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STABILITY OF (α, β, γ)-DERIVATIONS ON LIE C*-ALGEBRA ASSOCIATED TO A PEXIDERIZED QUADRATIC TYPE FUNCTIONAL EQUATION

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Abstract. In this article, we considered the stability of the following (α, β, γ)-derivation

$$\alpha D[x, y] = \beta[D(x), y] + \gamma[x, D(y)]$$

and homomorphisms associated to the quadratic type functional equation

$$f(kx + y) + f(kx + \sigma(y)) = 2kg(x) + 2g(y), \quad x, y \in A,$$

where σ is an involution of the Lie C*-algebra A and k is a fixed positive integer. The Hyers-Ulam stability on unbounded domains is also studied. Applications of the results for the asymptotic behavior of the generalized quadratic functional equation are provided.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [16] in 1940, concerning the stability of group homomorphisms: Let $(G_1, \cdot)$ be a group and $(G_2, *)$ be a metric group with metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

A C^*-algebra A endowed with the Lie product

$$[x, y] = xy - yx$$

on A is called a Lie C^*-algebra. Let A be a Lie C^*-algebra. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a Lie derivation of A if $D : A \to A$ satisfies

$$D[x, y] = [D(x), y] + [x, D(y)]$$

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for all $x, y \in A$. Following a $\mathbb{C}$-linear mapping $D : A \to A$ is called an (α, β, γ)-derivation of A if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\alpha D[x, y] = \beta[D(x), y] + \gamma[x, D(y)]$$

for all $x, y \in A$.

The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. Since then, a great deal of works has been published by a number of mathematicians for other functional equations (see for example [3], [4], [6], [7], [8], [9], [12] and [13]).

A Hyers-Ulam stability theorem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),\ x, y \in A$$

was proved by Skof [14] and later by Jung [10] on unbounded domains.

Recently, the functional equation

$$f(kx + y) + f(kx - y) = 2kf(x) + 2f(y),\ x, y \in A \tag{1.1}$$

was solved by Lee et al. [11]. Indeed, they proved the Hyers-Ulam-Rassias stability theorem of equation (1.1).

Throughout this paper, let k denote a fixed positive integer and $T^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let A be a Lie C^*-algebra and $\sigma : A \to A$ be an automorphism of A such that $\sigma(\sigma(x)) = x$ for all $x \in A$.

The purpose of the present paper is to extend the results mentioned due to Lee et al. [11] to the generalized quadratic functional equation

$$f(kx + y) + f(kx + \sigma(y)) = 2f(x) + 2f(y),\ x, y \in A. \tag{1.2}$$

It's clear that equation (1.2) is a proper extension of equation (1.1). The following equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y),\ x, y \in A$$

has been studied by Stetkaer [15] and the Hyera-Ulam-Rassias stability of this equation has been obtained by Bouikhalene et al. [1, 2].

2. Hyers-Ulam stability of Pexiderized quadratic type functional equation

Lemma 2.1. *Let X and Y be linear spaces and $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and any $\mu \in T^1$. Then the mapping f is $\mathbb{C}$-linear.*

Proof. See [5]. □

Theorem 2.2. *Let $f, g : A \to A$ be mappings with $f(0) = 0$ and $\varphi : A^7 \to [0, \infty)$ be a function satisfying*:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y, k^n u, k^n v, k^n w, k^n z, k^n t) = 0,\\ &\varphi(0, 0, 0, 0, 0, 0, x) \le \delta,\\ &\| f(kx + y) + f(kx + \sigma(y)) - 2kg(x) - 2g(y) \| \le \delta, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z) + f(t) - \frac{1}{k} f(kt) \| \\ &\le \varphi(x, y, u, v, w, z, t) \end{aligned} \tag{2.2}$$

for all $x, y, u, v, w, z, t \in A$ *and any* $\mu \in T^1$. *Then there exists a unique* (α, β, γ)*-derivation* $D : A \to A$*, such that*

$$\begin{aligned} \| f(x) - D(x) \| &\le \delta \frac{k^3 + 4k^2 + k - 2}{2k(k^2 - 1)}, \\ \| g(x) - D(x) \| &\le \frac{\delta}{2k} \frac{3k^2 + 3k - 2}{2k(k^2 - 1)}. \end{aligned} \tag{2.3}$$

