Commun. Korean Math. Soc. **31** (2016), No. 1, pp. 139–145
http://dx.doi.org/10.4134/CKMS.2016.31.1.139

FIXED POINT THEOREMS FOR CONDENSING MAPPINGS SATISFYING LERAY-SCHAUDER TYPE CONDITIONS

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Abstract. In this paper, some new fixed point theorems for condensing mappings are established based on a well known result of Petryshyn. We use several Leray-Schauder type conditions to prove new fixed point results. We also obtain generalizations of Altman's theorem and Petryshyn's theorem as well.

1. Introduction and preliminaries

Condensing mappings are arising extensively in differential and integral equations. Hence study of such operators are important part of nonlinear analysis and fixed point theorems related to condensing mappings are very useful in the study of existence of solutions of differential and integral equations.

In 1971, W. V. Petryshyn introduced several fixed point theorems for condensing mapping satisfying Leray-Schauder boundary condition.

Leray-Schauder boundary condition is formulated as follows:

A mapping $T : \overline{B} \to X$ is said to satisfy Leray-Schauder boundary condition if $T(x) \neq \lambda x,\ \forall x \in \partial B,\ \lambda > 1$, where B is the open ball about the origin (see [1]).

The following theorem is due to Petryshyn:

Let B be an open ball about the origin in a general Banach space X. If $T : \overline{B} \to X$ is a condensing mapping (and, in particular, a k-set contraction with $k < 1$) which satisfies the boundary condition $T(x) = \lambda x$ for some x in ∂B, $\lambda \leq 1$, then $F(T)$, the set of fixed points of T in $\overline{B}$, is nonempty and compact.

We need the following preliminary definitions:

Let D be a bounded subset of a metric space X. Define the *measure of noncompactness* $\alpha(D)$ of D by

$$\alpha(D) = \inf\{\varepsilon > 0 : \text{D admits a finite covering of subsets of diameter} < \varepsilon\}.$$

Received June 8, 2015.
2010 *Mathematics Subject Classification.* 47H08, 47H10.
Key words and phrases. condensing mappings, Leray-Schauder boundary condition, fixed point, Banach space.

It follows that $\alpha(A) \leq \alpha(B)$ whenever $A \subseteq B$, $\alpha(A \bigcup B) = \max\{\alpha(A), \alpha(B)\}$, $\alpha(co(A)) = \alpha(A)$, $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\alpha(\overline{A}) = \alpha(A)$, $\alpha(A+B) \leq \alpha(A) + \alpha(B)$ and $\alpha(A) = 0$ if and only if A is precompact.

Let $T : X \to X$ be a continuous mapping of a Banach space X. Then T is called a k- set contraction if for all $A \subseteq X$ with A bounded, $T(A)$ is bounded and $\alpha(T(A)) \leq k\alpha(A), 0 \leq k < 1$.

If $\alpha(T(A)) < \alpha(A)$ for all $\alpha(A) > 0$, then T is called densifying (or condensing).

In this paper we use various boundary conditions (see [3, 4]) to prove new fixed point results for condensing mappings. Some of which extends well known results of Altman and Petryshyn.

2. Fixed point theorems for mappings satisfying Leray-Schauder boundary conditions

Theorem 2.1 ([2, Theorem 1])**.** *Let B be an open ball about the origin in a general Banach space X. If $T : \overline{B} \to X$ is a condensing mapping (and, in particular, a k-set contraction with $k < 1$) which satisfies the boundary condition*

$$T(x) = \lambda x \quad \text{for some } x \text{ in } \partial B,$$

then $\lambda \leq 1$ and $F(T)$, the set of fixed points of T in $\overline{B}$, is nonempty and compact.

Theorem 2.2. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $1 \leq \alpha < \beta$ or $1 < \alpha \leq \beta$ such that*

$$\|Ax + x\|^{\alpha}\|x\|^{\beta} \geq \|Ax\|^{\alpha}\|Ax + x\|^{\beta} + \|Ax\|^{\alpha}\|x\|^{\beta}, \ \forall x \in \partial B. \tag{1}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. We shall prove that A satisfies Leray-Schauder condition. Suppose that A does not satisfy Leray-Schauder condition. Then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t+1)^{\alpha} - t^{\alpha}(t+1)^{\beta} - t^{\alpha}, \forall t > 1.$$

Then, f is strictly decreasing, and we have $f(t) < 0$, for $t > 1$.

