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REFINED ARITHMETIC-GEOMETRIC MEAN INEQUALITY AND NEW ENTROPY UPPER BOUND

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Abstract. In this paper, we establish a new refinement of the arithmetic-geometric mean inequality. Applying this result in information theory, we obtain a more precise upper bound for Shannon's entropy.

1. Introduction

For $n \geq 2$, let $p_1, \ldots, p_n$ be nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$. We denote by A_n and G_n the weighted arithmetic and geometric means of the positive real numbers $x_1, \ldots, x_n$, that is,

$$A_n = \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad G_n = \prod_{i=1}^{n} x_i^{p_i}.$$

It is well-known that

$$A_n \geq G_n$$

is called the arithmetic-geometric mean inequality.

The arithmetic-geometric mean inequality has found much interest among many mathematicians, and there are numerous new extensions, refinements, and applications of it. In 2003, Mercer [3] proved the following interesting refinement of arithmetic-geometric mean inequality,

$$c := \frac{1}{A_n} \sum_{i=1}^{n} \frac{p_i (x_i - A_n)^2}{x_i + \max(x_i, A_n)} \leq \log(A_n) - \log(G_n), \tag{1}$$

with equality occurring if and only if all x_i are equal.

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As is well known, some classic inequality such as AM-GM inequality [4], Jensen's inequality [6], Hölder inequality [8] play an important role in information sciences. Moreover, the Jensen's inequality is also an important cornerstone in information theory.

In 2009, Simic [6] obtained the following bound for the entropy ($H(X) := \sum_{i=1}^{n} p_i \log \frac{1}{p_i}$) by using refinement of Jensen's inequality,

$$0 \leq \mu \log\left(\frac{2\mu}{\mu+\nu}\right) + \nu \log\left(\frac{2\nu}{\mu+\nu}\right) \leq \log n - H(X),$$

where the probability distribution F is given by $P(X = i) = p_i$, $p_i > 0$, $1 \leq i \leq n$, with $\sum_{i=1}^{n} p_i = 1$ and where $\mu = \min_{1\leq i\leq n}(p_i)$ and $\nu = \max_{1\leq i\leq n}(p_i)$.

In 2012, Ţăpuş and Popescu [7] proved the following refinement of the Simic's result by using another refinement of Jensen's inequality,

(2)

$$H(X) \leq \log n - \max_{1\leq\mu_1<\mu_2<\cdots<\mu_{n-1}\leq n} \log\left[\left(\frac{n-1}{\sum_{k=1}^{n-1} p_{\mu_k}}\right)^{\sum_{k=1}^{n-1} p_{\mu_k}} \prod_{k=1}^{n-1} p_{\mu_k}^{p_{\mu_k}}\right].$$

For some related results, the reader is referred to papers [1, 2, 5] and references therein.

Recently, Parkash and Kakkar [4] obtained some inequalities, based on the arithmetic-geometric-harmonic mean inequality. They applied these inequalities to the entropy. Also the above bounds of the entropy become the particular cases of this result.

In this paper, we establish a new refinement of the inequality (1). Applying this result in information theory, we obtain a more precise upper bound for Shannon's entropy. In particular, our result refines the above bounds of the entropy.

2. Main results

In order to prove our main results, we need the following two lemmas.

Lemma 2.1. *Let* $f_a(x) := \frac{(x-a)^2}{a(x+\max\{x,a\})} + \log x$, $a > 0$. *Then* f_a *is a concave function on* $(0, +\infty)$.

Proof. In the case of $x \geq a$, $f_a(x) = \frac{(x-a)^2}{2ax} + \log x$. Direct computing yields

$$f_a''(x) = \frac{a-x}{x^3} \leq 0.$$

In the case of $0 < x < a$, $f_a(x) = \frac{(x-a)^2}{a(x+a)} + \log x$. Simple computations lead to

$$f_a''(x) = \frac{5ax^2 - x^3 - a^3 - 3a^2x}{x^2(x+a)^3} = \frac{(x-a)(a^2+4ax-x^2)}{x^2(x+a)^3} < 0.$$

Summing up, the function $f_a(x)$ is concave for $x > 0$. The proof is complete. □

Lemma 2.2. *Let f_a be as defined in Lemma 2.1 and let $k \in \{2, \dots, n-1\}$ and*

$$s_k := \max_{1 \le \mu_1 < \mu_2 < \cdots < \mu_k \le n} \left[\left(\sum_{i=1}^{k} p_{\mu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{k} p_{\mu_i}} \right) - \sum_{i=1}^{k} p_{\mu_i} f_{A_n}(x_{\mu_i}) \right].$$

