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ON GENERALIZED QUASI-CONFORMAL $N(k,\mu)$-MANIFOLDS

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Abstract. The object of the present paper is to introduce a new curvature tensor, named *generalized quasi-conformal curvature tensor* which bridges conformal curvature tensor, concircular curvature tensor, projective curvature tensor and conharmonic curvature tensor. Flatness and symmetric properties of *generalized quasi-conformal curvature tensor* are studied in the frame of (k,μ)-contact metric manifolds.

1. Introduction

In 1968, Yano and Sawaki [27] introduced the notion of quasi-conformal curvature tensor which contains both conformal curvature tensor as well as concircular curvature tensor, in the context of Riemannian geometry. In tune with Yano and Sawaki [27], the present paper attempts to introduce a new tensor field, named *generalized quasi-conformal curvature tensor.* The beauty of *generalized quasi-conformal curvature tensor* lies in the fact that it has the flavour of Riemann curvature tensor R, conformal curvature tensor C [8] conharmonic curvature tensor $\hat{C}$ [9], concircular curvature tensor E [26, p. 84], projective curvature tensor P [26, p. 84] and m-projective curvature tensor H [15], as particular cases. The *generalized quasi-conformal curvature tensor* is defined as

$$\begin{aligned}\mathcal{W}(X,Y)Z = {}& \frac{2n-1}{2n+1}\left[(1-b+2na)-\{1+2n(a+b)\}c\right] C(X,Y)Z \\ & + [1-b+2na]E(X,Y)Z + 2\,n\,(b-a)\,P(X,Y)Z \\ & + \frac{2\,n-1}{2\,n+1}(c-1)\{1+2\,n(a+b)\}\,\hat{C}(X,Y)Z \end{aligned} \tag{1.1}$$

for all $X,Y,Z\in\chi(M)$, the set of all vector field of the manifold M, where a, b and c are real constants. The above mentioned curvature tensors are defined

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as follows

$$
\begin{aligned}
C(X,Y)Z = R(X,Y)Z &- \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y] \\
&+ g(Y,Z)\ QX - g(X,Z)\ QY] \\
&+ \frac{r}{2n(2n-1)}[\ g(Y,Z)X - \ g(X,Z)Y],
\end{aligned} \tag{1.2}
$$

$$
E(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[\ g(Y,Z)X - \ g(X,Z)Y], \tag{1.3}
$$

$$
P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[\ S(Y,Z)X - \ S(X,Z)Y], \tag{1.4}
$$

$$
\begin{aligned}
\hat{C}(X,Y)Z = R(X,Y)Z &- \frac{1}{2n-1}[\ S(Y,Z)X - \ S(X,Z)Y] \\
&+ g(Y,Z)\ QX - \ g(X,Z)\ QY]
\end{aligned} \tag{1.5}
$$

for all $X, Y, Z \in \chi(M)$, where S, Q and r being Ricci tensor, Ricci operator and scalar curvature respectively. The *generalized quasi-conformal curvature tensor* $\mathcal{W}$ is reduced to be (1) Riemann curvature tensor R, if $a = b = c = 0$, (2) conformal curvature tensor C, if $a = b = -\frac{1}{2\ n-1}$, $c = 1$, (3) conharmonic curvature tensor $\hat{C}$, if $a = b = -\frac{1}{2\ n-1}$, $c = 0$, (4) concircular curvature tensor E, if $a = b = 0$ and $c = 1$, (5) projective curvature tensor P, if $a = -\frac{1}{2\ n}$, $b = 0$, $c = 0$ and (6) m-projective curvature tensor H, if $a = b = -\frac{1}{4\ n}$, $c = 0$. The m-projective curvature tensor is introduced by G. P. Pokhariyal and R. S. Mishra [15], which is defined as follows

$$
\begin{aligned}
H(X,Y)Z = R(X,Y) &- \frac{1}{4n}[\ S(Y,Z)X - \ S(X,Z)Y \\
&+ g(Y,Z)\ QX - \ g(X,Z)\ QY].
\end{aligned} \tag{1.6}
$$

Note that our generalized quasi-conformal curvature tensor $\mathcal{W}$ is not a generalized curvature tensor [7], [11], as it does not satisfy the condition

$$
\mathcal{W}(X_1, X_2, X_3, X_4) = \mathcal{W}(X_3, X_4, X_1, X_2),
$$

where $\mathcal{W}(X_1, X_2, X_3, X_4) = g(\mathcal{W}(X_1, X_2)X_3, X_4)$ for all X_1, X_2, X_3, X_4. Moreover our $\mathcal{W}$ is not a proper generalized curvature tensor [11], as it does not satisfy the second Bianchi identity

$$
(\nabla_{X_1}\mathcal{W})(X_2, X_3)X_4 + (\nabla_{X_2}\mathcal{W})(X_3, X_1)X_4 + (\nabla_{X_3}\mathcal{W})(X_1, X_2)X_4 = 0. \tag{1.7}
$$