Hence,

$$(t+1)^{\alpha} < t^{\alpha}(t+1)^{\beta} + t^{\alpha}, \forall t > 1. \tag{2}$$

Consequently, since $\|x_0\| \neq 0$ and $\lambda > 1$, we have

$$\begin{aligned}
\|Ax_0 + x_0\|^{\alpha}\|x_0\|^{\beta} &= \|\lambda x_0 + x_0\|^{\alpha}\|x_0\|^{\beta} \\
&= (\lambda+1)^{\alpha}\|x_0\|^{\alpha+\beta} \\
&< [\lambda^{\alpha}(\lambda+1)^{\beta} + \lambda^{\alpha}]\|x_0\|^{\alpha+\beta} \qquad \text{from (2)} \\
&= \|Ax_0\|^{\alpha}\|Ax_0 + x_0\|^{\beta} + \|Ax_0\|^{\alpha}\|x_0\|^{\beta}
\end{aligned}$$

which is a contradiction to (1), and hence Leray-Schauder condition is satisfied. Therefore from Theorem 2.1, A has a fixed point in $\overline{B}$. □

Theorem 2.3. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax - x\|^{\alpha+\beta}\|x\|^{\alpha} \geq \|Ax\|^{\alpha+\beta}\|Ax - x\|^{\alpha} + \|x\|^{2\alpha+\beta}, \ \forall x \in \partial B. \tag{3}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. Suppose that A does not satisfy Leray-Schauder condition, then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t-1)^{\alpha+\beta} - t^{\alpha+\beta}(t-1)^{\alpha} - 1, \forall t > 1.$$

Then, we can show that f is strictly decreasing. Hence

$$(t-1)^{\alpha+\beta} < t^{\alpha+\beta}(t-1)^{\alpha} + 1, \forall t > 1. \tag{4}$$

Consequently, since $\|x_0\| \neq 0$ and $\lambda > 1$, we have

$$\begin{aligned}
\|Ax_0 - x_0\|^{\alpha+\beta}\|x_0\|^{\alpha} &= \|\lambda x_0 - x_0\|^{\alpha+\beta}\|x_0\|^{\alpha} \\
&= (\lambda - 1)^{\alpha+\beta}\|x_0\|^{2\alpha+\beta} \\
&< [\lambda^{\alpha+\beta}(\lambda-1)^{\alpha} + 1]\|x_0\|^{2\alpha+\beta} \qquad \text{from (4)} \\
&= \|Ax_0\|^{\alpha+\beta}\|Ax_0 - x_0\|^{\alpha} + \|x_0\|^{2\alpha+\beta}
\end{aligned}$$

which is a contradiction to (3) and so, A satisfies Leray-Schauder condition. Therefore from Theorem 2.1, A has a fixed point in $\overline{B}$. □

Theorem 2.4. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax - x\|^{\alpha}\|x\|^{\beta} \leq \|Ax + x\|^{\beta}\|x\|^{\alpha} - \|Ax + x\|^{\alpha+\beta} \quad \forall\ x \in \partial B. \tag{5}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. Suppose that A does not satisfy Leray-Schauder condition, then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t-1)^{\alpha} - (t+1)^{\beta} + (t+1)^{\alpha+\beta}, \forall t > 1.$$

Then, we can show that f is strictly increasing. Hence

$$(t-1)^{\alpha} > (t+1)^{\beta} - (t+1)^{\alpha+\beta}, \forall t > 1. \tag{6}$$

Consequently, since $\|x_0\| \neq 0$ and $\lambda > 1$, we have

$$\begin{aligned}
\|Ax_0 - x_0\|^{\alpha}\|x_0\|^{\beta} &= \|\lambda x_0 - x_0\|^{\alpha}\|x_0\|^{\beta} \\
&= (\lambda - 1)^{\alpha}\|x_0\|^{\alpha+\beta} \\
&> [(\lambda+1)^{\beta} - (\lambda+1)^{\alpha+\beta}]\|x_0\|^{\alpha+\beta} \qquad \text{using (6)}
\end{aligned}$$

$$= \|Ax_0 + x_0\|^\beta \|x_0\|^\alpha - \|Ax_0 + x_0\|^{\alpha+\beta}$$

which is a contradiction to (5) and so, A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. □