Proof. By letting respectively $y = 0$ and $x = y = 0$ in (2.1), we get

$$\| g(x) - \frac{1}{k}\{f(kx) - g(0)\} \| \le \frac{\delta}{2k}, \ x \in A$$

and

$$\| g(0) \| \le \frac{\delta}{2(k + 1)}, \ x \in A.$$

So, we deduce that

$$\| g(x) - \frac{1}{k}\{f(kx)\} \| \le \frac{\delta}{2k} + \frac{\delta}{2k(k + 1)}. \tag{2.4}$$

By applying the inductive assumption we prove

$$\begin{aligned} \| g(x) - \frac{1}{k^n}\{f(k^n x)\} \| \le \delta[\frac{1}{2k} + \frac{1}{2k(k + 1)} + \frac{1}{k} \varphi(0, 0, 0, 0, 0, 0, kx) \\ + \cdots + \frac{1}{k^{(n-1)}} \varphi(0, 0, 0, 0, 0, 0, k^{n-1} x)] \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$. From (2.4) it follows that (2.5) is true for $n = 1$. Assume now that (2.5) holds for $n \in \mathbb{N}$. The inductive step must be demonstrated to hold for $n + 1$, that is

$$\begin{aligned} &\| g(x) - \frac{1}{k^{n+1}}\{f(k^{n+1} x)\} \| \\ \le \ &\| g(x) - \frac{1}{k^n}\{f(k^n x)\} \| + \frac{1}{k^n} \| g(k^n x) - \frac{1}{k}\{f(k^{n+1} x)\} \| \\ \le \ &\delta[\frac{1}{2k} + \frac{1}{2(k + 1)}) + \frac{1}{k} \varphi(0, 0, 0, 0, 0, 0, kx) \\ &+ \cdots + \frac{1}{k^{n-1}} \varphi(0, 0, 0, 0, 0, 0, k^{n-1} x)] + \frac{1}{k^n} \varphi(0, 0, 0, 0, 0, 0, k^n x). \end{aligned}$$

This proves the validity of the inequality (2.5). Let us define the sequence of functions

$$f_n(x) = \frac{1}{k^n}\{f(k^n x)\}, \ x \in A, \ n \in \mathbb{N}.$$

We will show that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in A$. By using (2.4), we have

$$\begin{aligned}\| f_{n+1}(x) - f_n(x) \| &= \| \frac{1}{k^{n+1}}\{f(k^{n+1}x)\} - \frac{1}{k^n}\{f(k^n x)\} \| \\ &= \frac{1}{k^n} \| \{f(k^n x)\} - \frac{1}{k}\{f(k^{n+1}x)\} \| \\ &\le \frac{\delta}{k^n}.\end{aligned}$$

It follows that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in A$. However, A is a complete normed space, thus the limit function $D(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in A$. Assume now that there exist two mappings $D_i : A \to A$ $(i = 1, 2)$ satisfying (1.2) and (2.3). By mathematical induction, we can easily verify that

$$D_i(k^n x) = k^n D_i(x), \ \ (i = 1, 2). \tag{2.6}$$

For all $x \in A$ and all $n \in \mathbb{N}$,we have

$$\begin{aligned}\| D_1(x) - D_2(x) \| &= \frac{1}{k^n} \| D_1(k^n x) - D_2(k^n x) \| \\ &\le \frac{1}{k^n} \| D_1(k^n x) - g(k^n x) \| + \frac{1}{k^n} \| D_2(k^n x) - g(k^n x) \| \\ &\le \frac{\delta}{k^{n+1}} \frac{3k^2 + 3k - 2}{k^2 - 1}.\end{aligned}$$

If we let $n \to +\infty$, we get $D_1(x) = D_2(X)$ for all $x \in A$. We show that $D : A \to A$ is (α, β, γ)-derivation. By setting $x = y = u = v = 0$ and using (2.2) we have

$$\| f(\mu w + z) - \mu f(w) - f(z) \| \le \varphi(0, 0, 0, 0, w, z, 0). \tag{2.7}$$

Replacing w, z in (2.7) by $k^n w$, $k^n z$ respectively, and divide both sides by k^n we obtain

$$D(\mu w + z) = \mu D(w) + D(z) \tag{2.8}$$

for any $\mu \in T^1$ and all $w, z \in A$. Letting $\mu = 1$ in (2.8), we conclude that D is additive. Set $z = 0$, we have $D(\mu w) = \mu D(w)$. Thus, Lemma 2.1 implies that D is $\mathbb{C}$-linear.

