \bar{a}$" is also an anti-isomorphism of ℓ.

We say an element $x \in \ell$ *vector* if it has the following form

$$x = x_0 + x_1 i_1 + \cdots + x_n i_n + \cdots \in \ell.$$

The set of all such vectors is denoted by ℓ_2 and $\overline{\ell_2} = \ell_2 \cup \{\infty\}$. For any $x \in \ell_2$, one can see that $x^* = x$ and $\bar{x} = x'$. For $x, y \in \ell_2$ the *inner product* $\langle x \cdot y \rangle$ of x and y is given by

$$\langle x \cdot y \rangle = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n + \cdots ,$$

where $x = x_0 + x_1 i_1 + \cdots + x_n i_n + \cdots , \ y = y_0 + y_1 i_1 + \cdots + y_n i_n + \cdots .$

It's easy to verify that any non-zero vector x is invertible in ℓ with $x^{-1} = \frac{\bar{x}}{|x|^2}$. Let Γ be the set of all elements in ℓ which can be expressed as a finite product of non-zero vectors.

Definition 2.1 ([4]). Suppose that a matrix $f = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ satisfies the following relations.

(i) $a, b, c, d \in \Gamma \cup \{0\}$;
(ii) $ad^* - bc^* = 1$; and
(iii) $ab^*, \ d^*b, \ cd^*, \ c^*a \in \ell_2$.

Then we say that f is a Clifford matrix in infinite dimension. The set of all such matrices is a multiplicative group which is denoted by $\mathbf{SL}(\Gamma)$ (cf. [8]).

For each $f = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{SL}(\Gamma)$, the corresponding mapping in ℓ_2

$$x \mapsto f(x) = (ax+b)(cx+d)^{-1}$$

is a bijection of $\overline{\ell_2}$ onto itself, which we call a *Möbius transformation in infinite dimension.* Correspondingly, the set of all such mappings is also a group, which is still denoted by $\mathbf{SL}(\Gamma)$.

Now, we give a classification of elements in $\mathbf{SL}(\Gamma)$ which is similar to the one of $\mathbf{SL}(2,\Gamma_n)$ as follows (see [8]):

Setting

$$f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathbf{SL}(\Gamma),$$

we say that

(i) f is *loxodromic* if it is conjugate to $\begin{bmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{bmatrix}$, where $r>1$, $\lambda\in\Gamma$ and $|\lambda|=1$, in particular, f is *hyperbolic* if $\lambda=\pm 1$;

(ii) f is *parabolic* if it is conjugate to $\begin{bmatrix} \lambda & \mu \\ 0 & \lambda' \end{bmatrix}$, where $\lambda,\mu\in\Gamma$, $|\lambda|=1,\mu\neq 0$ and $\lambda\mu=\mu\lambda'$, in particular, f is *strictly* if $\lambda=\pm1$; otherwise.

(iii) f is *elliptic*, in particular, f is *strictly* if it is conjugate to $\begin{bmatrix} \tau & 1 \\ -1 & 0 \end{bmatrix}$, where $\tau\in\mathbb{R}$ and $|\tau|<2$.

Let G be a subgroup of $\mathbf{SL}(\Gamma)$, we say that G is *non-elementary* if there are two loxodromic elements in G with distinct fixed points.

Definition 2.2. For any $f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathbf{SL}(\Gamma)$, we define the trace of f as

$$tr(f)=a+d^*.$$

For a non-trivial element $f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathbf{SL}(\Gamma)$, if $b^*=b$, $c^*=c$ and $tr(f)\in\mathbb{R}$, then we call f *vectorial*.

By straightforward calculations, one can verify the following:

Lemma 2.1. *Let $f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathbf{SL}(\Gamma)$ be vectorial. Then $tr(f)$ is invariant under conjugation.*

The following lemma is crucial for us:

Lemma 2.2. *Let $f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in \mathbf{SL}(\Gamma)$ be loxodromic. Then f is hyperbolic if and only if f is vectorial. If f is hyperbolic and $c\neq 0$, then the two fixed points of f are*

$$u,v=-\frac{1}{2}(c^{-1}d-ac^{-1})\pm\frac{1}{2}c^{-1}((a+d^*)^2-4)^{\frac{1}{2}}.$$

Proof. Suppose that $f=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is hyperbolic. By its definition, there exists $g=\begin{bmatrix} u & v \\ s & t \end{bmatrix}\in \mathbf{SL}(\Gamma)$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} u & v \\ s & t \end{bmatrix}\begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}\begin{bmatrix} t^* & -v^* \\ -s^* & u^* \end{bmatrix},$$

where $r>1$. A simple computation shows that

$$a=rut^*-r^{-1}vs^*,\ b=r^{-1}vu^*-ruv^*,$$

$$c=rst^*-r^{-1}ts^*,\ d=r^{-1}tu^*-rsv^*.$$

This implies that f is a vectorial since $uv^*,ts^*\in\ell_2$. For the converse, as f is loxodromic and vectorial, by Lemma 2.1, we see that f is hyperbolic. The fixed points of f follows from [8]. $\square$

3. The proofs of main results

In order to prove our results, we need the following lemma.