Then we have

$$0 \le s_2 \le s_3 \le \cdots \le s_{n-1}.$$

Proof. It is clear that $s_2 \ge 0$. Now we will show that for any $k \in \{2, \dots, n-2\}$, $s_k \le s_{k+1}$. Let us consider that the maximum of the expression

$$\left(\sum_{i=1}^{k} p_{\mu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{k} p_{\mu_i}} \right) - \sum_{i=1}^{k} p_{\mu_i} f_{A_n}(x_{\mu_i})$$

is obtained for $\mu_i = \nu_i,\ \nu_i \in \{1, \dots, n\},\ i = \{1, \dots, k\}$. Then it is enough to prove that

$$\begin{aligned} &\left(\sum_{i=1}^{k} p_{\nu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k} p_{\nu_i}} \right) - \sum_{i=1}^{k} p_{\nu_i} f_{A_n}(x_{\nu_i}) \\ \le\ &\left(\sum_{i=1}^{k+1} p_{\nu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k+1} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k+1} p_{\nu_i}} \right) - \sum_{i=1}^{k+1} p_{\nu_i} f_{A_n}(x_{\nu_i}) \end{aligned}$$

for any $\nu_{k+1} \in \{1, \dots, n\} \setminus \{\nu_1, \dots, \nu_k\}$. The above inequality is equivalent to

$$\begin{aligned} &p_{\nu_{k+1}} f_{A_n}(x_{\nu_{k+1}}) + \left(\sum_{i=1}^{k} p_{\nu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k} p_{\nu_i}} \right) \\ \le\ &\left(\sum_{i=1}^{k+1} p_{\nu_i} \right) f_{A_n} \left(\frac{\sum_{i=1}^{k+1} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k+1} p_{\nu_i}} \right). \end{aligned}$$

Multiplying by $\left(\sum_{i=1}^{k+1} p_{\nu_i} \right)^{-1}$, we obtain the inequality

$$\begin{aligned} &\frac{p_{\nu_{k+1}}}{\sum_{i=1}^{k+1} p_{\nu_i}} f_{A_n}(x_{\nu_{k+1}}) + \frac{\sum_{i=1}^{k} p_{\nu_i}}{\sum_{i=1}^{k+1} p_{\nu_i}} f_{A_n} \left(\frac{\sum_{i=1}^{k} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k} p_{\nu_i}} \right) \\ \le\ &f_{A_n} \left(\frac{\sum_{i=1}^{k+1} p_{\nu_i} x_{\nu_i}}{\sum_{i=1}^{k+1} p_{\nu_i}} \right), \end{aligned}$$

which follows from Jensen's inequality for the concave function $f_{A_n}(x)$. The lemma is proved. □

Theorem 2.3. *Let c, A_n, G_n be as defined the above, the following estimates hold*

(3) $$c \le c + s_2 \le c + s_3 \le \cdots \le c + s_{n-1} \le \log(A_n) - \log(G_n),$$

with equality occurring if and only if all x_i's are equal.

Proof. By Lemma 2.2, we have

$$c \leq c + s_2 \leq c + s_3 \leq \cdots \leq c + s_{n-1}.$$

We proceed now to prove the last inequality of (3). Choose arbitrary $x_{\mu_i} \in \{x_1, \ldots, x_n\}$, $1 \leq \mu_1 < \mu_2 < \cdots < \mu_{n-1} \leq n$, with corresponding weights $p_{\mu_i} \in \{p_1, \ldots, p_n\}$, and let $x_{\mu_n} = \{x_1, \ldots, x_n\} \setminus \{x_{\mu_1}, \ldots, x_{\mu_{n-1}}\}$. By the inequality (1), we get