By using the inequality (2.2) we get

$$\begin{aligned}&\| \alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)] \| \\ &= \| \alpha f(xy - yx) - \beta(f(x)y - yf(x)) - \gamma(xf(y) - f(y)x \| \\ &= \| \alpha f(xy) - \alpha f(yx) - \beta f(x)y + \beta yf(x) - \gamma xf(y) + \gamma f(y)x \| \\ &\le \| \alpha f(xy) - \beta f(x)y - \gamma xf(y) \| + \| \alpha f(yx) - \beta yf(x) - \gamma f(y)x \| \\ &\le \varphi(x, y, 0, 0, 0, 0, 0) + \varphi(0, 0, x, y, 0, 0, 0).\end{aligned} \tag{2.9}$$

Replacing x, y by $k^n x, k^n y$ respectively in (2.9), and divide both sides by k^{2n} we obtain

$$\alpha D[x,y] = \beta[D(x), y] + \gamma[x, D(y)]$$

for all $x, y \in A$. Hence D is a (α, β, γ)-derivations on A. □

Corollary 2.3. *Let $0 < q < 2$, $\eta > 0$ and $f, g : A \to A$ be mappings with $f(0) = 0$ satisfying*:

$$\| f(kx + y) + f(kx + y) - 2kg(x) - 2g(y) \| \le \delta,$$

and

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z) + f(t) - \frac{1}{k} f(kt) \| \\ &\le \eta \| x \|^{\frac{q}{7}} \| y \|^{\frac{q}{7}} \| u \|^{\frac{q}{7}} \| v \|^{\frac{q}{7}} \| w \|^{\frac{q}{7}} \| z \|^{\frac{q}{7}} \| t \|^{\frac{q}{7}} \end{aligned}$$

for all $x, y, u, v, w, z, t \in A$ and any $\mu \in T^1$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\begin{aligned} &\| f(x) - D(x) \| \le \delta \frac{k^3 + 4k^2 + k - 2}{2k(k^2 - 1)}, \\ &\| g(x) - D(x) \| \le \frac{\delta}{2k} \frac{3k^2 + +3k - 2}{k^2 - 1}, \ x \in A. \end{aligned}$$

Proof. It is a desired result of Theorem 2.2. □

Theorem 2.4. *Let $f, g : A \to A$ be mappings with $f(0) = g(0) = 0$ and $\varphi : A^7 \to [0, \infty)$ is a function satisfying*:

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y, k^n u, k^n v, k^n w, k^n z, k^n t) = 0, \\ &\varphi(0,0,0,0,0,0,x) \le \theta \| x \|^p, \\ &\| f(kx + y) + f(kx + \sigma(y)) - 2kg(x) - 2g(y) \| \le \theta(\| x \|^p + \| y \|^p) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z) + f(t) - \frac{1}{k} f(kt) \| \\ &\le \varphi(x, y, u, v, w, z, t) \end{aligned} \tag{2.11}$$

for some $\theta \ge 0$, $p \in (0, 1)$ and for all $x, y, w, z, u, v, t \in A$ and any $\mu \in T^1$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\begin{aligned} &\| f(x) - D(x) \| \le \frac{\theta}{2} \frac{2k^{p+1} - k^p + k^{2-p}}{k^2 - k^{p+1}} \| x \|^p, \\ &\| g(x) - D(x) \| \le \frac{\theta}{2k} \frac{k^{1-p} + 2k - 1}{k^{1-p} - 1} \| x \|^p, \ x \in A. \end{aligned} \tag{2.12}$$

