

## FRACTIONAL DIFFERENTIATION OF THE PRODUCT OF APPELL FUNCTION $F_3$ AND MULTIVARIABLE $H$ -FUNCTIONS

JUNESANG CHOI, JITENDRA DAIYA, DINESH KUMAR, AND RAM KISHORE SAXENA

ABSTRACT. Fractional calculus operators have been investigated by many authors during the last four decades due to their importance and usefulness in many branches of science, engineering, technology, earth sciences and so on. Saigo *et al.* [9] evaluated the fractional integrals of the product of Appell function of the third kernel  $F_3$  and multivariable  $H$ -function. In this sequel, we aim at deriving the generalized fractional differentiation of the product of Appell function  $F_3$  and multivariable  $H$ -function. Since the results derived here are of general character, several known and (presumably) new results for the various operators of fractional differentiation, for example, Riemann-Liouville, Erdélyi-Kober and Saigo operators, associated with multivariable  $H$ -function and Appell function  $F_3$  are shown to be deduced as special cases of our findings.

### 1. Introduction and preliminaries

The multivariable  $H$ -function which was introduced and investigated by Srivastava and Panda [18, p. 271, Eq. (4.1)] in terms of a multiple Mellin-Barnes type contour integrals is defined by

$$\begin{aligned}
 (1) \quad & H[z_1, \dots, z_r] \\
 &= H_{p, q; p_1, q_1; \dots; p_r, q_r}^{0, n; m_1, n_1; \dots; m_r, n_r} \left[ \begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \left| \begin{array}{c} (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1, p} : (c_j^{(1)}, \gamma_j^{(1)})_{1, p_1} \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1, q} : (d_j^{(1)}, \delta_j^{(1)})_{1, q_1} \dots; (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{array} \right. \right] \\
 &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \phi_i(\xi_i) Z_i^{\xi_i} \right\} d\xi_1 \cdots d\xi_r,
 \end{aligned}$$

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where  $\omega = \sqrt{-1}$ ,

$$(2) \quad \psi(\xi_1, \dots, \xi_r) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r \zeta_j^{(i)} \xi_i)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r \zeta_j^{(i)} \xi_i) \prod_{j=1}^q \Gamma(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i)},$$

$$(3) \quad \phi_i(\xi_i) = \frac{\prod_{l=1}^{m_i} \Gamma(d_l^{(i)} - \delta_l^{(i)} \xi_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i) \prod_{l=m_i+1}^{q_i} \Gamma(1 - d_l^{(i)} + \delta_l^{(i)} \xi_i)},$$

and  $L_j = L_{\omega\tau_j\infty}$ ,  $\omega = (-1)^{\frac{1}{2}}$  represents the contours which start at the point  $\tau_j - \omega\infty$  and terminate at the points  $\tau_j + \omega\infty$  with  $\tau_j \in \mathbb{R} = (-\infty, \infty)$  ( $j = 1, \dots, r$ ), such that all the poles of  $\Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i)$  ( $j = 1, \dots, m_i; i = 1, \dots, r$ ) are separated from those of  $\Gamma(1 - c_j^{(i)} - \gamma_j^{(i)} \xi_i)$  ( $j = 1, \dots, n_i; i = 1, \dots, r$ ) and  $\Gamma(1 - a_j + \sum_{i=1}^r \zeta_j^{(i)} \xi_i)$  ( $j = 1, \dots, n$ ). For a detailed definition, convergence and existence conditions of the multivariable  $H$ -function, the reader may be referred to the original paper due to Srivastava and Panda [18] (also see Srivastava *et al.* [19], Mathai *et al.* [5] and Samko *et al.* [11]).

For  $n = p = q = 0$ , the multivariable  $H$ -function breaks up into  $r$  product of  $H$ -functions as follows (see Mathai *et al.* [5, 6]):

$$(4) \quad H_{0,0; p_1, q_1; \dots; p_r, q_r}^{0,0; m_1, n_1; \dots; m_r, n_r} \left[ \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \left| \begin{matrix} (c_j^{(1)}, \gamma_j^{(1)})_{1, p_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \\ (d_j^{(1)}, \delta_j^{(1)})_{1, q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{matrix} \right. \right]$$

$$= \prod_{i=1}^r H_{p_i, q_i}^{m_i, n_i} \left[ z \left| \begin{matrix} (c_j^{(i)}, \gamma_j^{(i)})_{1, p_i} \\ (d_j^{(i)}, \delta_j^{(i)})_{1, q_i} \end{matrix} \right. \right],$$

where  $H_{p,q}^{m,n}(\cdot)$  are the  $H$ -functions.

Fractional integration of multivariable  $H$ -function and a general class of polynomials associated with Saigo-Maeda operators was investigated by Saigo *et al.* [10]. In the sequel we aim at deriving the generalized fractional differentiation of the product of Appell function  $F_3$  and multivariable  $H$ -function under Saigo-Maeda fractional derivative. Two lemmas generalizing the results given by Kilbas and Sebastian [3] are established. The results obtained here are shown to be certain extensions of several earlier results given, for example, by Kilbas [1], Kilbas and Saigo [2], Saigo and Kilbas [8], and so on.