A contact metric manifold with ξ belonging to (k, μ)-nullity distribution (we denote such manifold by $N(k, \mu)$-manifold), is said to be semi-symmetry type (respectively Ricci semi-symmetry type) if the generalized quasi-conformal curvature tensor $\mathcal{W}$(respectively Ricci tensor S) obeys the condition

$$
R(X,Y) \cdot \mathcal{W} = 0, \ \text{respectively } \mathcal{W}(X,Y) \cdot S = 0 \text{ for any } X, Y \text{ on } M, \tag{1.8}
$$

where the dot means that $R(X,Y)$ acts on $\mathcal{W}$ (respectively on S) as derivation. In particular, manifold satisfying the condition $R(X,Y) \cdot R = 0$ (obtained from

(1.8) by setting $a = b = c = 0$) is said to be semi-symmetric, in the sense of Cartan [6, p. 265]. A full classification of such space is given by Z. I. Szabő ([22], [23], [24]). This type of the manifolds have been studied by several authors such as Sekigawa and Tanno [18], Papantoniou [13], Perrone [14], Kowalski [10], and the references therein. Our work is structured as follows. Section 2 of the present paper is concerned with some basic results of $N(k,\mu)$-manifold. In Section 3, we have studied generalized quasi-conformally flat $N(k,\mu)$-manifold and it is found that such a manifold is either an Einstein space or an η -Einstein space or *locally isometric to the Riemann sphere* $E^{2n+1}(1)$. It is also proved that every 3-dimensional non-Sasakian $N(k,\mu)$-manifold with vanishing generalized quasi-conformal curvature tensor is necessarily $N(k)$-manifold. Section 4 is devoted to the study of $N(k,\mu)$-manifold with divergent free *generalized quasi-conformal curvature tensor* and observed that the Ricci tensor of such manifold is Codazzi tensor. $N(k,\mu)$-manifold with $\mathcal{W}\cdot S = 0$ is discussed in Section 5 and it is pointed that the relations -(a) M is an Einstein space, (b) M is Ricci symmetric, i.e., $\nabla S = 0$, (c) $P(\xi.X)\cdot S = 0$ (or $E(\xi,X)\cdot S = 0$) are equivalent. In the next section, we have investigated $N(k,\mu)$-manifold satisfying $R(\xi,X)\cdot\mathcal{W} = 0$ and based on it, the nature of the Ricci tensors for different semi-symmetry type conditions are obtained and tabled. Furthermore, we bring out that a $N(k,\mu)$-manifold satisfying the relation $R(\xi,X)\cdot C = 0$ (resp. $R(\xi,X)\cdot\hat{C} = 0$) is either conformally (resp. conharmonically) flat or *locally isometric to* $S^{2n+1}(1)$.

2. Preliminaries

In this section, we recall some basic results which will be used later. A $(2n+1)$-dimensional differential manifold M^{2n+1} is called a contact manifold if it carries a global differentiable 1-form η such that $\eta\Lambda(d\eta)^n \neq 0$ everywhere on M^{2n+1}. This 1-form η is called the contact form on M^{2n+1}. A Riemannian metric g is said to be associated with a contact manifold if there exist a $(1,1)$ tensor field ϕ and a contravariant global vector field ξ, called the characteristic vector field of the manifold such that

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta\circ\phi = 0, \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X,\phi Y) = -g(Y,\phi X),\ g(X,\xi) = \eta(X),\ g(X,\phi Y) = d\eta(X,Y) \tag{2.3}$$

for all vector fields X, Y on M. In a contact metric manifold we define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie differentiation. Then h is symmetric and satisfies $h\phi = -\ \phi h$. We have $Trh = Tr\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X\xi = -\phi X - \phi hX, \tag{2.4}$$

holds in a contact metric manifold. D. E. Blair, T. Koufogiorgos and B. J. Papantioniou [4] considered the (k,μ)-nullity condition on a contact metric manifold and gave several reasons for studying it. The (k,μ)-nullity distribution $N(k,\mu)$ ([2], [13], [25]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k,\mu): p \to N_P(k,\mu) = U \in \quad & T_P M \mid R(X,Y)Z \\ &= \{(kI+\mu h)g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.5}$$

for all $X, Y \in TM$, where $(k,\mu) \in R^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k,\mu)$ is called a (k,μ)-contact metric manifold, which we denote by $N(k,\mu)$-manifold. We have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