Proof. Suppose that f satisfies the inequality (2.10). Letting $x = y = 0$ in (2.10), we get $f(0) = 0$. Putting $y = 0$ in (2.10), we get

$$\| 2f(kx) - 2kg(x) \| \leq \theta \| x \|^p \tag{2.13}$$

for all $x \in A$. So

$$\| g(x) - \frac{1}{k} f(kx) \| \leq \frac{\theta}{2k} \| x \|^p \tag{2.14}$$

for all $x \in A$. By mathematical induction we verify that

$$\| g(x) - \frac{1}{k^n} f(k^n x) \| \leq \theta[\frac{1}{2k} + \frac{1}{k^{1-p}} + \cdots + \frac{1}{k^{(n-1)(1-p)}}] \| x \|^p \tag{2.15}$$

holds for all $n \in \mathbb{N}$. Next, we will show that the sequence of functions $g_n(x) = \frac{1}{k^n} g(k^n x)$ is a Cauchy sequence for every $x \in A$. By using the inequality (2.14), we get

$$\begin{aligned} \| g_{n+1}(x) - g_n(x) \| &= \| \frac{1}{k^{n+1}} f(k^{n+1}x) - \frac{1}{k^n} f(k^n x) \| \\ &= \frac{1}{k^n} \| f(k^n x) - \frac{1}{k} f(k^{n+1} x) \| \\ &\leq \frac{\theta}{k^{n(1-p)}} \| x \|^p . \end{aligned}$$

Consequently, $\{g_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for all $x \in A$. Since A is a complete normed space, the limit function $D(x) = \lim_{n \to \infty} g_n(x)$ exists for every $x \in A$. By using the same method as in the proof of Theorem 2.2, D is a unique (α, β, γ)-derivation. □

Corollary 2.5. *Let $0 < q < 2$, $\eta > 0$ and $f, g : A \to A$ be mappings such that $f(0) = 0$ and*

$$\| f(kx + y) + f(kx - y) - 2kg(x) - 2g(y) \| \leq \theta(\| x \|^p + \| y \|^p),$$

and also

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z) + f(t) - \frac{1}{k} f(kt) \| \\ &\leq \eta \| x \|^{\frac{q}{7}} \| y \|^{\frac{q}{7}} \| u \|^{\frac{q}{7}} \| v \|^{\frac{q}{7}} \| w \|^{\frac{q}{7}} \| z \|^{\frac{q}{7}} \| t \|^{\frac{q}{7}} \end{aligned}$$

for some $\theta \geq 0$, $p \in (0, 1)$ and for all $x, y, u, v, w, z, t \in A$. then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\begin{aligned} \| f(x) - D(x) \| &\leq \frac{\theta}{2} \frac{2k^{p+1} - k^p + k^{2-p}}{k^2 - k^{p+1}} \| x \|^p, \\ \| g(x) - D(x) \| &\leq \frac{\theta}{2k} \frac{k^{1-p} + 2k - 1}{k^{1-p} - 1} \| x \|^p, \ x \in A. \end{aligned}$$

Proof. It is a desired result of Theorem 2.4. □

Theorem 2.6. *Let $f, g : A \to A$ be mappings with $f(0) = 0$ and $\varphi : A^7 \to [0,\infty)$ is a function satisfying:*

$$\begin{aligned} &\lim_{n\to\infty} k^{2n}\varphi(\frac{x}{k^n},\frac{y}{k^n},\frac{u}{k^n},\frac{v}{k^n},\frac{w}{k^n},\frac{z}{k^n},\frac{t}{k^n}) = 0,\\ &\varphi(0,0,0,0,0,0,kx) \le \theta \parallel x \parallel^p,\\ &\parallel f(kx+y) + f(kx+\sigma(y)) - 2kg(x) - 2g(y) \parallel \le \theta(\parallel x \parallel^p + \parallel y \parallel^p), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\parallel \alpha g(xy) - \beta g(x)y - \gamma x g(y) + \alpha g(uv) - \beta u g(v) - \gamma g(u)v\\ &\quad + g(\mu w + z) - \mu g(w) - g(z) + g(t) - kg(\tfrac{t}{k}) \parallel\\ &\le \varphi(x,y,u,w,w,z,t) \end{aligned} \tag{2.17}$$

for some $\theta \ge 0$, $p > 1$ and for all $x, y, u, v, w, z, t \in A$. Then there exists a unique (α,β,γ)-derivation $D : A \to A$, such that

$$\begin{aligned} &\parallel f(x) - D(x) \parallel \le \frac{\theta}{2}\frac{1 - k^{1-p} + 2k}{k^p - k} \parallel x \parallel^p,\\ &\parallel g(x) - D(x) \parallel \le \frac{\theta}{2}\frac{k^p - k^{2-p} + 2k^2}{k^{p+1} - k^2} \parallel x \parallel^p, \ x \in A. \end{aligned} \tag{2.18}$$