A general class of multivariable polynomials of real or complex variables  $x_1, \dots, x_s$  was defined and studied by Srivastava and Garg [17] in the following form:

$$(5) \quad S_L^{h_1, \dots, h_s}(x_1, \dots, x_s)$$

$$= \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \frac{x_1^{k_1}}{k_1!} \dots \frac{x_s^{k_s}}{k_s!},$$

where  $L, h_1, \dots, h_s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\mathbb{N}$  being the set of positive integers, and the coefficients  $A(L; k_1, \dots, k_s)$  ( $k_j \in \mathbb{N}_0; j = 1, \dots, s$ ) are real or complex constants which are arbitrarily chosen.

## 2. Fractional calculus operators

For  $\alpha, \beta, \eta \in \mathbb{C}$ ,  $\mathbb{C}$  being the set of complex numbers, and  $\Re(\alpha) > 0, x > 0$ , the generalized fractional calculus operators defined by Saigo [7] are given as follows:

$$\begin{aligned} (6) \quad & \left( I_{0+}^{\alpha, \beta, \eta} f \right) (x) \\ &= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right) f(t) dt \\ &= \left( \frac{d}{dx} \right)^k \left( I_{0+}^{\alpha+k, \beta-k, \eta-k} f \right) (x) \quad (\Re(\alpha) < 0; k = [\Re(-\alpha)] + 1); \end{aligned}$$

$$\begin{aligned} (7) \quad & \left( I_-^{\alpha, \beta, \eta} f \right) (x) \\ &= \frac{1}{\Gamma(\alpha)} \int_x^\infty t^{-\alpha-\beta} (t-x)^{\alpha-1} {}_2F_1 \left( \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right) f(t) dt \\ &= \left( -\frac{d}{dx} \right)^k \left( I_-^{\alpha+k, \beta-k, \eta} f \right) (x) \quad (\Re(\alpha) < 0; k = [\Re(-\alpha)] + 1); \end{aligned}$$

$$\begin{aligned} (8) \quad & \left( D_{0+}^{\alpha, \beta, \eta} f \right) (x) \\ &= \left( I_{0+}^{-\alpha, -\beta, \alpha+\eta} f \right) (x) \quad [\Re(\alpha) > 0] \\ &= \left( \frac{d}{dx} \right)^k \left( I_{0+}^{-\alpha+k, -\beta-k, \alpha+\eta-k} f \right) (x) \quad (\Re(\alpha) > 0; k = [\Re(\alpha)] + 1); \end{aligned}$$

$$\begin{aligned} (9) \quad & \left( D_-^{\alpha, \beta, \eta} f \right) (x) \\ &= \left( I_-^{-\alpha, -\beta, \alpha+\eta} f \right) (x) \quad [\Re(\alpha) > 0] \\ &= \left( -\frac{d}{dx} \right)^k \left( I_-^{-\alpha+k, -\beta-k, \alpha+\eta} f \right) (x) \quad (\Re(\alpha) > 0; k = [\Re(\alpha)] + 1). \end{aligned}$$

If  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$  and  $\Re(\gamma) > 0, x > 0$ , the generalized fractional derivative operators involving Appell function  $F_3$  given by Saigo and Maeda [9] are defined by

$$(10) \quad \left( I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x)$$

$$\begin{aligned}
&= \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt \\
&= \left( \frac{d}{dx} \right)^k \left( I_{0+}^{\alpha, \alpha' + k, \beta', \gamma + k} f \right) (x) \quad (\Re(\gamma) \leq 0; k = [-\Re(\gamma)] + 1); \\
(11) \quad & \left( I_-^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x) \\
&= \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty t^{-\alpha} (t-x)^{\gamma-1} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) dt \\
&= \left( -\frac{d}{dx} \right)^k \left( I_-^{\alpha, \alpha', \beta, \beta' + k, \gamma + k} f \right) (x) \quad (\Re(\gamma) \leq 0; k = [-\Re(\gamma)] + 1); \\
(12) \quad & \left( D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x) \\
&= \left( I_{0+}^{-\alpha', -\alpha, -\beta', -\beta, -\gamma} f \right) (x) \quad (\Re(\gamma) > 0) \\
&= \left( \frac{d}{dx} \right)^k \left( I_{0+}^{-\alpha', -\alpha, -\beta' + k, -\beta, -\gamma + k} f \right) (x) \quad (\Re(\gamma) > 0; k = [\Re(\gamma)] + 1) \\
&= \frac{1}{\Gamma(n-\gamma)} \left( \frac{d}{dx} \right)^n (x)^{\alpha'} \int_0^x (x-t)^{n-\gamma-1} t^\alpha \\
&\quad \times F_3 \left( -\alpha', -\alpha, n-\beta', -\beta, n-\gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) f(t) dt; \\
(13) \quad & \left( D_-^{\alpha, \alpha', \beta, \beta', \gamma} f \right) (x) \\
&= \left( I_-^{-\alpha', -\alpha, -\beta', -\beta, -\gamma} f \right) (x) \quad (\Re(\gamma) > 0) \\
&= \left( -\frac{d}{dx} \right)^k \left( I_-^{-\alpha', -\alpha, -\beta', -\beta + k, -\gamma + k} f \right) (x) \quad (\Re(\gamma) > 0; k = [\Re(\gamma)] + 1) \\
&= \frac{1}{\Gamma(n-\gamma)} \left( -\frac{d}{dx} \right)^n (x)^\alpha \int_x^\infty (t-x)^{n-\gamma-1} t^{\alpha'} \\
&\quad \times F_3 \left( -\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) f(t) dt.
\end{aligned}$$