Theorem 2.5. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $1 \leq \alpha < \beta$ or $1 < \alpha \leq \beta$ such that*

$$\|Ax - x\|^\alpha \|x\|^\beta \geq \|Ax\|^\alpha \|Ax + x\|^\beta - \|Ax\|^\alpha \|x\|^\beta, \ \forall x \in \partial B \ (\textit{see } [2]). \tag{7}$$

Then A has at least one fixed point in $\overline{B}$

Proof. Suppose that A does not satisfy Leray-Schauder condition. Then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t-1)^\alpha - t^\alpha (t+1)^\beta + t^\alpha, \forall t > 1.$$

Then f is strictly decreasing. Thus

$$(t-1)^\alpha < t^\alpha (t+1)^\beta - t^\alpha. \tag{8}$$

Since $\|x_0\| \neq 0$ for $x_0 \in \partial B$ and $\lambda > 1$, we have

$$\begin{aligned} \|Ax_0 - x_0\|^\alpha \|x_0\|^\beta &= \|\lambda x_0 - x_0\|^\alpha \|x_0\|^\beta \\ &= (\lambda - 1)^\alpha \|x_0\|^{\alpha+\beta} \\ &< (\lambda^\alpha (\lambda + 1)^\beta - \lambda^\alpha) \|x_0\|^{\alpha+\beta} \qquad \text{using (8)} \\ &= \|Ax_0\|^\alpha \|Ax_0 + x_0\|^\beta - \|Ax_0\|^\alpha \|\|x_0\|^\beta \end{aligned}$$

which is a contradiction to (7). Hence A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. □

Theorem 2.6. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax - x\|^\alpha \|x\|^\beta \geq \|Ax\|^\alpha \|Ax + x\|^\beta - \|Ax\|^\beta \|x\|^\alpha \quad \forall \ x \in \partial B. \tag{9}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. Suppose that A does not satisfy Leray-Schauder condition. Then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t-1)^\alpha - t^\alpha (t+1)^\beta + t^\beta, \forall t > 1.$$

Then f is strictly decreasing, and so $f(t) < f(1) \leq 0$ for $t > 1$.

Thus

$$(t-1)^\alpha < t^\alpha (t+1)^\beta - t^\beta. \tag{10}$$

Since $\|x_0\| \neq 0$ for $x_0 \in \partial B$ and $\lambda > 1$, we have

$$\|Ax_0 - x_0\|^\alpha \|x_0\|^\beta = \|\lambda x_0 - x_0\|^\alpha \|x_0\|^\beta$$

$$
\begin{aligned}
&= (\lambda - 1)^{\alpha}\|x_0\|^{\alpha+\beta} \\
&< [\lambda^{\alpha}(\lambda+1)^{\beta} - \lambda^{\beta}]\|x_0\|^{\alpha+\beta} \qquad \text{using 10} \\
&= \|Ax_0\|^{\alpha}\|Ax_0 + x_0\|^{\beta} - \|Ax_0\|^{\beta}\|x_0\|^{\alpha}
\end{aligned}
$$

which is a contradiction to (9). Hence A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. ☐

Corollary 2.1. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping such that*

$$\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2 \quad \forall x \in \partial B. \tag{11}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. With $\alpha = 2$ and $\beta = 0$ in the above theorem, the result follows. This is the famous Altman's condition. ☐

Theorem 2.7. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax\|^{\alpha}\|Ax + x\|^{\beta} \leq \|Ax\|^{\beta}\|Ax - x\|^{\alpha}, \ \forall x \in \partial B. \tag{12}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. If A does not satisfy Leray-Schauder condition, then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = t^{\alpha}(t+1)^{\beta} - t^{\beta}(t-1)^{\alpha}, \forall t > 1.$$

Then f is strictly increasing, and so $f(t) > f(1) > 0$ for $t > 1$.