Proof. Suppose that f satisfies the inequality (2.16). Letting $x = y = 0$ in (2.16), we get $f(0) = 0$. Putting $y = 0$ in (2.16), we get

$$\parallel 2f(kx) - 2kg(x) \parallel \le \theta \parallel x \parallel^p,$$

and

$$\parallel f(x) - kg(\tfrac{x}{k}) \parallel \le \frac{\theta}{2k^p} \parallel x \parallel^p \tag{2.19}$$

for all $x \in A$. By mathematical induction we verify that

$$\parallel f(x) - k^n g(\tfrac{x}{k^n}) \parallel \le \theta[\frac{1}{2k^p} + k^{1-p} + \cdots + k^{(n-1)(1-p)}] \parallel x \parallel^p \tag{2.20}$$

holds for all $n \in \mathbb{N}$. Next, we will show that the sequence of functions $g_n(x) = k^n g(\frac{x}{k^n})$ is a Cauchy sequence for every $x \in A$. By using the inequality (2.19), we get

$$\begin{aligned} \parallel g_{n+1}(x) - g_n(x) \parallel &= \parallel k^{n+1} g(\tfrac{x}{k^{n+1}}) - k^n g(\tfrac{x}{k^n}) \parallel\\ &= k^n \parallel g(\tfrac{x}{k^n}) - kg(\tfrac{x}{k^{n+1}}) \parallel\\ &\le \frac{\theta}{k^{n(p-1)}} \parallel x \parallel^p . \end{aligned}$$

Consequently, $\{g_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for all $x \in A$. Since A is a complete normed space, the limit function $D(x) = \lim_{n\to\infty} g_n(x)$ exists for every $x \in A$. Assume now that there exist two mappings $D_i : A \to A$ $(i = 1, 2)$ satisfying (2.18). By mathematical induction, we can easily verify that

$$D_i(x) = k^n D_i(\tfrac{x}{k^n}), \ (i = 1, 2). \tag{2.21}$$

For all $x \in A$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned}\| D_1(x) - D_2(x) \| &= k^n \| D_1(\tfrac{x}{k^n}) - D_2(\tfrac{x}{k^n}) \| \\ &\leq k^n \| D_1(\tfrac{x}{k^n}) - f(\tfrac{x}{k^n}) \| + k^n \| D_2(\tfrac{x}{k^n}) - f(\tfrac{x}{k^n}) \| \\ &\leq \frac{\theta}{k^{n(p-1)}} \frac{1 - k^{1-p} + 2k}{k^p - k} \| x \|^p .\end{aligned}$$

If we let $n \to +\infty$, we get $D_1(x) = D_2(X)$ for all $x \in A$. We show that $D : A \to A$ is (α, β, γ)-derivation. By setting $x = y = u = v = 0$ and equation (2.17) we have

$$\| g(\mu w + z) - \mu g(w) - g(z) \| \leq \varphi(0, 0, 0, 0, w, z, 0). \tag{2.22}$$

Replacing w, z in (2.22) by $\frac{w}{k^n}$, $\frac{z}{k^n}$ respectively, and divide both sides by k^n we obtain

$$D(\mu w + z) = \mu D(w) + D(z) \tag{2.23}$$

for any $\mu \in T^1$ and all $w, z \in A$. Letting $\mu = 1$ in (2.23), we conclude that D is additive. By setting $z = 0$, we have $D(\mu w) = \mu D(w)$. Thus, Lemma 2.1 implies that D is $\mathbb{C}$-linear. By using the inequality (2.17) we get

$$\begin{aligned}&\| \alpha g[x, y] - \beta[g(x), y] - \gamma[x, g(y)] \| \\ &= \| \alpha g(xy - yx) - \beta(g(x)y - yg(x)) - \gamma(xg(y) - g(y)x \| \\ &= \| \alpha g(xy) - \alpha g(yx) - \beta g(x)y + \beta yg(x) - \gamma xg(y) + \gamma g(y)x \| \\ &\leq \| \alpha g(xy) - \beta g(x)y - \gamma xg(y) \| + \| \alpha g(yx) - \beta yg(x) - \gamma g(y)x \| \\ &\leq \varphi(x, y, 0, 0, 0, 0, 0) + \varphi(0, 0, x, y, 0, 0, 0).\end{aligned} \tag{2.24}$$