The following results are also required (see [11, p. 727, Eq. (5.4.51.2)]; see also Saigo *et al.* [10]):

$$\begin{aligned}
(14) \quad & \int_0^x t^{\rho-1} (x-t)^{c-1} F_3 \left( a, a', b, b'; c; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) dt \\
&= \frac{\Gamma(c) \Gamma(\rho + a') \Gamma(\rho + b') \Gamma(\rho + c - a - b)}{\Gamma(\rho + a' + b') \Gamma(\rho + c - a) \Gamma(\rho + c - b)} x^{\rho+c-1} \\
&(x > 0, \Re(c) > 0, \Re(\rho) > \max \{ \Re(-a'), \Re(-b'), \Re(a + b - c) \})
\end{aligned}$$

and

$$(15) \quad \int_x^\infty t^{\rho-1} (t-x)^{c-1} F_3 \left( a, a', b, b'; c; 1 - \frac{x}{t}, 1 - \frac{t}{x} \right) dt \\ = \frac{\Gamma(c) \Gamma(1+a'-c-\rho) \Gamma(1+b'-c-\rho) \Gamma(1-a-b-\rho)}{\Gamma(1+a'+b'-c-\rho) \Gamma(1-a-\rho) \Gamma(1-b-\rho)} x^{\rho+c-1}$$

$$(x > 0, \Re(c) > 0, \Re(\rho) < 1 + \min \{ \Re(a'-c), \Re(b'-c), \Re(-a-b) \}).$$

The generalized fractional calculus operators due to Saigo-Maeda defined in [11, 12] reduce to the following generalized fractional calculus operators due to Saigo [7]:

$$(16) \quad \left( I_{0+}^{\alpha, 0, \beta, \beta', \gamma} f \right) (x) = \left( I_{0+}^{\gamma, \alpha-\gamma, -\beta} f \right) (x) \quad (\gamma \in \mathbb{C}; x > 0);$$

$$(17) \quad \left( I_{-}^{\alpha, 0, \beta, \beta', \gamma} f \right) (x) = \left( I_{-}^{\gamma, \alpha-\gamma, -\beta} f \right) (x) \quad (\gamma \in \mathbb{C}; x > 0);$$

$$(18) \quad \left( D_{0+}^{0, \alpha', \beta, \beta', \gamma} f \right) (x) = \left( D_{0+}^{\gamma, \alpha'-\gamma, \beta'-\gamma} f \right) (x) \quad (\gamma \in \mathbb{C}; x > 0);$$

$$(19) \quad \left( D_{-}^{0, \alpha', \beta, \beta', \gamma} f \right) (x) = \left( D_{-}^{\gamma, \alpha'-\gamma, \beta'-\gamma} f \right) (x) \quad (\Re(\gamma) > 0; x > 0).$$

Further from [9, p. 394, Eqs. (4.18) and (4.19)], we have:

**Lemma 2.1.** *The following formula holds true: For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ ,  $\Re(\alpha) > 0$ ,*

$$(20) \quad \left( I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ = \frac{\Gamma(\rho) \Gamma(\rho + \gamma - \alpha - \alpha' - \beta) \Gamma(\rho + \beta' - \alpha')}{\Gamma(\rho + \gamma - \alpha - \alpha') \Gamma(\rho + \gamma - \alpha' - \beta) \Gamma(\rho + \beta')} x^{\rho - \alpha - \alpha' + \gamma - 1},$$

where  $\Re(\gamma) > 0$ ,  $\Re(\rho) > \max \{ 0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta') \}$ , and  $x > 0$ .

In particular, if  $\alpha' = 0$ ,  $\beta = -\eta$ ,  $\gamma = \alpha$ , and  $\alpha$  is replaced by  $\alpha + \beta$  in (20), then the Saigo-Maeda operator becomes the Saigo operator as follows (see [9]):

$$(21) \quad \left( I_{0+}^{\alpha, \beta, \eta} t^{\rho-1} \right) (x) = \frac{\Gamma(\rho) \Gamma(\rho + \eta - \beta)}{\Gamma(\rho - \beta) \Gamma(\rho + \alpha + \eta)} x^{\rho - \beta - 1} \quad (x > 0).$$

**Lemma 2.2.** *The following formula holds true: For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ ,  $\Re(\alpha) > 0$ ,*

$$(22) \quad \left( I_{-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ = \frac{\Gamma(1 + \alpha + \alpha' - \gamma - \rho) \Gamma(1 + \alpha + \beta' - \gamma - \rho) \Gamma(1 - \beta - \rho)}{\Gamma(1 - \rho) \Gamma(1 + \alpha + \alpha' + \beta' - \gamma - \rho) \Gamma(1 + \alpha - \beta - \rho)} x^{\rho - \alpha - \alpha' + \gamma - 1},$$

where  $\Re(\gamma) > 0$ ,  $x > 0$ ,  $\Re(\rho) < 1 + \min \{ \Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma) \}$ .

In particular, if  $\alpha' = 0$ ,  $\beta = -\eta$ ,  $\gamma = \alpha$ ,  $\alpha$  is replaced by  $\alpha + \beta$  in (22), then the generalized fractional differentiation of  $t^{\rho-1}$  is given by

$$(23) \quad \left( I_-^{\alpha, \beta, \eta} t^{\rho-1} \right) (x) = \frac{\Gamma(1 + \beta - \rho) \Gamma(1 + \eta - \rho)}{\Gamma(1 - \rho) \Gamma(1 + \alpha + \beta + \eta - \rho)} x^{\rho - \beta - 1} \quad (x > 0).$$

We present two more formulas asserted by the subsequent lemmas.