Also, in a (k,μ)-contact metric manifold, the following relations hold [3]:

$$h^2 = (k-1)\phi^2, k \le 1, \tag{2.7}$$

$$(\nabla_X \phi)(Y) = g(X+hX,Y)\xi - \eta(Y)(X+hX), \tag{2.8}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\quad + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \end{aligned} \tag{2.9}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.10}$$

$$R(\xi,X)\xi = k[\eta(X)\xi - X] - \mu hX, \tag{2.11}$$

$$\begin{aligned} S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1)+\mu]g(hX,Y) \\ &\quad + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y). \end{aligned} \tag{2.12}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.13}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2+\mu)g(hX,Y), \tag{2.14}$$

where S is the Ricci tensor of type $(0,2)$ of the manifold.

$$(\nabla_X \eta)(Y) = g(X+hX,\phi Y), \tag{2.15}$$

$$\begin{aligned} (\nabla_X h)(Y) &= \{(1-k)g(X,\phi Y) + g(X,h\phi Y)\}\xi \\ &\quad + \eta(Y)\{h(\phi X + \phi hX)\} - \mu\eta(X)\phi hY. \end{aligned} \tag{2.16}$$

If $\mu = 0$, the (k,μ)-nullity distribution is reduced to k-nullity distribution [25]. The (k,μ)-contact metric manifolds are studied by several geometers (see [5], [3], [19], [20], [21], etc.).

Proposition 2.1 ([3]). *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a (k,μ)-contact metric manifold. Then the relation*

$$Q\phi - \phi Q = 2[2(n-1)+\mu]h\phi \quad \textit{holds.} \tag{2.17}$$

Lemma 2.2 ([3]). *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $N(k,\mu)$-manifold with harmonic curvature tensor. Then M is either* (i) *Einstein Sasakian manifold, or* (ii) *η-Einstein manifold, or* (iii) *locally isometric to the Riemannian product $E^{2n+1} \times S(4)$ including a flat contact metric structure for $n = 1$.*

3. Generalized quasi-conformally flat $N(k,\mu)$-manifolds

In this section, we study the flatness of the *generalized quasi-conformal curvature tensor in* $N(k,\mu)$-manifold. In consequence of (1.2)-(1.5) and (1.1), the *generalized quasi-conformal curvature tensor* $\mathcal{W}$ takes the form

$$\begin{aligned}\mathcal{W}(X,Y)Z = {} & R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] \\ & + b[g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

Definition. A $N(k,\mu)$-manifold is said to be η-Einstein if its Ricci tensor S satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \forall X,Y \in \chi(M) \tag{3.2}$$

for some real constants α and β.

Such notion was first introduced and studied by Okumura [12] and named by Sasaki [17] in his lecture notes 1965. In particular, if $\beta = 0$, we say that $N(k,\mu)$-manifold is Einstein. Suppose that $M^{2n+1}(\phi,\xi,\eta,g)$ is a generalized quasi-conformally flat $N(k,\mu)$-manifold. Then from (3.1), we obtain

$$\begin{aligned} R(X,Y)Z = {} & -a[S(Y,Z)X - S(X,Z)Y] - b[g(Y,Z)QX - g(X,Z)QY] \\ & + \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.3}$$

Taking inner product on both sides of (3.3) by ξ and then using (2.9) and (2.13), we get

$$\begin{aligned} & \left[k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right]\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \\ & + \mu\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\} \\ & + a\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\} = 0. \end{aligned} \tag{3.4}$$

Substituting Y by ξ in (3.4), we obtain by virtue of (2.1) and (2.13), that

$$S(X,Z) = \alpha g(X,Z) - \frac{\mu}{a}g(hX,Z) + \beta\eta(X)\eta(Z) \tag{3.5}$$

for all X and Z, provided $a \neq 0$, where

$$\alpha = -\frac{1}{a}\left[k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n} + a + b\right)\right] \text{ and}$$

$$\beta = \frac{1}{a}\{1 + 2n(a+b)\}\left\{k - \frac{cr}{2n(2n+1)}\right\}.$$

In view of (2.1), (2.7), (2.12) and (3.5), we have

$$S(X,Z) = \left[\frac{a\{2(n-1)+\mu\}\alpha + \mu\{2(n-1) - n\mu\}}{\mu(a+1) + 2a(n-1)}\right]g(X,Z)$$

$$+\left[\frac{\beta\{2(n-1)+\mu\}a+\mu\{2(1-n)+n(2k+\mu)\}}{\mu(a+1)+2a(n-1)}\right]\eta(X)\eta(Z). \tag{3.6}$$

Again, for a $N(k,\mu)$-manifold with $R=0$ or $E=0$ (i.e., for the case $a=0$ and $b=0$), one can easily determine that such manifold is an Einstein. This leads to the followings:

Theorem 3.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a generalized quasi-conformally flat $N(k,\mu)$-manifold. Then M is either an Einstein manifold or an η-Einstein manifold or isometric to the Riemann sphere $S^{2n+1}(1)$.*

Again, in view of (3.5), we have

$$QX = \alpha X - \frac{\mu}{a}hX + \beta\eta(X)\xi \tag{3.7}$$

which gives

$$Q\phi - \phi Q = -\frac{2\mu}{a}h\phi \ \text{ as } \phi h + h\phi = 0. \tag{3.8}$$

By virtue of (2.17) and (3.8), we have

$$\mu = -\left[\frac{2a(n-1)}{a+1}\right],\ a \neq -1. \tag{3.9}$$

Thus for $n=1$, we can state the following corollary

Theorem 3.2. *Every three dimensional non-Sasakian $N(k,\mu)$-manifold with vanishing generalized quasi-conformal curvature tensor is necessarily $N(k)$-manifold.*

Theorem 3.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a generalized quasi-conformally flat $N(k,\mu)$-manifold. Then grad r and the characteristic vector field ξ are co-directional.*

Proof. Differentiating (3.7), we obtain

$$(\nabla_Y Q)X = -\frac{\mu}{a}(\nabla_Y h)X + \beta[(\nabla_Y \eta)X + \eta(X)\nabla_Y \xi]. \tag{3.10}$$

In consequence of (2.4), (2.15) and (2.16), the relations (3.10) reduces to

$$\begin{aligned} g(\nabla_Y Q)X,U) = &-\frac{\mu}{a}[\{(1-k)g(y,\phi X)+g(Y,h\phi X)\}\eta(U) \\ &+\eta(X)\{g(\phi Y,hU)+g(\phi hY,hU)\}+\mu\eta(Y)g(\phi hX\ ,U)] \\ &+\beta[g(Y+hY,\phi X)\eta(U)-\eta(X)g(\phi Y,U)-\eta(X)g(\phi hY,U)]. \end{aligned} \tag{3.11}$$

Let $\{e_1,e_2,e_3,\dots,e_{2n},e_{2n+1}=\xi\}$ be an orthonormal basis of the tangent space at any point of the manifold. Contracting Y over U, we get from the above

$$(divQ)X = -\Psi\left[\beta - \frac{\mu(k-1)}{a}\right]\eta(X) \ \text{ for all } X, \tag{3.12}$$

where $\Psi = \text{trace}(\phi)$. This gives

$$X = \xi, \; gradr = 2\Psi\left[\frac{\mu(k-1)}{a} - \beta\right]\xi. \tag{3.13}$$

This completes the proof. □

Again, for $\Psi = 0$ and Lemma 2.1 we can easily bring out the following theorem.

Theorem 3.4 ([13])**.** *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n > 1)$ be a generalized quasi-conformally flat $N(k,\mu)$-manifold with trace$(\phi) = 0$. Then M is either* (i) *Einstein Sasakian manifold, or* (ii) *η-Einstein manifold, or* (iii) *locally isometric to the Riemannian product $E^{n+1}\times S^n(4)$ including a flat contact metric structure for $n = 1$.*

4. $N(k,\mu)$-manifold with divergent free generalized quasi-conformal curvature tensor

Theorem 4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n > 1)$ be a $N(k,\mu)$-manifold with divergent free generalized quasi-conformal curvature tensor. Then the Ricci tensor S is a Codazzi tensor.*

Proof. Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $N(k,\mu)$-manifold satisfying the condition

$$(div\mathcal{W})(X,Y)Z = 0, \tag{4.1}$$

i.e.,

$$\begin{aligned} &divR(X,Y)Z + a[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ &+\frac{b}{2}[dr(X)g(Y,Z) - dr(Y)g(X,Z)] \\ &-\frac{c}{2n+1}\left[\frac{1}{2n} + a + b\right][dr(X)g(Y,Z) - dr(Y)g(X,Z)] = 0, \end{aligned} \tag{4.2}$$

i.e.,

$$\begin{aligned} &(1+a)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ &+\left[\frac{b}{2} - \frac{c}{2n+1}\left(\frac{1}{2n} + a + b\right)\right][dr(X)g(Y,Z) - dr(Y)g(X,Z)] = 0. \end{aligned} \tag{4.3}$$

Putting $X = Z = e_i$ in (4.3) and then taking summation over i, $1 \le i \le 2n+1$, we get

$$(1+a)\left[\frac{1}{2}dr(Y) - dr(Y)\right] + \frac{b}{2}[dr(Y) - (2n+1)dr(Y)] = 0,$$

i.e.,

$$[1 + a + 2nb]dr(Y) = 0, \tag{4.4}$$

hence $dr(Y) = 0$ for all Y, provided $1 + a + 2nb \neq 0$.