Lemma 3.1 ([4]). *If $a \in \Gamma$ and $x \in \ell_2$, then the map $\rho(a)x = axa^* \in \ell_2$.*

Proof of Theorem 1.4. Let $f \in G$ be a loxodromic element fixing 0 and ∞. In terms of matrices, we can write

$$f = \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix},$$

where $r > 1$. Since G is non-elementary, there exists a loxodromic element $g \in G$ such that

$$fix(g) \cap fix(f) = \emptyset.$$

It follows that

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $bc \neq 0$. As for each positive integer n, $f^n g \in G$, we have

$$a + d^* \in \mathbb{R},$$

and

$$r^n a + r^{-n} d^* \in \mathbb{R}.$$

This implies that $a, d \in \mathbb{R}$ and $b, c \in \ell_2$. If $b \in \mathbb{R}$, then $c \in \mathbb{R}$. For $b \notin \mathbb{R}$, if $\mu = \frac{|b| - \bar{b}}{2}$, $q = \mu/|\mu|$ and

$$h = \begin{bmatrix} q & 0 \\ 0 & q' \end{bmatrix},$$

then we have

$$hfh^{-1} = f, \; hgh^{-1} = \begin{bmatrix} a & -|b| \\ k|b| & d \end{bmatrix}, k \in \mathbb{R}.$$

Observe that for each element $w \in G$, $tr(w) = tr(hwh^{-1})$. So in what follows, we always assume that $f, g \in \mathbf{SL}(2, \mathbb{R})$.

Let

$$p = \begin{bmatrix} u & v \\ s & t \end{bmatrix} \in \mathbf{SL}(\Gamma)$$

be any non-trivial element in G. Noting that $tr(f^n p) \in \mathbb{R}$ for each integer n, we obtain that $u, t \in \mathbb{R}$.

Since

$$gp = \begin{bmatrix} au + bs & av + bt \\ cu + ds & cv + dt \end{bmatrix},$$

and $au + bs, cv + dt \in \mathbb{R}$, we conclude that $v, s \in \mathbb{R}$.

Thus $p \in \mathbf{SL}(2, \mathbb{R})$. The proof is completed. □

The following result is analogous to the one of Apanasov in the setting of $\mathbf{SL}(\Gamma)$.

Corollary 3.1. *If G is a non-elementary subgroup of $\mathbf{SL}(\Gamma)$ of which each non-trivial element is either hyperbolic, strictly parabolic, or strictly elliptic then G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{R})$.*

Proof. Since each non-trivial element of G is either hyperbolic, strictly parabolic or strictly elliptic, we see that for each $f \in G$, $tr(f)$ is invariant under conjugation by Lemma 2.1. Hence, by Theorem 1.4, G is conjugate to a subgroup of $\mathbf{SL}(2,\mathbb{R})$. □

Proof of Theorem 1.5. Without loss of generality, we assume that G contains two loxodromic elements

$$f = \begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix},\ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(\Gamma),$$

with $k \in \mathbb{C}, |k| > 1$, $tr(g) = a + d^* \in \mathbb{C}$, and $abc \neq 0$. For each integer n, $f^n g \in G$, which yields $a, d \in \mathbb{C}$.

Since $a = a^*, d = d^*$, $ba^*, a^*c \in \ell_2$, we can write $b = a^*a'\mu a'^*$ and $c = a'\nu$, where $\mu, \nu \in \ell_2$. It follows from Lemma 3.1 that $a'\mu a'^* \in \ell_2$.

Setting $p = a'\mu a'^*$ and $q = \nu$, then g has the following form

$$g = \begin{bmatrix} a & ap \\ a'q & d \end{bmatrix},$$

where $p, q \in \ell_2$.

By Definition 2.1, $ad^* - ap(a'q)^* = 1$, which implies that $p \in \mathbb{C}$ if and only if $q \in \mathbb{C}$. Since $dap \in \ell_2$, it deduces that $ad \in \mathbb{R}$ or $p \in \mathbb{C}$. We claim that $p \in \mathbb{C}$. If $ad \in \mathbb{R}$ and $p \notin \mathbb{C}$, then $d = k_1 a'$, $q = k_2 p'$, where $k_1, k_2 \in \mathbb{R}$, i.e.,

$$g = \begin{bmatrix} a & ap \\ k_2 a'p' & k_1 a' \end{bmatrix}.$$

Under the conjugation of a suitable element in $\mathbf{SL}(2,\mathbb{R})$, we may assume that

$$f = \begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix},\ g = \begin{bmatrix} a & ap \\ \varepsilon a'p' & ra' \end{bmatrix} \in \mathbf{SL}(\Gamma),$$

with $k \in \mathbb{C}, |k| > 1$, $r \in \mathbb{R}$ and $\varepsilon = \pm 1$.