**Lemma 2.3.** *The following formula holds true: For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ ,  $\Re(\alpha) > 0$ ,*

$$(24) \quad \begin{aligned} & \left( D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ &= \frac{\Gamma(\rho) \Gamma(\rho + \alpha - \beta) \Gamma(\rho + \alpha + \alpha' + \beta' - \gamma)}{\Gamma(\rho - \beta) \Gamma(\rho + \alpha + \alpha' - \gamma) \Gamma(\rho + \alpha + \beta' - \gamma)} x^{\rho + \alpha + \alpha' - \gamma - 1}, \end{aligned}$$

where  $\Re(\rho) > 0$ ,  $\Re(\rho + \alpha - \beta) > 0$ ,  $\Re(\rho + \alpha + \alpha' + \beta' - \gamma) > 0$ , and  $x > 0$ .

*Proof.* Using (12) and (14), we have

$$\begin{aligned} & \left( D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ &= \frac{1}{\Gamma(n - \gamma)} \left( \frac{d}{dx} \right)^n \left( x^{\alpha'} \right) t^{\rho-1} \\ & \quad \times \int_0^x (x-t)^{n-\gamma-1} t^\alpha F_3 \left( -\alpha', -\alpha, n - \beta', -\beta, n - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) dt \\ &= \frac{1}{\Gamma(n - \gamma)} \left( \frac{d}{dx} \right)^n \left( x^{\alpha'} \right) \int_0^x (x-t)^{n-\gamma-1} t^{\rho+\alpha-1} \\ & \quad \times F_3 \left( -\alpha', -\alpha, n - \beta', -\beta, n - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right) dt \\ &= \frac{x^{\alpha'}}{\Gamma(n - \gamma)} \left( -\frac{d}{dx} \right)^n \frac{\Gamma(n - \gamma) \Gamma(\rho + \alpha - \alpha)}{\Gamma(\rho + \alpha - \alpha - \beta) \Gamma(\rho + \alpha + n - \gamma + \alpha')} \\ & \quad \times \frac{\Gamma(\rho + \alpha - \beta) \Gamma(\rho + \alpha + n - \gamma + \alpha' - n + \beta')}{\Gamma(\rho + \alpha + n - \gamma - n + \beta')} x^{\rho + \alpha + n - \gamma - 1}. \end{aligned}$$

An  $n$ -times differentiation of the last expression gives the formula (24). This completes the proof.  $\square$

In particular, if  $\alpha' = 0$ ,  $\beta = -\eta$ ,  $\gamma = \alpha$ , and  $\alpha$  is replaced by  $\alpha + \beta$  in (24), then the Saigo-Maeda operator becomes the Saigo operator (see [3]):

$$(25) \quad \left( D_{0+}^{\alpha, \beta, \eta} t^{\rho-1} \right) (x) = \frac{\Gamma(\rho) \Gamma(\rho + \alpha + \beta + \eta)}{\Gamma(\rho + \eta) \Gamma(\rho + \beta)} x^{\rho + \beta - 1}.$$

**Lemma 2.4.** *The following formula holds true: For  $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ ,  $\Re(\alpha) > 0$ ,*

$$(26) \quad \left( D_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x)$$

$$= (-1)^n \frac{\Gamma(1 - \alpha - \alpha' + \gamma - \rho)}{\Gamma(1 - \rho)} \times \frac{\Gamma(1 - \alpha' - \beta + \gamma - \rho) \Gamma(1 + \beta' - \rho)}{\Gamma(1 - \alpha - \alpha' - \beta + \gamma - \rho) \Gamma(1 - \alpha' + \beta' - \rho)} x^{\rho + \alpha + \alpha' - \gamma - 1},$$

where  $\Re(1 + \beta - \rho) > 0$ ,  $\Re(1 - \rho - \alpha' - \beta + \gamma) > 0$ ,  $\Re(1 - \rho - \alpha - \alpha' + \gamma) > 0$ ,  $n = [\Re(\gamma) + 1]$ , and  $x > 0$ .

*Proof.* Using (13) and (15), we have

$$\begin{aligned} & \left( D_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \right) (x) \\ &= \frac{1}{\Gamma(n - \gamma)} \left( -\frac{d}{dx} \right)^n (x^\alpha) t^{\rho-1} \\ & \quad \times \int_x^\infty (t-x)^{n-\gamma-1} t^{\alpha'} F_3(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}) f(t) dt \\ &= \frac{1}{\Gamma(n - \gamma)} \left( -\frac{d}{dx} \right)^n (x^\alpha) \int_x^\infty (t-x)^{n-\gamma-1} t^{\rho+\alpha'-1} \\ & \quad \times F_3\left(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t) dt \\ &= \frac{1}{\Gamma(n - \gamma)} \left( -\frac{d}{dx} \right)^n \frac{\Gamma(n - \gamma) \Gamma(1 - \alpha - \alpha' - n + \gamma - \rho)}{\Gamma(1 - \alpha - \alpha' - \beta + \gamma - \rho) \Gamma(1 - \rho)} \\ & \quad \times \frac{\Gamma(1 - \alpha' - \beta + \gamma - \rho) \Gamma(1 + \beta' - \rho)}{\Gamma(1 - \alpha' + \beta' - \rho)} x^{\rho + \alpha + \alpha' + n - \gamma - 1}. \end{aligned}$$

An  $n$ -times differentiation of the last expression gives the formula (26). This completes the proof.  $\square$

