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ON A FUNCTIONAL EQUATION ARISING FROM PROTH IDENTITY

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Abstract. We determine the general solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation $f(ux-vy, uy+v(x+y)) = f(x,y)f(u,v)$ for all $x, y, u, v \in \mathbb{R}$. We also investigate both bounded and unbounded solutions of the functional inequality $|f(ux-vy, uy+v(x+y)) - f(x,y)f(u,v)| \leq \phi(u,v)$ for all $x, y, u, v \in \mathbb{R}$, where $\phi : \mathbb{R}^2 \to \mathbb{R}_+$ is a given function.

1. Introduction

The simple identity

$$x^4 + y^4 + (x+y)^4 = 2\left(x^2 + xy + y^2\right)^2 \tag{1.1}$$

is known as the *Proth identity* and was first published in 1878 (see [2]). It can be easily established by expanding the right hand side and using the binomial theorem. If we denote the quadratic form on the right hand side of (1.1) by

$$f(x,y) = x^2 + xy + y^2, \tag{1.2}$$

then it can be easily verified that

$$f(ux - vy, uy + v(x+y)) = f(x,y)\, f(u,v) \tag{1.3}$$

for all $x, y, u, v \in \mathbb{R}$. However, it is not so obvious that the function f given in (1.2) is the only solution of (1.3).

The Proth identity means that if we represent a given positive integer n as $f(x,y)$, where x and y are integers, then we have a representation of $2n^2$ as a sum of three fourth powers, i.e., $2n^2 = x^4 + y^4 + (x+y)^4$. The functional equation (1.3) says that if two integers have representations of the form $f(x,y)$, then so does their product. Using the facts, the authors of [2] gave a simple proof of the following result: *For every positive integer m, there is an integer n with at least m proper representations as the sum of three fourth powers.* As an example, suppose $m = 4$, then one can find an integer n, namely $n = 1729$

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such that $2 \cdot 1729^2$ can be expressed as the sum of three fourth powers in four ways:

$$2(1729)^2 = 8^4+37^4+45^4 = 3^4+40^4+43^4 = 23^4+25^4+48^4 = 15^4+32^4+47^4.$$

About the integer 1729, once Ramanujan said to Hardy that the number 1729 is an interesting number because it is the smallest number expressible as the sum of two cubes in two different ways (namely $9^3 + 10^3 = 1^3 + 12^3 = 1729$).

In [3], the authors determined the general solution $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation (1.3) (see Theorem 2.4 in [3]). In the general solution, a factor was missing from the solution. The interested reader should refer to the books [5] and [4] on the subject of functional equations and stabilities. In this paper, first we determine the general solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation (1.3) for all $x, y, u, v \in \mathbb{R}$ by a simple but different method than the one used in [3] and correct the solution in Theorem 2.4 of [3]. Then, by finding a condition (see (3.2)) for the solution f of the following functional inequality to be unbounded we investigate both the bounded and unbounded solutions of the functional inequality

$$|f(ux - vy, uy + v(x + y)) - f(x, y)\, f(u, v)| \le \phi(u, v) \tag{1.4}$$

for all $x, y, u, v \in \mathbb{R}$, where $\phi : \mathbb{R}^2 \to \mathbb{R}_+$ is a given function.

2. General solution of the equation (1.3)

In this section, we find the general solutions of the functional equation (1.3) for all $x, y, u, v \in \mathbb{R}$. Throughout this paper we denote by $\mathbb{R}_+$ the set of all positive real numbers and $\mathbb{R}_0^2 = \mathbb{R}^2 \setminus \{(0, 0)\}$. For a complex number z of the form $x + iy$, we denote $\arctan\left(\frac{y}{x}\right) = \arg(x + iy) \in [0, 2\pi)$.

Theorem 2.1. *The general solution* $f : \mathbb{R}^2 \to \mathbb{R}$ *of the equation* (1.3) *is given by* $f \equiv 1$ *or*

$$f(x, y) = M(x^2 + xy + y^2)\, e^{B\left(\arctan\left(\frac{\sqrt{3}y}{2x+y}\right)\right)}, \quad f(0, 0) = 0 \tag{2.1}$$

for all $x, y \in \mathbb{R}_0^2$*, where* $M : \mathbb{R}_+ \to \mathbb{R}$ *is a multiplicative function and* $B : [0, 2\pi) \to \mathbb{R}$ *is an additive function such that* $B(x+\pi) = B(x)$ *for all* $x \in [0, \pi)$.