Let $p = p_0 + \sum_{s=2}^{\infty} p_s i_s$, where $p_0 \in \mathbb{C}$ and $p_s \in \mathbb{R}$. Since $fix(g) \cap \mathbb{C} \neq \emptyset$ and g is a loxodromic element, we have $p_0 \neq 0$ and $a' = -\varepsilon a$. Since g^2 is loxodromic, by a simple calculation, $pa'p' \in \mathbb{C}$, which implies that $a = a'$, i.e. $a \in \mathbb{R}$. By Lemma 2.2, g is hyperbolic and its fixed points are

$$x, y = -\frac{1}{2}((\varepsilon a'p')^{-1}d - a(\varepsilon a'p')^{-1}) \pm \frac{1}{2}(\varepsilon a'p')^{-1}((a + ra')^2 - 4)^{\frac{1}{2}}.$$

It follows immediately that $p \in \mathbb{C}$ since $a \in \mathbb{R}$.

So

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2,\mathbb{C}).$$

Let

$$h = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathbf{SL}(\Gamma)$$

be any non-trivial element in G. It follows from a discussion similar to that of Theorem 1.4 that $h \in \mathbf{SL}(2, \mathbb{C})$, which completes the proof. □

Corollary 3.2. *Let $G < \mathbf{SL}(\Gamma)$ be non-elementary. If G is conjugate in $\mathbf{SL}(\Gamma)$ to G' which satisfying properties* (1) *and* (2) *as in Theorem 1.5, then G is discrete if and only if each subgroup generated by two elements of G is discrete.*

Proof of Theorem 1.6. Sufficiency. In view of the assumptions, $f, g \in G$ can be written as

$$f = \begin{bmatrix} k\lambda & 0 \\ 0 & k^{-1}\lambda' \end{bmatrix}, \ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(\Gamma),$$

where $k > 1, \lambda \in \Gamma_m$ and $bc \neq 0$. Let $w = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ be any non-trivial element in G. By considering $tr(f^n w)$ for each integer n, we see that $\alpha, \delta \in \Gamma_m$. Since g is hyperbolic, $a, d \in \Gamma_m$ and $fix(f) \cap fix(g) = \emptyset$, by Lemma 2.2, we have $b, c \in \Gamma_m$. Hence, $a, b, c, d \in \Gamma_m$. By using a similar method as in the proof of Theorem 1.4, we conclude that G is a subgroup of $\mathbf{SL}(2, \Gamma_m)$.

Necessity. Since G is conjugate to a subgroup of $\mathbf{SL}(2, \Gamma_m)$, it's easy to find an element $h \in \mathbf{SL}(\Gamma)$ such that $hGh^{-1} = G'$ which satisfies the conditions (1) and (2).

The proof is complete. □

Remark 3.1. Obviously, $\mathbf{SL}(2, \Gamma_n)$ can be viewed as a subgroup of $\mathbf{SL}(\Gamma)$. Hence, the conclusions still hold if we replace $\mathbf{SL}(\Gamma)$ by $\mathbf{SL}(2, \Gamma_n)$ $(n > m)$ in Theorems 1.4 $\sim$ 1.6.

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References

[1] L. V. Ahlfors, *On the fixed points of Möbius transformations in* $\bar{\mathbb{R}}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 15–27.

[2] B. N. Apanasov, *The Geometry of Discrete Groups and Manifolds*, Nauka, Moscow, 1991.

[3] M. Chen, *The extension of Möbius groups*, Complex Var. Theory Appl. **47** (2002), no. 3, 225–228.

[4] M. Frunză, *Möbius transformations in infinite dimension*, Rev. Roumaine Math. Pures Appl. **36** (1991), no. 7-8, 369–376.

[5] X. Fu and B. Xie, *A characterization of Fuchsian groups in* $\mathbf{SU}(n, 1)$, Complex Var. Elliptic Equ. **59** (2014), no. 5, 723–731.

[6] J. Kim, *Quaternionic hyperbolic Fuchsian groups*, Linear Algebra Appl. **438** (2013), no. 9, 3610–3617.

[7] L. Li, *Ball-preserving Möbius transformations in infinite dimension*, Complex Var. Elliptic Equ. **54** (2009), no. 7, 697–703.
[8] ———, *A generalization of Jørgensen's inequality to infinite dimension*, New York J. Math. **17** (2011), 41–49.
[9] ———, *An inequality on parabolic Möbius groups in infinite dimension and its application*, Acta Math. Sci. Ser. A Chin. Ed. **32** (2012), no. 2, 349–355.
[10] ———, *Discreteness of Möbius groups in infinite dimension*, Complex Var. Elliptic Equ. **58** (2013), no. 1, 109–112.
[11] B. Maskit, *Kleinian Groups*, Springer-Verlag, New York, 1987.
[12] X. Wang and W. Yang, *Generating systems of subgroups in* $\mathbf{PSL}(2, \Gamma_n)$, Proc. Edinb. Math. Soc. (2) **45** (2002), no. 1, 49–58.
[13] P. Waterman, *Möbius transformations in several dimensions*, Adv. Math. **101** (1993), no. 1, 87–113.

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