In particular, if  $\alpha' = 0$ ,  $\beta = -\eta$ ,  $\gamma = \alpha$ , and  $\alpha$  is replaced by  $\alpha + \beta$  in (26), then we obtain another known result (see [3]):

$$(27) \quad \left( D_-^{\alpha, \beta, \eta} t^{\rho-1} \right) (x) = (-1)^n \frac{\Gamma(1 - \beta - \rho) \Gamma(1 + \alpha + \eta - \rho)}{\Gamma(1 - \rho) \Gamma(1 + \eta - \beta - \rho)} x^{\rho + \beta - 1} \\ (n = [\Re(\gamma) + 1] \quad \text{and} \quad x > 0).$$

*Remark 2.5.* A detailed account of the operators of fractional integration and their applications can be found in a survey-cum-expository paper by Srivastava and Saxena [20], which contains a fairly comprehensive bibliography of as many as 190 *further* references on the subject.

In what follows, the following notations are used throughout this paper:

$$(28) \quad \xi = \min_{1 \leq j \leq m_i, 1 \leq i \leq r} \left[ \frac{\mu_i \left( \Re \left( d_j^{(i)} \right) \right)}{\delta_j^{(i)}} \right];$$

$$(29) \quad \eta = \max_{1 \leq j \leq n_i, 1 \leq i \leq r} \left[ \frac{\mu_i \left( \Re \left( c_j^{(i)} - 1 \right) \right)}{\gamma_j^{(i)}} \right];$$

$$(30) \quad \begin{aligned} \Omega_i &= \sum_{j=1}^n \alpha_j^{(i)} + \sum_{j=n+1}^p \alpha_j^{(i)} - \sum_{j=1}^q \beta_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} \\ &- \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \gamma_j^{(i)} > 0, \quad (i = 1, \dots, r). \end{aligned}$$

### 3. Left-sided generalized fractional differentiation of the multivariable $H$ -functions

Here the left-sided generalized fractional differentiation defined in (12) for the multivariable  $H$ -functions is investigated.

**Theorem 3.1.** *The left-sided generalized fractional derivative  $D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma}$  of the multivariable  $H$ -function is given as follows: For  $x > 0$ ,*

(31)

$$\begin{aligned} & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\ &= x^{\rho + \alpha + \alpha' - \gamma - 1} \sum_{\substack{h_1 k_1 + \dots + h_s k_s \leq L \\ k_1, \dots, k_s = 0}} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\ & \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+3, q+3; p_1, q_1; \dots, p_r, q_r}^{0, n+3; m_1, n_1; \dots, m_r, n_r} \\ & \left[ \begin{array}{l} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{array} \left( \begin{array}{l} (1-\rho - \sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1-\rho - \sum_{i=1}^s \lambda_i k_i + \beta - \alpha; \mu_1, \dots, \mu_r), \\ (1-\rho - \sum_{i=1}^s \lambda_i k_i + \beta; \mu_1, \dots, \mu_r), (1-\rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \beta'; \mu_1, \dots, \mu_r), \end{array} \right. \right. \\ & \left. \left. (1-\rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha' - \beta'; \mu_1, \dots, \mu_r), (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1,p} : (c_j^{(1)}, \zeta_j^{(1)})_{1,p_1} ; \dots ; (c_j^{(r)}, \zeta_j^{(r)})_{1,p_r} \right. \right. \\ & \left. \left. (1-\rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha'; \mu_1, \dots, \mu_r), (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} : (d_j^{(1)}, \delta_j^{(1)})_{1,q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \right] \right. \end{aligned}$$

Eq. (31) is true provided (in addition to the appropriate convergence and existence conditions) that the following conditions are satisfied:

- (i)  $\alpha, \alpha', \beta, \beta', \gamma, \lambda \in \mathbb{C}$ , with  $\Re(\gamma) > 0$ ,  $\mu_i \in \mathbb{R}_+$ ,  $\mathbb{R}_+$  being the set of positive real numbers;
- (ii)  $\xi < \Re(\lambda) + \min \{0, \Re(\alpha - \beta), \Re(\alpha' - \beta' + \alpha - \gamma)\}$ ;
- (iii)  $|\arg z_j| < \frac{\pi}{2} \Omega_j$  ( $j = 1, \dots, r$ ), where  $\Omega_j$  is the same as in (30);
- (iv)  $\Re(\rho + \Xi) > 0$ ,  $\Re(\rho + \alpha + \alpha' + \beta' - \gamma + \Xi) > 0$ ,  $\Re(\rho + \alpha - \beta + \Xi) > 0$ , where  $\Xi := \sum_{i=1}^r \lambda_i k_i + \sum_{j=1}^r \mu_j \tau_j$ .