Using (4.4) in (4.3), we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0,$$

i.e.,

$$(\nabla_X Q)(Y) = (\nabla_Y Q)(X). \tag{4.5}$$

This completes the proof. □

The above theorem implies that the curvature tensor R of the manifold is harmonic. Consequently we state the following theorem.

Theorem 4.2 ([3]). *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a $N(k,\mu)$-manifold with divergent free generalized quasi-conformal curvature tensor. Then M is either* (i) *Einstein Sasakian manifold, or* (ii) *η-Eintein manifold, or* (iii) *locally isometric to the Riemannian product $E^{2n+1}\times S(4)$ including a flat contact metric structure for $n=1$.*

5. $N(k,\mu)$-manifold with $\mathcal{W}\cdot S = 0$

Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a generalized quasi-conformal $N(k,\mu)$-manifold satisfying the condition

$$\mathcal{W}(\xi,Y)\cdot S = 0, \tag{5.1}$$

i.e., $\quad \mathcal{W}\ (\xi,Y)S(Z,U) - S(\mathcal{W}\ (\xi,Y)Z,U) - S(Z,\mathcal{W}(\xi,Y)U) = 0,$

i.e.,

$$S(\mathcal{W}\ (\xi,Y)Z,U) + S(Z,\mathcal{W}(\xi,Y)U) = 0. \tag{5.2}$$

Taking $U=\xi$ in (5.2) and using (2.13), we get

$$2nk\eta(\mathcal{W}(\xi,Y)Z) + S(Z,\mathcal{W}(\xi,Y)\xi) = 0. \tag{5.3}$$

In view of (2.10), (2.13) and (3.1), we have

$$\begin{aligned}\eta(\mathcal{W}(\xi,Y)Z) =\ & \left[k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right] g(Y,Z)\\ & - \left[k(1+2na+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]\eta(Y)\eta(Z)\\ & + \mu g(hY,Z) + aS(Y,Z) + aS(Y,Z),\end{aligned} \tag{5.4}$$

$$\begin{aligned}S(Z,\mathcal{W}(\xi,Y)\xi) =\ & -\left[k(1+2na) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right] S(Y,Z)\\ & + 2nk\left[k(1+2na+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right]\eta(Y)\eta(Z)\\ & - \mu S(hY,Z) - bS^2(Y,Z).\end{aligned} \tag{5.5}$$

By virtue of (5.4) and (5.5), (5.3) yields

$$\begin{aligned}S^2(Y,Z) =\ & \frac{2nk}{b}\left[k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right] g(Y,Z)\\ & - \frac{1}{b}\left[k - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right] S(Y,Z)\end{aligned}$$

$$+\frac{2nk\mu}{b}g(hY,Z)-\frac{\mu}{b}S(hY,Z) \tag{5.6}$$

for all Y,Z provided $b\neq 0$, where $S^2(X,Y)=S(QX,Y)$.

Theorem 5.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a generalized quasi-conformal $N(k,\mu)$-manifold with $\mathcal{W}\cdot S=0$. Then the Ricci tensor S admits the relation* (5.6) *provided $b\neq 0$.*

Now for $b=0$, the equation (5.6) reduces to

$$\begin{aligned}&\left[k-\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)\right]S(Y,Z)\\ &=2nk\left[k-\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)\right]g(Y,Z)+2nk\mu g(hY,Z)-\mu S(hY,Z).\end{aligned} \tag{5.7}$$

Replacing Y by hY in the above equation and using (2.1) and (2.7), we get

$$\begin{aligned}&\left[k-\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)\right]S(hY,Z)\\ &=2nk\left[k-\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)\right]g(hY,Z)\\ &\quad+\mu(k-1)S(Y,Z)-2nk\mu(k-1)g(Y,Z).\end{aligned} \tag{5.8}$$

By virtue of (5.8), the equation (5.7) becomes

$$\frac{\bar{A}_1^2+\mu^2(k-1)}{\bar{A}_1}S(Y,Z)=2nk\left\{\frac{\bar{A}_1^2+\mu^2(k-1)}{\bar{A}_1}\right\}g(Y,Z),$$

i.e.,

$$S(Y,Z)=2nkg(Y,Z),\text{ or } \bar{A}_1^2+\mu^2(k-1)=0, \tag{5.9}$$

where $\bar{A}_1=\left[k-\frac{cr}{2n+1}\left(\frac{1}{2n}+a\right)\right]$. From the equations (5.8) and (5.9) one can easily point out the following theorem.