Proof. Replacing (x, y) by $(x - \frac{1}{\sqrt{3}}y,\ \frac{2}{\sqrt{3}}y)$ and (u, v) by $(u - \frac{1}{\sqrt{3}}v,\ \frac{2}{\sqrt{3}}v)$ in (1.3) we have

$$\begin{aligned} &f\left(ux - vy - \frac{1}{\sqrt{3}}(uy + vx), \frac{2}{\sqrt{3}}(uy + vx)\right) \\ &= f\left(x - \frac{1}{\sqrt{3}}y, \frac{2}{\sqrt{3}}y\right) f\left(u - \frac{1}{\sqrt{3}}v, \frac{2}{\sqrt{3}}v\right) \end{aligned} \tag{2.2}$$

for all $x, y, u, v \in \mathbb{R}$. Defining $g : \mathbb{C} \to \mathbb{R}$ by

$$g(x + iy) = f\left(x - \frac{1}{\sqrt{3}}y, \frac{2}{\sqrt{3}}y\right), \quad x, y \in \mathbb{R},$$

and using (2.2), we obtain

$$g(zw) = g(z)\, g(w) \tag{2.3}$$

for all $z, w \in \mathbb{C}$. Putting $z = w = 0$ in (2.3) we have $g(0) = 0$ or $g(0) = 1$. If $g(0) = 1$, then putting $w = 0$ in (2.3), we have $g(z) = 1$ for all $z \in \mathbb{C}$. It remains to consider the case when $g(0) = 0$. Assume that $g \not\equiv 0$. If $g(z_0) = 0$ for some $z_0 \neq 0$, then $g(z) = g(zz_0^{-1})g(z_0) = 0$ for all $z \in \mathbb{C}$. Thus, $g(z) \neq 0$ for all $z \neq 0$. Replacing w by z in (2.3) we have $g(z^2) = g(z)^2 > 0$ for all $z \neq 0$, which implies $g(z) > 0$ for all $z \neq 0$. Using polar form of complex numbers, we define $h : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}$ by

$$h(r, \alpha) = \log g(z), \quad z = e^r e^{i\alpha}, \quad 0 \leq \alpha < 2\pi. \tag{2.4}$$

Then from (2.3) and (2.4) it is easy to see that

$$h(r, \alpha) + h(s, \beta) = h(r + s,\, \alpha + \beta) \tag{2.5}$$

for all $r, s \in \mathbb{R}$, $\alpha, \beta, \alpha + \beta \in [0, 2\pi)$, and

$$h(r, \alpha) + h(s, \beta) = h(r + s, \alpha + \beta - 2\pi) \tag{2.6}$$

for all $r, s \in \mathbb{R}$, $\alpha, \beta \in [0, 2\pi)$, $\alpha + \beta \geq 2\pi$. Now, we can write

$$h(r, \alpha) = h(r, 0) + h(0, \alpha) := A(r) + B(\alpha) \tag{2.7}$$

for all $r \in \mathbb{R}$, $\alpha \in [0, 2\pi)$, where A is an additive function on $\mathbb{R}$ and $B : [0, 2\pi) \to \mathbb{R}$ satisfies

$$B(\alpha) + B(\beta) = B(\alpha + \beta), \quad \alpha + \beta < 2\pi \tag{2.8}$$

$$B(\alpha) + B(\beta) = B(\alpha + \beta - 2\pi), \quad \alpha + \beta \geq 2\pi \tag{2.9}$$

for all α, $\beta \in [0, 2\pi)$. Note that the equations (2.8) and (2.9) clearly imply that $B(\pi) + B(\pi) = B(0) = 0$. Therefore $B(\pi) = 0$ and hence

$$B(x + \pi) = B(x) + B(\pi) = B(x) \tag{2.10}$$

for all $x \in [0, \pi)$. Now, we can write

$$g(re^{i\alpha}) = e^{h(\log r, \alpha)} = e^{A(\log r) + B(\alpha)}$$

for all $r > 0$, $\alpha \in [0, 2\pi)$. Let $M : (0, \infty) \to \mathbb{R}$ by $M(t) = e^{A(\log \sqrt{t})}$. Then M is a multiplicative function on $(0, \infty)$. Now, we have

$$f(x, y) = g\left(x + \frac{1}{2}y + i\frac{\sqrt{3}}{2}y\right) = M(x^2 + xy + y^2)\, e^{B\left(\arctan\left(\frac{\sqrt{3}y}{2x+y}\right)\right)} \tag{2.11}$$

for all $(x, y) \in \mathbb{R}_0^2$ and $f(0, 0) = g(0) = 0$. The proof of the theorem is now complete. □