*Proof.* Using (1), (5), and Lemma 2.3 and taking  $n = [\Re(\gamma)] + 1$  and  $q = \sum_{j=1}^s \lambda_j k_j + \sum_{j=1}^r \mu_j \xi_j$ , we obtain

$$\begin{aligned}
 & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\
 = & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \frac{[y_1 t^{\lambda_1}]^{k_1}}{k_1!} \dots \frac{[y_s t^{\lambda_s}]^{k_s}}{k_s!} \right. \\
 & \left. \cdot \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \phi_i(\xi_i) [z_i t^{\mu_i}]^{\xi_i} \right\} d\xi_1, \dots, d\xi_r \right\} (x) \\
 = & \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} \frac{1}{(2\pi\omega)^r} \\
 & \times \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \varphi_i(\xi_i) [z_i]^{\xi_i} \right\} d\xi_1, \dots, d\xi_r \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho+q-1} \right\} (x), \\
 = & \left( \frac{d}{dx} \right)^n \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} \\
 & \times \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \varphi_i(\xi_i) [z_i]^{\xi_i} \right\} d\xi_1, \dots, d\xi_r \\
 & \times \frac{\Gamma(\rho+q)\Gamma(\rho+q-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+q-\beta+\alpha)}{\Gamma(\rho+q-\gamma+n+\alpha+\alpha')\Gamma(\rho+q-\gamma+\alpha+\beta')\Gamma(\rho+q-\beta)} x^{\rho+q+\alpha+\alpha'-\gamma+n-1} \\
 = & x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} \\
 & \times \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \varphi_i(\xi_i) [z_i]^{\xi_i} \right\} d\xi_1, \dots, d\xi_r \\
 & \times \frac{\Gamma(\rho+q)\Gamma(\rho+q-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+q-\beta+\alpha)}{\Gamma(\rho+q-\gamma+\alpha+\alpha')\Gamma(\rho+q-\gamma+\alpha+\beta')\Gamma(\rho+q-\beta)} x^q \\
 = & x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
 & \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \left\{ \prod_{i=1}^r \varphi_i(\xi_i) [z_i x^{\mu_i}]^{\xi_i} \right\} \\
 & \times \frac{\Gamma(\rho+q)\Gamma(\rho+q-\gamma+\alpha+\alpha'+\beta')\Gamma(\rho+q-\beta+\alpha)}{\Gamma(\rho+q-\gamma+\alpha+\alpha')\Gamma(\rho+q-\gamma+\alpha+\beta')\Gamma(\rho+q-\beta)} d\xi_1, \dots, d\xi_r.
 \end{aligned}$$

Now, interpreting the above result by means of (1), we arrive at the desired result (31). This completes the proof.  $\square$

**Corollary 3.2.** *A special case of Theorem 3.1 when  $n = p = q = 0$  and  $x > 0$  gives the following result which is expressed in terms of  $r$  product of*

*H*-functions:

$$\begin{aligned}
(32) \quad & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) \prod_{i=1}^r H_{p_i, q_i}^{m_i, n_i} \left[ z_i t^{\mu_i} \left[ \begin{matrix} (c_j^{(i)}, \gamma_j^{(i)})_{1, p_i} \\ (d_j^{(i)}, \delta_j^{(i)})_{1, q_i} \end{matrix} \right] \right] \right\} (x) \\
& = x^{\rho + \alpha + \alpha' - \gamma - 1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
& \quad \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{3, 3; p_1, q_1; \dots, p_r, q_r}^{0, 3; m_1, n_1; \dots, m_r, n_r} \\
& \quad \left[ z_i x^{\mu_i} \left[ \begin{matrix} (1 - \rho - \sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \beta - \alpha; \mu_1, \dots, \mu_r), \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \beta; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \beta'; \mu_1, \dots, \mu_r), \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha' - \beta'; \mu_1, \dots, \mu_r), (c_j^{(1)}, \gamma_j^{(1)})_{1, p_i} \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha'; \mu_1, \dots, \mu_r), (d_j^{(1)}, \delta_j^{(1)})_{1, q_i} \end{matrix} \right] \right],
\end{aligned}$$

which holds true under the same conditions as given in Theorem 3.1.

Further, if we set  $r = 1$ , then (32) reduces to the result for a single *H*-function given by Saxena and Saigo [16].

**Corollary 3.3.** *A special case of Theorem 3.1 when  $\alpha' = 0$ ,  $\beta = -\eta$ ,  $\alpha = \alpha + \beta$ ,  $\gamma = \alpha$  and  $x > 0$  reduces to the following result for the left-sided Saigo fractional differentiation of the multivariable *H*-function:*

$$\begin{aligned}
(33) \quad & \left\{ D_{0+}^{\alpha, \beta, \eta} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\
& = x^{\rho + \beta - 1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
& \quad \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+2, q+2; p_1, q_1; \dots, p_r, q_r}^{0, n+2; m_1, n_1; \dots, m_r, n_r} \\
& \quad \left[ \begin{matrix} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{matrix} \left[ \begin{matrix} (1 - \rho - \sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i - \alpha - \beta - \eta; \mu_1, \dots, \mu_r), \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i - \beta; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i - \eta; \mu_1, \dots, \mu_r), \end{matrix} \right. \right. \\
& \quad \left. \left. \begin{matrix} (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1, p} : (c_j^{(1)}, \gamma_j^{(1)})_{1, p_1} \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1, p_r} \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1, q} : (d_j^{(1)}, \delta_j^{(1)})_{1, q_1} \dots; (d_j^{(r)}, \delta_j^{(r)})_{1, q_r} \end{matrix} \right] \right],
\end{aligned}$$

which holds true provided that (in addition to the appropriate convergence and existence condition) the following conditions are satisfied:

- (i)  $\alpha, \beta, \eta, \lambda \in \mathbb{C}$  with  $\Re(\alpha) > 0$  and  $\mu_i \in \mathbb{R}_+$ ;
- (ii)  $\xi < \Re(\lambda) + \min\{0, \Re(\alpha + \beta + \eta)\}$ ;
- (iii)  $|\arg z_j| < \frac{\pi}{2} \Omega_j$  ( $j = 1, \dots, r$ ), where  $\Omega_j$  is the same as in (30);

(iv)  $\Re(\rho + \Xi) > 0, \Re(\rho + \alpha + \beta + \eta + \Xi) > 0$ , where  $\Xi$  is the same as in Theorem 3.1.