Theorem 5.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a $N(k,\mu)$-manifold with $\mathcal{W}\cdot S=0$. Then the following conditions are equivalent*:

(a) *M is an Einstein space.*

(b) *M is Ricci symmetric, i.e., $\nabla S=0$.*

(c) *$P(\xi.X)\cdot S=0$ (or $E(\xi,X)\cdot S=0$) for all $X\in\chi(M)$.*

Theorem 5.3. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a $N(k,\mu)$-manifold. If M satisfies $P(\xi,X)\cdot S=0$ (or $E(\xi,X)\cdot S=0$), M is locally isometric to $E^{n+1}\times S^n(4)$ or is Einstein-Sasakian.*

6. Generalized quasi-conformally semi-symmetric $N(k,\mu)$-manifold

Definition. A $(2n+1)$-dimensional $(n>1)$ $N(k,\mu)$-manifold is said to be semi-symmetric type [22] if the condition $R(X,Y)\cdot\mathcal{W}=0$ holds, for any vector fields X, Y on the manifold where $R(X,Y)$ acts on $\mathcal{W}$ as derivation.

Let us consider a $(2n+1)$-dimensional $N(k,\mu)$-manifold M, satisfying the condition

$$(R(\xi,X)\circ\mathcal{W})(Y,Z)U = 0 \tag{6.1}$$

which yields

$$\begin{aligned} &g(R(\xi,X)\mathcal{W}(Y,Z)U,\xi) - g(\mathcal{W}(R(\xi,X)Y,Z)U,\xi)\\ &-g(\mathcal{W}(Y,R(\xi,X)Z)U,\xi) - g(\mathcal{W}(Y,Z)R(\xi,X)U,\xi) = 0. \end{aligned} \tag{6.2}$$

By virtue of (2.5) the above equation reduces to

$$\begin{aligned} &k[g(X,\mathcal{W}(Y,Z)U) - \eta(X)\eta(\mathcal{W}(Y,Z)U) - g(X,Y)\eta(\mathcal{W}(\xi,Z)U)\\ &+\eta(Y)\eta(\mathcal{W}(X,Z)U) + g(X,Z)\eta(\mathcal{W}(\xi,Y)U) - \eta(Z)\eta(\mathcal{W}(X,Y)U)\\ &+\eta(U)\eta(\mathcal{W}(Y,Z)X)] + \mu[g(hX,\mathcal{W}(Y,Z)U) - g(X,Y)\eta(\mathcal{W}(\xi,Z)U)\\ &+\eta(Y)\eta(\mathcal{W}(hX,Z)U) + g(hX,Z)\eta(\mathcal{W}(\xi,Y)U) - \eta(Z)\eta(\mathcal{W}(hX,Y)U)\\ &+\eta(U)\eta(\mathcal{W}(Y,Z)hX)] = 0. \end{aligned} \tag{6.3}$$

Replacing X by hX in (6.3), we obtain

$$\begin{aligned} &k[g(hX,\mathcal{W}(Y,Z)U) - g(hX,Y)\eta(\mathcal{W}(\xi,Z)U) + \eta(Y)\eta(\mathcal{W}(hX,Z)U)\\ &+g(hX,Z)\eta(\mathcal{W}(\xi,Y)U) - \eta(Z)\eta(\mathcal{W}(hX,Y)U) + \eta(U)\eta(\mathcal{W}(Y,Z)hX)]\\ &-\mu(k-1)[g(X,\mathcal{W}(Y,Z)U) - g(X,Y)\eta(\mathcal{W}(\xi,Z)U) + \eta(Y)\eta(\mathcal{W}(X,Z)U)\\ &+g(X,Z)\eta(\mathcal{W}(\xi,Y)U) - \eta(Z)\eta(\mathcal{W}(X,Y)U) + \eta(U)\eta(\mathcal{W}(Y,Z)X)\\ &-\eta(X)\eta(\mathcal{W}(Y,Z)U)] = 0. \end{aligned} \tag{6.4}$$

Using (6.3) and (6.4), we can easily bring out

$$\begin{aligned} &[k^2+\mu^2(k-1)][g(X,\mathcal{W}(Y,Z)U) - g(X,Y)\eta(\mathcal{W}(\xi,Z)U)\\ &+\eta(Y)\eta(\mathcal{W}(X,Z)U) + g(X,Z)\eta(\mathcal{W}(\xi,Y)U) - \eta(Z)\eta(\mathcal{W}(X,Y)U)\\ &+\eta(U)\eta(\mathcal{W}(Y,Z)X) - \eta(X)\eta(\mathcal{W}(Y,Z)U)] = 0. \end{aligned} \tag{6.5}$$