Remark 2.2. We can find a nonzero function $B : [0, 2\pi) \to \mathbb{R}$ satisfying (2.8) and (2.9). Let H be a basis (Hamel basis) of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers such that $\pi \in H$. Let $B_0 : H \to \mathbb{R}$ be a nonzero function such that $B_0(\pi) = 0$. Then, there exists a unique extension $B : \mathbb{R} \to \mathbb{R}$ of B_0

such that $B(x+y) = B(x)+B(y)$ for all $x, y \in \mathbb{R}$. Now, the restriction $B|_{[0,2\pi)}$ of B on $[0, 2\pi)$ is a nonzero function satisfying (2.8) and (2.9).

In particular, if f is Lebesgue measurable, then both A and B are Lebesgue measurable. Thus, we have $M(t) = t^a$ and $B(x) = bx$ for some $a, b \in \mathbb{R}$. From (2.10) we have $B \equiv 0$. Thus, as a direct consequence of Theorem 2.1 we have the following corollary.

Corollary 2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lebesgue measurable function. Then f satisfies the functional equation* (1.3) *for all $x, y, u, v \in \mathbb{R}$ if and only if either $f \equiv 0$, $f \equiv 1$ or there exists $a \in \mathbb{R}$ such that*

$$f(x, y) = (x^2 + xy + y^2)^a, \quad f(0, 0) = 0 \tag{2.12}$$

for all $(x, y) \in \mathbb{R}_0^2$.

3. Stability of functional equation (1.3)

In this section, we consider the stability of the equation (1.3), by determining the solution of the functional inequality

$$|f(ux - vy, uy + v(x + y)) - f(x, y)f(u, v)| \le \phi(u, v) \tag{3.1}$$

for all $x, y, u, v \in \mathbb{R}$ and for some given $\phi : \mathbb{R}^2 \to \mathbb{R}_+$.

Theorem 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy the functional inequality* (3.1)*. Assume that there exist $(x_0, y_0) \in \mathbb{R}^2$ and $(u_0, v_0) \in \mathbb{R}_0^2$ such that*

$$\big|f(x_0, y_0)\,(|f(u_0, v_0)| - 1)\big| > \phi(u_0, v_0). \tag{3.2}$$

Then f is unbounded and has the form

$$f(x, y) = M(x^2 + xy + y^2)\, e^{B\left(\arctan\left(\frac{\sqrt{3}y}{2x+y}\right)\right)}, \quad f(0, 0) = 0 \tag{3.3}$$

for all $(x, y) \in \mathbb{R}_0^2$, where $M : \mathbb{R}_+ \to \mathbb{R}$ is a multiplicative function and the function $B : [0, 2\pi) \to \mathbb{R}$ is an additive function satisfying $B(x + \pi) = B(x)$ for all $x \in [0, \pi)$.

Proof. Putting $(u, v) = (u_0, v_0)$ in (3.1), letting $\tau(x, y) = (u_0x - v_0y, v_0x + (u_0 + v_0)y)$ and using the triangle inequality on the resulting inequality we have

$$|f(\tau(x, y))| \ge |f(x, y)|\, |f(u_0, v_0))| - \phi(u_0, v_0) \tag{3.4}$$

for all $x, y \in \mathbb{R}$.

Let us assume that $|f(u_0, v_0)| > 1$. Then we have

$$|f(x_0, y_0)| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} > 0. \tag{3.5}$$

Subtracting $\frac{\phi(u_0,v_0)}{|f(u_0,v_0)|-1}$ from both sides of (3.4) we have

$$|f(\tau(x, y))| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} \ge |f(u_0, v_0))| \left(|f(x, y)| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1}\right) \tag{3.6}$$

for all $x, y \in \mathbb{R}$. Replacing (x, y) by $\tau^{(n)}(x_0, y_0), n = 0, 1, 2, \ldots,$ in (3.6) we have

$$\begin{aligned}(3.7)\qquad & |f(\tau^{(n+1)}(x_0, y_0))| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} \\ & \geq |f(u_0, v_0)| \left(|f(\tau^{(n)}(x_0, y_0))| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} \right)\end{aligned}$$

for all $n = 1, 2, 3, \ldots$. Thus, from (3.7) we have

$$\begin{aligned}(3.8)\qquad & |f(\tau^{(n)}(x_0, y_0))| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} \\ & \geq |f(u_0, v_0)|^n \left(|f(x_0, y_0)| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} \right)\end{aligned}$$

for all $n = 1, 2, 3, \ldots$. Since $|f(u_0, v_0)| > 1$ and $|f(x_0, y_0)| - \frac{\phi(u_0, v_0)}{|f(u_0, v_0)| - 1} > 0$, it follows from (3.8) that

$$|f(\tau^{(n)}(x_0, y_0))| \to \infty$$

as $n \to \infty$.