**Corollary 3.4.** *A further special case of Corollary 3.3 when  $\beta = -\alpha$  gives the following result for the left-sided Riemann-Liouville fractional differentiation:*

$$\begin{aligned}
 (34) \quad & \left\{ {}_0 D_x^\alpha t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\
 &= x^{\rho-\alpha-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
 &\times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+1, q+1; p_1, q_1; \dots, p_r, q_r}^{0, n+1; m_1, n_1; \dots, m_r, n_r} \\
 &\left[ \begin{array}{l} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{array} \left| \begin{array}{l} (1-\rho-\sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1,p} : \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i + \alpha; \mu_1, \dots, \mu_r), (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} : \\ (c_j^{(1)}, \gamma_j^{(1)})_{1,p_1} \dots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)})_{1,q_1} \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right. \right],
 \end{aligned}$$

whose conditions are easily modified from those in Corollary 3.3.

**4. Right-sided generalized fractional differentiation of the multivariable  $H$ -functions**

The right-sided generalized fractional differentiation defined in (13) for the multivariable  $H$ -functions is investigated.

**Theorem 4.1.** *The right-sided generalized fractional derivative  $D_-^{\alpha, \alpha', \beta, \beta', \gamma}$  of the multivariable  $H$ -function for  $x > 0$  is given as follows:*

$$\begin{aligned}
 (35) \quad & \left\{ D_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\
 &= (-1)^{[\Re(\gamma)]+1} x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
 &\times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+3, q+3; p_1, q_1; \dots, p_r, q_r}^{0, n+3; m_1, n_1; \dots, m_r, n_r} \\
 &\left[ \begin{array}{l} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{array} \left| \begin{array}{l} (1-\rho-\sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1-\rho-\sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha' - \beta; \mu_1, \dots, \mu_r), \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i + \beta'; \mu_1, \dots, \mu_r), (1-\rho-\sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha'; \mu_1, \dots, \mu_r), \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i - \alpha' + \beta'; \mu_1, \dots, \mu_r), (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1,p} : (c_j^{(1)}, \zeta_j^{(1)})_{1,p_1} \dots; (c_j^{(r)}, \zeta_j^{(r)})_{1,p_r} \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i + \gamma - \alpha' - \beta; \mu_1, \dots, \mu_r), (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} : (d_j^{(1)}, \delta_j^{(1)})_{1,q_1} \dots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right. \right],
 \end{aligned}$$

provided (in addition to the appropriate convergence and existence conditions) that the following conditions are satisfied:

- (i)  $\alpha, \alpha', \beta, \beta', \gamma, \lambda \in \mathbb{C}$  with  $\Re(\gamma) > 0, \mu_i \in \mathbb{R}_+$ ;
- (ii)  $\xi + 1 > \Re(\lambda) + \min\{0, \Re(\gamma - \alpha - \alpha' - \beta), \Re(-\alpha' - \beta + \gamma), -\Re(\beta')\}$ ;
- (iii)  $|\arg z_j| < \frac{\pi}{2}\Omega_j$  ( $j = 1, \dots, r$ ) where  $\Omega_j$  are the same as in (30);
- (iv)  $\Re(\rho + \Xi) > 0, \Re(\rho + \alpha + \alpha' + \beta - \gamma + \Xi) > 0, \Re(\rho + \alpha' - \beta' + \Xi) > 0$ , where  $\Xi$  is the same as in Theorem 3.1.

*Proof.* A similar argument in the proof of Theorem 3.1 can be used to establish (35). So a detailed account of its proof is omitted.  $\square$

**Corollary 4.2.** *A special case of Theorem 4.1 when  $n = p = q = 0$  and  $x > 0$  gives the following result which is expressed in terms of  $r$  product of  $H$ -functions:*

$$\begin{aligned}
 (36) \quad & \left\{ D_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) \prod_{i=1}^r H_{p_i, q_i}^{m_i, n_i} \left[ z_i t^{\mu_i} \left| \begin{array}{c} (c_j^{(i)}, \gamma_j^{(i)})_{1, p_i} \\ (d_j^{(i)}, \delta_j^{(i)})_{1, q_i} \end{array} \right. \right] \right\} (x) \\
 &= (-1)^{[\Re(\gamma)]+1} x^{\rho + \alpha + \alpha' - \gamma - 1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
 &\quad \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{3, 3; p_1, q_1; \dots, p_r, q_r}^{0, 3; m_1, n_1; \dots, m_r, n_r} \\
 &\quad \left[ z_i x^{\mu_i} \left| \begin{array}{c} (1 - \rho - \sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha' - \beta; \mu_1, \dots, \mu_r), \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \beta'; \mu_1, \dots, \mu_r), (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha - \alpha'; \mu_1, \dots, \mu_r), \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i - \alpha' + \beta'; \mu_1, \dots, \mu_r), (c_j^{(1)}, \gamma_j^{(1)})_{1, p_i} \\ (1 - \rho - \sum_{i=1}^s \lambda_i k_i + \gamma - \alpha' - \beta; \mu_1, \dots, \mu_r), (d_j^{(1)}, \delta_j^{(1)})_{1, q_i} \end{array} \right. \right],
 \end{aligned}$$

which holds true under the same conditions as given in Theorem 4.1.