For a non-Sasakian $N(k,\mu)$-manifold, we have $[k^2+\mu^2(k-1)] \neq 0$. Hence, contracting X over Y, we get

$$\begin{aligned} &\sum_{i=1}^{2n+1} \bar{\mathcal{W}}(e_i,Z,U,e_i) - 2n\eta(\mathcal{W}(\xi,Z)U)\\ &+\sum_{i=1}^{2n+1} \eta(U)\eta(\mathcal{W}(e_i,Z),e_i) - \sum_{i=1}^{2n+1} \eta(Z)\eta(\mathcal{W}(e_i,e_i)U) = 0. \end{aligned} \tag{6.6}$$

Again, from (3.1), we have

$$\begin{aligned} \sum_{i=1}^{2n+1} \bar{\mathcal{W}}(e_i,Z,U,e_i) = {}&(1-a+2nb)S(Z,U)\\ &+\left\{ar - \frac{2ncr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}g(Z,U), \end{aligned} \tag{6.7}$$

$$\sum_{i=1}^{2n+1} \eta(\mathcal{W}(e_i, Z), e_i) = -2nk(1-a+2nb)\eta(Z) - \left\{ar - \frac{2ncr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}\eta(Z). \tag{6.8}$$

In view of (6.7) and (6.8), we have

$$2n\eta(\mathcal{W}(\xi, Z)U) = (1-a+2nb)S(Z,U) - 2nk\eta(Z)\eta(U) + \left\{ar - \frac{2ncr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right\}\{g(Z,U)-\eta(Z)\eta(Z)\}. \tag{6.9}$$

Using (2.9) and (2.13) in (3.1), we obtain

$$\begin{aligned}&\eta(\mathcal{W}(Y,Z)U)\\ &= \left[k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right)\right][g(Z,U)\eta(Y) - g(Y,U)\eta(Z)]\\ &+ \mu[g(hZ,U)\eta(Y) - g(hY,U)\eta(Z)] + a[S(Z,U)\eta(Y) - S(Y,U)\eta(Z)].\end{aligned} \tag{6.10}$$

Comparing (6.9) and (6.10), we get

$$\begin{aligned}S(Z,U) = &\left\{\frac{2nk(1+2nb)-ar}{1-(2n+1)a+2nb}\right\}g(Z,U)\\ &+\left\{\frac{ar-2nka(1+2n)}{1-(2n+1)a+2nb}\right\}\eta(U)\eta(Z)\\ &+\frac{2n\mu}{1-(2n+1)a+2nb}g(hZ,U).\end{aligned} \tag{6.11}$$

In view of (2.12) and (6.11), we have

$$\begin{aligned}&\left[\tfrac{\{2(n-1)+\mu\}\{2nk(1+2nb)-ar\}-2n\mu\{2(n-1)-n\mu\}}{1-(2n+1)a+2nb}\right]g(Z,U)\\ &+\left[\tfrac{\{2(n-1)+\mu\}\{ar-2nka(1+2n)\}-2n\mu\{2(1-n)+n(2k+\mu)\}}{1-(2n+1)a+2nb}\right]\eta(U)\eta(Z)\\ &= \left\{2(n-1)+\mu-\tfrac{2n\mu}{1-(2n+1)a+2nb}\right\}S(Z,U).\end{aligned} \tag{6.12}$$

Using (6.11) in (6.10), we get

$$\begin{aligned}&\eta(W(Y,Z)U)\\ &= \left\{k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right) + \frac{2nka(1+2nb)-a^2r}{1-(2n+1)a+2nb}\right\}\\ &[g(Z,U)\eta(Y) - g(Y,U)\eta(Z)]\\ &+\frac{\mu(1-a+2nb)}{1-(2n+1)a+2nb}[g(hZ,U)\eta(Y) - g(hY,U)\eta(Z)].\end{aligned} \tag{6.13}$$

In view of (6.13) equation (6.5) becomes

$$\bar{\mathcal{W}}(Y,Z,U,X)$$

$$= \left\{k(1+2nb) - \frac{cr}{2n+1}\left(\frac{1}{2n}+a+b\right) + \frac{2nka(1+2nb)-a^2r}{1-(2n+1)a+2nb}\right\}$$
$$[g(Z,U)g(X,Y) - g(Y,U)g(X,Z)]$$
$$+\frac{\mu(1-a+2nb)}{1-(2n+1)a+2nb}[g(hZ,U)g(X,Y) - g(hY,U)g(X,Z)]. \tag{6.14}$$

From (6.5) and (6.14), we can easily bring out the followings theorem and corollary.