Next, assume that $|f(u_0, v_0)| < 1$. Putting $(u, v) = (u_0, v_0)$ in (3.1) and using the triangle inequality on the resulting inequality we have

$$(3.9)\qquad |f(x, y)||f(u_0, v_0))| \geq |f(\tau(x, y))| - \phi(u_0, v_0)$$

for all $x, y \in \mathbb{R}$. Since $(u_0, v_0) \neq (0, 0)$ we have

$$\tau^{-1}(x, y) = \left(\frac{(u_0 + v_0)x + v_0 y}{u_0^2 + u_0 v_0 + v_0^2}, \frac{u_0 y - v_0 x}{u_0^2 + u_0 v_0 + v_0^2} \right).$$

Replacing (x, y) by $\tau^{-1}(x, y)$ in (3.9) we have

$$(3.10)\qquad |f(\tau^{-1}(x, y))|\,|f(u_0, v_0))| \geq |f(x, y)| - \phi(u_0, v_0)$$

for all $x, y \in \mathbb{R}$. If $f(u_0, v_0) = 0$, putting $(u, v) = (u_0, v_0)$ and $(x, y) = \tau^{-1}(x_0, y_0)$ in (3.1) we have

$$(3.11)\qquad |f(x_0, y_0)| \leq \phi(u_0, v_0)$$

which contradicts (3.2). Thus, $f(u_0, v_0) \neq 0$. Dividing (3.10) by $|f(u_0, v_0)|$, subtracting $\frac{\phi(u_0, v_0)}{1 - |f(u_0, v_0)|}$ from the result and using the same procedure as in the case when $|f(u_0, v_0)| > 1$ we have

$$\begin{aligned}(3.12)\qquad & |f((\tau^{-1})^{(n)}(x_0, y_0))| - \frac{\phi(u_0, v_0)}{1 - |f(u_0, v_0)|} \\ & \geq |f(u_0, v_0)|^{-n} \left(|f(x_0, y_0)| - \frac{\phi(u_0, v_0)}{1 - |f(u_0, v_0)|} \right)\end{aligned}$$

for all $n = 1, 2, 3, \ldots$. Since $0 < |f(u_0, v_0)| < 1$ and $|f(x_0, y_0)| - \frac{\phi(u_0, v_0)}{1 - |f(u_0, v_0)|} > 0$, it follows from (3.12) that

$$|f((\tau^{-1})^{(n)}(x_0, y_0))| \to \infty$$

as $n \to \infty$. Thus, f is unbounded. Choose a sequence $(p_n, q_n) \in \mathbb{R}^2$, $n = 1, 2, 3, \ldots$, such that $|f(p_n, q_n)| \to \infty$ as $n \to \infty$. Replacing (x, y) by (p_n, q_n) in (3.1), dividing the result by $|f(p_n, q_n)|$ and letting $n \to \infty$ in the result we have

$$f(u,v) = \lim_{n\to\infty} \frac{f(up_n - vq_n, vp_n + (u+v)q_n)}{f(p_n, q_n)} \tag{3.13}$$

for all $u, v \in \mathbb{R}$. Multiplying both sides of (3.13) by $f(x, y)$ and using (3.1) we have

$$\begin{aligned}
&f(u,v)\, f(x,y) \\
&= \lim_{n\to\infty} \frac{f(up_n - vq_n, vp_n + (u+v)q_n) f(x,y)}{f(p_n, q_n)} \\
&= \lim_{n\to\infty} \frac{f((ux-vy)p_n - (uy+v(x+y))q_n, (uy+v(x+y))p_n - (vx+u(x+y))q_n)}{f(p_n, q_n)} \\
&= f(ux - vy,\ uy + v(x+y))
\end{aligned}$$

for all $u, v, x, y \in \mathbb{R}$. Thus, by Theorem 2.1 we obtain the asserted result and the proof of the theorem is now complete. □

As a consequence of Theorem 3.1 one can describe all bounded functions f satisfying (3.1).