Thus

$$t^{\alpha}(t+1)^{\beta} > t^{\beta}(t-1)^{\alpha}. \tag{13}$$

Since $\|x_0\| \neq 0$ for $x_0 \in \partial B$ and $\lambda > 1$, we have

$$
\begin{aligned}
\|Ax_0\|^{\alpha}\|Ax_0 + x_0\|^{\beta} &= \|\lambda x_0\|^{\alpha}\|\lambda x_0 + x_0\|^{\beta} \\
&= \lambda^{\alpha}(\lambda+1)^{\beta}\|x_0\|^{\alpha+\beta} \\
&> \lambda^{\beta}(\lambda-1)^{\alpha}\|x_0\|^{\alpha+\beta} \qquad \text{using (13)} \\
&= \|Ax_0\|^{\beta}\|Ax_0 - x_0\|^{\alpha}
\end{aligned}
$$

which is a contradiction to (12). Hence A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. ☐

Remark 2.1. When $\alpha = 1$ and $\beta = 0$ in Theorem 2.7, we have

$$\|Ax\| \leq \|Ax - x\| \ \forall x \in \partial B$$

which is the famous Petryshyn's Condition. Thus Theorem 2.7 extends Petryshyn's Theorem.

Theorem 2.8. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax - x\|^{\alpha}\|x\|^{\alpha+\beta} \geq \|Ax\|^{\alpha}\|Ax + x\|^{\alpha+\beta} - \|x\|^{2\alpha+\beta},\ \forall x \in \partial B. \tag{14}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. Suppose that A does not satisfy Leray-Schauder condition. Then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t-1)^{\alpha} - t^{\alpha}(t+1)^{\alpha+\beta} + 1, \forall t > 1.$$

Then, f is strictly decreasing. Therefore $f(t) < f(1) < 0$ for $t > 1$. Thus

$$(t-1)^{\alpha} < t^{\alpha}(t+1)^{\alpha+\beta} - 1, \forall t > 1. \tag{15}$$

Since $\|x_0\| \neq 0$ for $x_0 \in \partial B$ and $\lambda > 1$, we have

$$\begin{aligned}
\|Ax_0 - x_0\|^{\alpha}\|x_0\|^{\alpha+\beta} &= (\lambda-1)^{\alpha}\|x_0\|^{2\alpha+\beta} \\
&< [\lambda^{\alpha}(\lambda+1)^{\alpha+\beta} - 1]\|x_0\|^{2\alpha+\beta} \qquad \text{using (15)} \\
&= \|Ax_0\|^{\alpha}\|Ax_0 + x_0\|^{\alpha+\beta} - \|x_0\|^{2\alpha+\beta}
\end{aligned}$$

which is a contradiction to (14). Hence A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. □

Theorem 2.9. *Let X be a Banach space, B be an open ball about origin. Let $A : \overline{B} \to X$ be a condensing mapping. Suppose that there exist $\alpha \geq 1$ and $\beta \geq 0$ such that*

$$\|Ax + x\|^{\alpha+\beta} \leq \|Ax\|^{\beta}\|Ax - x\|^{\alpha} + \|x\|^{\alpha+\beta},\ \forall x \in \partial B. \tag{16}$$

Then A has at least one fixed point in $\overline{B}$.

Proof. Suppose that A does not satisfy Leray-Schauder condition. Then there exists $x_0 \in \partial B$ such that $Ax_0 = \lambda x_0$ for $\lambda > 1$. Consider the function defined by

$$f(t) = (t+1)^{\alpha+\beta} - t^{\beta}(t-1)^{\alpha} - 1,\ \forall t > 1.$$

Then f is strictly increasing. Therefore $f(t) > f(1) > 0$ for $t > 1$.
i.e.,

$$(t+1)^{\alpha+\beta} > t^{\beta}(t-1)^{\alpha} + 1. \tag{17}$$

Consequently, since $\|x_0\| \neq 0$ and $\lambda > 1$ we have,

$$\begin{aligned}
\|Ax_0 + x_0\|^{\alpha+\beta} = \|\lambda x_0 + x_0\|^{\alpha+\beta} &= (\lambda+1)^{\alpha+\beta}\|x_0\|^{\alpha+\beta} \\
&> [\lambda^{\beta}(\lambda-1)^{\alpha} + 1]\|x_0\|^{\alpha+\beta} \text{ using (17)} \\
&= \|Ax_0\|^{\beta}\|Ax_0 - x_0\|^{\alpha} + \|x_0\|^{\alpha+\beta}
\end{aligned}$$

which is a contradiction to (16). Hence A satisfies Leray-Schauder condition. Therefore by Theorem 2.1, A has a fixed point in $\overline{B}$. □

Conclusion. In this paper, we used several Leray-Schauder type conditions to prove new fixed point results for condensing mappings based on a well known result of Petryshyn. We have obtained generalizations of Altman's theorem and Petryshyn's theorem as well.

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