A further special case of (36) when  $r = 1$  reduces to a known result for a single  $H$ -function given by Saxena and Saigo [16].

**Corollary 4.3.** *A special case of Theorem 4.1 when  $\alpha' = 0, \beta = -\eta, \alpha = \alpha + \beta, \gamma = \alpha$  and  $x > 0$  reduces to the following result for the right-sided Saigo fractional differentiation of the multivariable  $H$ -function:*

$$\begin{aligned}
 (37) \quad & \left\{ D_-^{\alpha, \beta, \eta} t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\
 &= (-1)^{[\Re(\gamma)]+1} x^{\rho + \beta - 1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\
 &\quad \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+2, q+2; p_1, q_1; \dots, p_r, q_r}^{0, n+2; m_1, n_1; \dots, m_r, n_r}
 \end{aligned}$$

$$\left[ \begin{array}{l} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{array} \left| \begin{array}{l} (1-\rho-\sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (1-\rho-\sum_{i=1}^s \lambda_i k_i - \beta + \eta; \mu_1, \dots, \mu_r), \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i - \beta; \mu_1, \dots, \mu_r), (1-\rho-\sum_{i=1}^s \lambda_i k_i + \alpha + \eta; \mu_1, \dots, \mu_r), \end{array} \right. \right. \\ \left. \left. \begin{array}{l} (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1,p} : (c_j^{(1)}, \gamma_j^{(1)})_{1,p_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} : (d_j^{(1)}, \delta_j^{(1)})_{1,q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right] , \right.$$

provided that (in addition to the appropriate convergence and existence condition) the following conditions are satisfied:

- (i)  $\alpha, \beta, \eta, \lambda \in \mathbb{C}$  with  $\Re(\alpha) > 0$ ;
- (ii)  $\xi + 1 < \Re(\lambda) + \min \{0, \Re(\beta) + \Re(\alpha) + 1, -\Re(\alpha + \eta)\}$ ;
- (iii)  $|\arg z_j| < \frac{\pi}{2} \Omega_j$  ( $j = 1, \dots, r$ ), where  $\Omega_j$  are the same as in (30);
- (iv)  $\Re(\rho + \Xi) > 0, \Re(\rho + \beta - \eta + \Xi) > 0$ , where  $\Xi$  is the same as in Theorem 3.1.

**Corollary 4.4.** A further special case of Corollary 4.3 when  $\beta = -\alpha$  gives the following result for the right-sided Riemann-Liouville fractional differentiation of the multivariable  $H$ -function:

$$(38) \quad \left\{ {}_x D_\infty^\alpha t^{\rho-1} S_L^{h_1, \dots, h_s} (y_1 t^{\lambda_1}, \dots, y_s t^{\lambda_s}) H [z_1 t^{\mu_1}, \dots, z_r t^{\mu_r}] \right\} (x) \\ = (-1)^{[\Re(\gamma)]+1} x^{\rho-\alpha-1} \sum_{k_1, \dots, k_s=0}^{h_1 k_1 + \dots + h_s k_s \leq L} (-L)_{h_1 k_1 + \dots + h_s k_s} A(L; k_1, \dots, k_s) \\ \times \frac{y_1^{k_1}}{k_1!} \dots \frac{y_s^{k_s}}{k_s!} x^{\lambda_1 k_1 + \dots + \lambda_s k_s} H_{p+1, q+1; p_1, q_1; \dots, p_r, q_r}^{0, n+1; m_1, n_1; \dots, m_r, n_r} \\ \left[ \begin{array}{l} z_1 x^{\mu_1} \\ \vdots \\ z_r x^{\mu_r} \end{array} \left| \begin{array}{l} (1-\rho-\sum_{i=1}^s \lambda_i k_i; \mu_1, \dots, \mu_r), (a_j; \zeta_j^{(1)}, \dots, \zeta_j^{(r)})_{1,p} : \\ (1-\rho-\sum_{i=1}^s \lambda_i k_i + \alpha; \mu_1, \dots, \mu_r), (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} : \\ (c_j^{(1)}, \gamma_j^{(1)})_{1,p_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (d_j^{(1)}, \delta_j^{(1)})_{1,q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{array} \right] , \right.$$

whose conditions are easily modified from those in Corollary 4.3.

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JUNESANG CHOI  
 DEPARTMENT OF MATHEMATICS  
 DONGGUK UNIVERSITY  
 GYEONGJU 780-714, KOREA  
 E-mail address: junesang@mail.dongguk.ac.kr

JITENDRA DAIYA  
 DEPARTMENT OF MATHEMATICS AND STATISTICS  
 JAI NARAIN VYAS UNIVERSITY  
 JODHPUR-342005, INDIA  
 E-mail address: jitendra.daiya25@gmail.com

DINESH KUMAR  
DEPARTMENT OF MATHEMATICS AND STATISTICS  
JAI NARAIN VYAS UNIVERSITY  
JODHPUR-342005, INDIA  
*E-mail address:* [dinesh\\_dino03@yahoo.com](mailto:dinesh_dino03@yahoo.com)

RAM KISHORE SAXENA  
DEPARTMENT OF MATHEMATICS AND STATISTICS  
JAI NARAIN VYAS UNIVERSITY  
JODHPUR-342005, INDIA  
*E-mail address:* [ram.saxena@yahoo.com](mailto:ram.saxena@yahoo.com)