Replacing x, y in (2.24) by $\frac{x}{k^n}$, $\frac{y}{k^n}$ respectively and dividing both sides by k^{2n} we obtain

$$\alpha D[x, y] = \beta[D(x), y] + \gamma[x, D(y)]$$

for all $x, y \in A$. Hence D is a (α, β, γ)-derivations on A. □

Corollary 2.7. *Let $0 < q < 2$, $\eta > 0$ and $f, g : A \to A$ are functions such that $f(0) = 0$ and*

$$\| f(kx + y) + f(kx - y) - 2kg(x) - 2g(y) \| \leq \theta(\| x \|^p + \| y \|^p),$$

and also

$$\begin{aligned}&\| \alpha g(xy) - \beta g(x)y - \gamma xg(y) + \alpha g(uv) - \beta ug(v) - \gamma g(u)v \\ &\quad + g(\mu w + z) - \mu g(w) - g(z) + g(t) - kg(\tfrac{t}{k}) \| \\ &\leq \eta \| x \|^{\frac{q}{7}} \| y \|^{\frac{q}{7}} \| u \|^{\frac{q}{7}} \| v \|^{\frac{q}{7}} \| w \|^{\frac{q}{7}} \| z \|^{\frac{q}{7}} \| t \|^{\frac{q}{7}}\end{aligned}$$

for some $\theta \geq 0$, $p > 1$ $x, y, u, v, w, z, t \in A$ and $\mu \in T^1$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\| f(x) - D(x) \| \leq \frac{\theta}{2} \frac{1 - k^{1-p} + 2k}{k^p - k} \| x \|^p$$

and

$$\| g(x) - D(x) \| \leq \frac{\theta}{2} \frac{k^p - k^{2-p} + 2k^2}{k^{p+1} - k^2} \| x \|^p, \ x \in A.$$

Proof. It is a desired result of Theorem 2.6 □

3. Hyers-Ulam stability of quadratic equation on unbounded domains

In this section, we investigate the Hyers-Ulam stability of equation (1.2) on unbounded domains $\{(x, y) \in A^2 : \| x \| + \| y \| \geq d\}$.

Theorem 3.1. *Let $d > 0$ be given. Assume that mappings $f : A \to A$ and $\varphi : A^6 \to [0, \infty)$ satisfy the following*:

$$\lim_{n\to\infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y, k^n u, k^n v, k^n w, k^n z) = 0, \tag{3.1}$$

$$\| f(kx + y) + f(kx + \sigma(y)) - 2kf(x) - 2f(y) \| \leq \delta, \tag{3.2}$$

and

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z) \| \\ &\leq \varphi(x, y, u, v, w, z) \end{aligned} \tag{3.3}$$

for all $x, y, u, v, w, z \in A$ with $\| x \| + \| y \| \geq d$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\| f(x) - D(x) \| \leq \frac{2\delta}{k} \frac{k+1}{k-1}, \ x \in A. \tag{3.4}$$

Proof. Let $x, y \in A$ such that $0 < \| x \| + \| y \| < d$. We choose $z = 2^n x$ if $x \neq 0$ or $z = 2^n y$ if $y \neq 0$. At first we have

$$\| \frac{z}{k} \| + \| kx + y \| \geq d, \| \frac{z}{k} \| + \| kx + \sigma(y) \| \geq d, \| x \| + \| z + \sigma(y) \| \geq d,$$

$$\| x \| + \| y + z \| \geq d, \| \frac{z}{k} \| + \| y \| \geq d, \| kx + \sigma(y) + \sigma(z) \| \geq d.$$