Corollary 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded function satisfying the functional inequality* (3.1) *for all $x, y, u, v \in \mathbb{R}$. Then f satisfies*

$$|f(x,y)| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x,y)}\,\big) \tag{3.14}$$

for all $(x, y) \in \mathbb{R}^2$. Also, if $K = \{(x, y) \in \mathbb{R}_0^2 : \phi(x, y) < \frac{1}{4}\} \ne \emptyset$, then f satisfies either

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x,y)}\,\big) \le |f(x,y)| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x,y)}\,\big) \tag{3.15}$$

for all $(x, y) \in K$, or else

$$|f(x,y)| \le \frac{1}{2}\big(1 - \sqrt{1 - 4\phi(x,y)}\,\big) \tag{3.16}$$

for all $(x, y) \in K$.

Proof. By Theorem 3.1, every bounded solution of (3.1) satisfies

$$|f(x,y)(|f(u,v)| - 1)| \le \phi(u,v) \tag{3.17}$$

for all $(x, y) \in \mathbb{R}^2$, $(u, v) \in \mathbb{R}_0^2$. Replacing (u, v) by (x, y) in (3.17) and solving the resulting inequality we obtain (3.14). Also, replacing (u, v) by $(x, y) \in K$ in (3.17) we have, for each $(x, y) \in K$,

$$\frac{1}{2}\big(1 + \sqrt{1 - 4\phi(x,y)}\,\big) \le |f(x,y)| \le \frac{1}{2}\big(1 + \sqrt{1 + 4\phi(x,y)}\,\big) \tag{3.18}$$

or

$$|f(x,y)| \le \frac{1}{2}\big(1-\sqrt{1-4\phi(x,y)}\,\big). \tag{3.19}$$

Assume that there exist $(x_0,y_0),(u_0,v_0)\in K$ such that

$$|f(u_0,v_0)| \le \frac{1}{2}(1-\sqrt{1-4\phi(u_0,v_0)}) \text{ and } |f(x_0,y_0)| \ge \frac{1}{2}(1+\sqrt{1-4\phi(x_0,y_0)}).$$

Then, putting $(x,y)=(x_0,y_0)$, $(u,v)=(u_0,v_0)$ in (3.17) we arrive at the contradiction

$$\frac{1}{4} \le |f(x_0,y_0)(|f(u_0,v_0)|-1)| \le \phi(u_0,v_0) < \frac{1}{4}.$$

Thus, f satisfies (3.18) for all $(x,y)\in K$, or (3.19) for all $(x,y)\in K$. This completes the proof. □

If $\phi(x)\equiv\epsilon<\frac{1}{4}$ is constant, then we can give a more transparent description of the bounded functions f satisfying (3.1) (see [1]).

Corollary 3.3. *Let $\epsilon<\frac{1}{4}$ and let $f:\mathbb{R}^2\to\mathbb{R}$ be a bounded function satisfying*

$$|f(ux-vy, uy+v(x+y)) - f(x,y)f(u,v)| \le \epsilon \tag{3.20}$$

for all $x,y,u,v\in\mathbb{R}$. Then f satisfies either

$$\frac{1}{2}(1+\sqrt{1-4\epsilon}) \le f(x,y) \le \frac{1}{2}(1+\sqrt{1+4\epsilon}) \tag{3.21}$$

for all $(x,y)\in\mathbb{R}^2_0$, or else

$$-\epsilon \le f(x,y) \le \frac{1}{2}(1-\sqrt{1-4\epsilon}) \tag{3.22}$$

for all $(x,y)\in\mathbb{R}^2_0$.

Proof. Replacing (x,y) by (u,v) in (3.20) and using the triangle inequality we have

$$f(u^2-v^2,\ 2uv+v^2) \ge f(u,v)^2-\epsilon \ge -\epsilon \tag{3.23}$$

for all $(u,v)\in\mathbb{R}^2$. It is easy to check that $\{(u^2-v^2,\ 2uv+v^2) : u,v\in\mathbb{R}\}=\mathbb{R}^2$. Thus, it follows that

$$f(x,y) \ge -\epsilon \tag{3.24}$$

for all $(x,y)\in\mathbb{R}^2$. Since $\epsilon<\frac{1}{2}(1-\sqrt{1-4\epsilon})$, from (3.15) and (3.24) we get (3.21), and from (3.16) and (3.24) we get (3.22). This completes the proof □

Remark 3.4. The value $f(0,0)$ may not belong to the range of $f(x,y)$ for $(x,y)\in\mathbb{R}^2_0$. As a simple example, let $f(x,y)\approx 1$ for all $(x,y)\in\mathbb{R}^2_0$ and $f(0,0)\approx 0$. Then f satisfies (3.20).

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