Theorem 6.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a $N(k,\mu)$-manifold. If M admit $R(\xi,X)\cdot C = 0$ (resp. $R(\xi,X)\cdot\hat{C} = 0$). Then M is either* (i) *conformllly flat* (*resp. conharmonically flat*) *or* (iii) *locally isometric to the Riemannian product $E^{n+1}\times S^{2n}(4)$.*

Theorem 6.2. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ $(n>1)$ be a non-Sasakian $N(k,\mu)$-manifold. Then for respective semi-symmetry type conditions, the Ricci tensor of the manifold M takes the respective forms as follows*:

Curvature condition	Expression for Ricci tensor
$\mathcal{R}(\xi,X)\cdot\mathcal{R}=0$ (Obtain by $a=b=c=0$)	$S=\frac{2n[k\{2(n-1)+\mu\}-\mu\{2(n-1)-n\mu\}]}{2(n-1)+\mu(1-2n)}g + \frac{2n\mu\{2(n-1)-n(2k+\mu)\}}{2(n-1)+\mu(1-2n)}\eta\otimes\eta$ η-Einstein manifold
$\mathcal{R}(\xi,X)\cdot C=0$ (Obtain by $c=1$ & $a=b=-\frac{1}{2n-1}$)	$S=\left[\frac{\{2(n-1)+\mu\}\{r-2nk\}}{4n(n-1)(1-\mu)} - \frac{2n\mu(2n-1)\{2(n-1)-n\mu\}}{4n(n-1)(1-\mu)}\right]g - \left[\frac{\{2(n-1)+\mu\}\{r-2nk(1+2n)\}}{4n(n-1)(1-\mu)} + \frac{2n(2n-1)\mu\{2(1-n)+n(2k+\mu)\}}{4n(n-1)(1-\mu)}\right]\eta\otimes\eta$ η-Einstein manifold
$\mathcal{R}(\xi,X)\cdot\hat{C}=0$, (Obtain by $c=0$ & $a=b=-\frac{1}{2n-1}$)	$S=\left[\frac{\{2(n-1)+\mu\}\{r-2nk\}}{4n(n-1)(1-\mu)} - \frac{2n\mu(2n-1)\{2(n-1)-n\mu\}}{4n(n-1)(1-\mu)}\right]g - \left[\frac{\{2(n-1)+\mu\}\{r-2nk(1+2n)\}}{4n(n-1)(1-\mu)} + \frac{2n(2n-1)\mu\{2(1-n)+n(2k+\mu)\}}{4n(n-1)(1-\mu)}\right]\eta\otimes\eta$ η-Einstein manifold
$\mathcal{R}(\xi,X)\cdot E=0$ (Obtain by $a=b=0, c=1$)	$S=\frac{2n[k\{2(n-1)+\mu\}-\mu\{2(n-1)-n\mu\}]}{2(n-1)+\mu(1-2n)}g + \frac{2n\mu\{2(n-1)-n(2k+\mu)\}}{2(n-1)+\mu(1-2n)}\eta\otimes\eta$ η-Einstein manifold
$\mathcal{R}(\xi,X)\cdot P=0$ (Obtain by $c=0$ $a=-\frac{1}{2n}, b=0$)	$S=\left[\frac{\{2(n-1)+\mu\}\left\{r+4n^2k\right\}}{(4n+1)\{2(n-1)+\mu\}-4n^2\mu} - \frac{4n^2\mu(2n-1)\{2(n-1)-n\mu\}}{(4n+1)\{2(n-1)+\mu\}-4n^2\mu}\right]g + \left[\frac{\{2(n-1)+\mu\}\{2nk(1+2n)-r\}}{(4n+1)\{2(n-1)+\mu\}-4n^2\mu} + \frac{4n^2\mu\{2(1-n)+n(2k+\mu)\}}{(4n+1)\{2(n-1)+\mu\}-4n^2\mu}\right]\eta\otimes\eta$ η-Einstein manifold
$\mathcal{R}(\xi,X)\cdot H=0$ (Obtain by $c=0$ $a=b=-\frac{1}{2n}$)	$S=\frac{r\{2(n-1)+\mu\}-4n^2\mu\{2(n-1)-n\mu\}}{(2n+1)\{2(n-1)+\mu\}-4n^2\mu}g + \left[\frac{\{2nk(2n+1)-r\}\{2(n-1)+\mu\}}{(2n+1)\{2(n-1)+\mu\}-4n^2\mu} - \frac{4n^2\mu\{2(1-n)+n(2k+\mu)\}}{(2n+1)\{2(n-1)+\mu\}-4n^2\mu}\right]\eta\otimes\eta$ η-Einstein manifold

Remark 6.3. In a $N(k,\mu)$-manifold $R(\xi,X)\cdot\mathcal{W}=0$ and $R(\xi,X)\cdot E=0$ are equivalent.

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