From inequality (3.1) we get

$$\begin{aligned} &2[f(kx + y) + f(kx + \sigma(y)) - 2kf(x) - 2f(y) \\ = &- [f(z + kx + y) + f(z + \sigma(kx) + \sigma(y)) - 2kf(\tfrac{z}{k}) - 2f(kx + y)] \\ &- [f(z + kx + \sigma(y)) + f(z + \sigma(kx) + y) - 2kf(\tfrac{z}{k}) - 2f(kx + \sigma(y))] \\ &+ [f(kx + z + \sigma(y)) + f(kx + \sigma(z) + \sigma(y)) - 2kf(x) - 2f(z + \sigma(y))] \\ &+ [f(kx + y + z) + f(kx + \sigma(y) + \sigma(z)) - 2kf(x) - 2f(y + z)] \\ &+ 2[f(z + y) + f(z + \sigma(y)) - 2kf(\tfrac{z}{k}) - 2f(y)] \\ &+ [f(z + \sigma(kx) + \sigma(y)) - f(kx + y + \sigma(z)) - 2kf(0)] \\ &+ [f(\sigma(kx) + z + y) - f(kx + \sigma(y) + \sigma(z)) - 2kf(0)]. \end{aligned}$$

So

$$\| f(kx+y)+f(kx+\sigma(y))-2kf(x)-2f(y) \| \leq 4\delta$$

for $x, y \in A$ with $x \neq 0$ and $y \neq 0$. Now, if $x = y = 0$, we use the following relation with an arbitrary $z \in A$ such that $\| z \| = kd$

$$\begin{aligned} &2[f(0)+f(0)-2kf(0)-2f(0)] \\ &= [f(z)+f(\sigma(z))-2kf(0)-2f(z)]+[f(z)-f(\sigma(z))-2kf(0)] \end{aligned}$$

to obtain

$$\| 2kf(0) \| \leq \delta.$$

Consequently, the inequality

$$\| f(kx+y)+f(kx+\sigma(y))-2kf(x)-2f(y) \| \leq 4\delta$$

holds for all $x, y \in A$. By letting $y = 0$ (resp. $x = y = 0$) in (3.2), we get

$$\| f(x) - \frac{1}{k}\{f(kx)-f(0)\} \| \leq \frac{\delta}{2k}, \ x \in A.$$

and

$$\| f(0) \| \leq \frac{\delta}{2k}, \ x \in A.$$

So, we deduce that

$$\| f(x) - \frac{1}{k}\{f(kx)\} \| \leq \frac{\delta}{2k} + \frac{\delta}{2k^2}, \ x \in A. \tag{3.5}$$

By applying the inductive assumption we prove

$$\| f(x) - \frac{1}{k^n}\{f(k^n x)\} \| \leq \frac{\delta}{2k}(1+\frac{1}{k})[1+\frac{1}{k}+\cdots+\frac{1}{k^{(n-1)}}] \tag{3.6}$$

for all $n \in \mathbb{N}$. From (3.5) it follows that (3.6) is true for $n = 1$. Assume now that (3.6) holds for $n \in \mathbb{N}$. The inductive step must be demonstrated to hold for $n+1$, that is

$$\begin{aligned} &\| f(x) - \frac{1}{k^{n+1}}\{f(k^{n+1}x)\} \| \\ &\leq \| f(x) - \frac{1}{k^n}\{f(k^n x)\} \| + \frac{1}{k^n} \| f(k^n x) - \frac{1}{k}\{f(k^{n+1}x)\} \| \\ &\leq \frac{\delta}{2k}(1+\frac{1}{k})[1+\frac{1}{k}+\cdots+\frac{1}{k^{(n-1)}}] + \frac{1}{k^n}\frac{\delta}{2k}(1+\frac{1}{k}) \\ &= \frac{\delta}{2k}(1+\frac{1}{k})[1+\frac{1}{k}+\cdots+\frac{1}{k^n}]. \end{aligned}$$

This proves the validity of the inequality (3.6). Let us define the sequence of functions

$$f_n(x) = \frac{1}{k^n}\{f(k^n x)\}, \ x \in A, \ n \in \mathbb{N}.$$

We will show that $\{f_n(x)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in A$. By using (3.5), we have

$$\| f_{n+1}(x) - f_n(x) \| = \| \frac{1}{k^{n+1}}\{f(k^{n+1}x)\} - \frac{1}{k^n}\{f(k^n x)\} \|$$

$$
\begin{aligned}
&= \frac{1}{k^n} \parallel \{f(k^n x)\} - \frac{1}{k}\{f(k^{n+1}x)\} \parallel \\
&\leq \frac{\delta}{2k}(1+\frac{1}{k})\frac{1}{k^n}.
\end{aligned}
$$

It follows that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in A$. However, A is a complete normed space, thus the limit function $D(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in A$. Assume now that there exist two mappings $D_i : A \to A$ $(i = 1, 2)$ satisfying (3.1) and (3.4). By mathematical induction, we can easily verify that

$$
D_i(k^n x) = k^n D_i(x), \ (i = 1, 2). \tag{3.7}
$$

For all $x \in A$ and all $n \in \mathbb{N}$,we have

$$
\begin{aligned}
\parallel D_1(x) - D_2(x) \parallel &= \frac{1}{k^n} \parallel D_1(k^n x) - D_2(k^n x) \parallel \\
&\leq \frac{1}{k^n} \parallel D_1(k^n x) - f(k^n x) \parallel + \frac{1}{k^n} \parallel D_2(k^n x) - f(k^n x) \parallel \\
&\leq \frac{\delta}{k^{n+1}} \frac{k+1}{k-1}.
\end{aligned}
$$

If we let $n \to +\infty$, we get $D_1(x) = D_2(x)$ for all $x \in A$. We show that $D : A \to A$ is an (α, β, γ)-derivation. By setting $x = y = u = v = 0$ and using (3.3) we have

$$
\parallel f(\mu w + z) - \mu f(w) - f(z) \parallel \leq \varphi(0, 0, 0, 0, w, z). \tag{3.8}
$$

Replacing w, z in (3.8) by $k^n w$, $k^n z$ respectively, and divide both sides by k^n we obtain

$$
D(\mu w + z) = \mu D(w) + D(z) \tag{3.9}
$$

for any $\mu \in T^1$ and all $w, z \in A$. Letting $\mu = 1$ in (3.9), we conclude that D is additive. Set $z = 0$, we have $D(\mu w) = \mu D(w)$. Thus, Lemma 2.1 implies that D is $\mathbb{C}$-linear. By using the inequality (3.3) we get

$$
\begin{aligned}
&\parallel \alpha f[x, y] - \beta[f(x), y] - \gamma[x, f(y)] \parallel \\
&= \parallel \alpha f(xy - yx) - \beta(f(x)y - yf(x)) - \gamma(xf(y) - f(y)x \parallel \\
&= \parallel \alpha f(xy) - \alpha f(yx) - \beta f(x)y + \beta y f(x) - \gamma x f(y) + \gamma f(y)x \parallel \\
&\leq \parallel \alpha f(xy) - \beta f(x)y - \gamma x f(y) \parallel + \parallel \alpha f(yx) - \beta y f(x) - \gamma f(y)x \parallel \\
&\leq \varphi(x, y, 0, 0, 0, 0) + \varphi(0, 0, x, y, 0, 0).
\end{aligned} \tag{3.10}
$$

Replacing x, y by $k^n x$, $k^n y$ respectively in (3.10), and divide both sides by k^{2n} and then try taking the limit as $n \to \infty$, we obtain

$$
\alpha D[x, y] = \beta[D(x), y] + \gamma[x, D(y)]
$$

for all $x, y \in A$. Hence D is a (α, β, γ)-derivations on A. □

Corollary 3.2. *Let $d > 0$, $q > 1$, $\eta > 0$ and $f : A \to A$ is a function such that*

$$\| f(kx+y) + f(kx+y) - 2kf(x) - 2f(y) \| \leq \delta,$$

and

$$\begin{aligned} &\| \alpha f(xy) - \beta f(x)y - \gamma x f(y) + \alpha f(uv) - \beta u f(v) - \gamma f(u)v \\ &\quad + f(\mu w + z) - \mu f(w) - f(z)|| \\ &\leq \eta ||x||^{\frac{q}{7}} ||y||^{\frac{q}{7}} ||u||^{\frac{q}{7}} ||v||^{\frac{q}{7}} ||w||^{\frac{q}{7}} ||z||^{\frac{q}{7}} ||t||^{\frac{q}{7}} \end{aligned}$$

for all $x, y, u, v, w, z \in A$ and any $\mu \in T^1$ with $\| x \| + \| y \| \geq d$. Then there exists a unique (α, β, γ)-derivation $D : A \to A$, such that

$$\| f(x) - D(x) \| \leq \frac{2\delta}{k} \frac{k+1}{k-1}, \ x \in A.$$

Proof. It is a desired result of Theorem 3.1 □

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