$$\begin{aligned}
\log(A_n) &= \log\left(\sum_{i=1}^{n} p_i x_i\right) = \log\left(p_{\mu_n} x_{\mu_n} + \left(\sum_{i=1}^{n-1} p_{\mu_i}\right) \frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}}\right) \\
&\geq \frac{1}{A_n} \frac{p_{\mu_n}(x_{\mu_n} - A_n)^2}{x_{\mu_n} + \max(x_{\mu_n}, A_n)} + \frac{1}{A_n} \frac{\left(\sum_{i=1}^{n-1} p_{\mu_i}\right)\left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}} - A_n\right)^2}{\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}} + \max\left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}}, A_n\right)} \\
&\quad + \log\left(x_{\mu_n}^{p_{\mu_n}} \left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}}\right)^{\sum_{i=1}^{n-1} p_{\mu_i}}\right) \\
&= \frac{1}{A_n} \sum_{i=1}^{n} \frac{p_i(x_i - A_n)^2}{x_i + \max(x_i, A_n)} - \frac{1}{A_n} \sum_{i=1}^{n-1} \frac{p_{\mu_i}(x_{\mu_i} - A_n)^2}{x_{\mu_i} + \max(x_{\mu_i}, A_n)} \\
&\quad + \log(G_n) - \sum_{i=1}^{n-1} p_{\mu_i} \log x_{\mu_i} + \left(\sum_{i=1}^{n-1} p_{\mu_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}}\right) \\
&= \log(G_n) + c + \left(\sum_{i=1}^{n-1} p_{\mu_i}\right) f_{A_n}\left(\frac{\sum_{i=1}^{n-1} p_{\mu_i} x_{\mu_i}}{\sum_{i=1}^{n-1} p_{\mu_i}}\right) - \sum_{i=1}^{n-1} p_{\mu_i} f_{A_n}(x_{\mu_i}).
\end{aligned}$$

Since x_{μ_i}, $i = \{1, \ldots, k\}$ are arbitrary, the last inequality of (3) follows. The theorem is proved. □

By using Theorem 2.3, we get the following proposition.

Proposition 2.4. *We have*

(4)

$$H(X) \leq \log n - \frac{1}{n} \sum_{k=1}^{n} \frac{(1 - np_k)^2}{1 + \max(1, np_k)} - \max_{1 \leq \mu_1 < \mu_2 < \cdots < \mu_{n-1} \leq n} \{L(\mu) + M(\mu)\},$$

where

$$L(\mu) := \log\left[\left(\frac{n-1}{\sum_{k=1}^{n-1} p_{\mu_k}}\right)^{\sum_{k=1}^{n-1} p_{\mu_k}} \prod_{k=1}^{n-1} p_{\mu_k}^{p_{\mu_k}}\right],$$

and

$$M(\mu) := \frac{(n - 1 - n\sum_{k=1}^{n-1} p_{\mu_k})^2}{n(n - 1 + \max(n - 1, n\sum_{k=1}^{n-1} p_{\mu_k}))} - \frac{1}{n} \sum_{k=1}^{n-1} \frac{(1 - np_{\mu_k})^2}{1 + \max(1, np_{\mu_k})}.$$

Proof. Applying the last inequality of (3) with $x_i = 1/p_i$, $1 \leq i \leq n$, after some calculations the desired result follows. □

Remark 2.5. It is easy to see that $g(x) := \frac{(x-a)^2}{a(x+\max\{x,a\})}$, $a > 0$ is convex for $x > 0$. Hence, by Jensen's inequality, $M(\mu) \leq 0$.

The next proposition demonstrates that the estimation is better than (2).

Proposition 2.6. *The estimation* (4) *is better than* (2), *i.e.,*

$$\max_{1\leq\mu_1<\mu_2<\cdots<\mu_{n-1}n}\{L(\mu)\}$$
$$\leq \frac{1}{n}\sum_{k=1}^{n}\frac{(1-np_k)^2}{1+\max(1,np_k)} + \max_{1\leq\mu_1<\mu_2<\cdots<\mu_{n-1}\leq n}\{L(\mu)+M(\mu)\}.$$

Proof. Let us consider that the maximum of $L(\mu)$ is obtained for $\mu_i = \nu_i$, $\nu_i \in \{1,\ldots,n\}$, $i = \{1,\ldots,n-1\}$ and let $p_{\nu_n} = \{p_1,\ldots,p_n\} \setminus \{p_{\nu_1},\ldots,p_{\nu_{n-1}}\}$. Then we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\frac{(1-np_k)^2}{1+\max(1,np_k)} + \max_{1\leq\mu_1<\mu_2<\cdots<\mu_{n-1}\leq n}\{L(\mu)+M(\mu)\}\\
&\quad - \max_{1\leq\mu_1<\mu_2<\cdots<\mu_{n-1}\leq n}\{L(\mu)\}\\
&\geq \frac{1}{n}\sum_{k=1}^{n}\frac{(1-np_k)^2}{1+\max(1,np_k)} + M(\nu)\\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{(1-np_k)^2}{1+\max(1,np_k)} + \frac{(n-1-n\sum_{k=1}^{n-1}p_{\nu_k})^2}{n(n-1+\max(n-1,n\sum_{k=1}^{n-1}p_{\nu_k}))}\\
&\quad - \frac{1}{n}\sum_{k=1}^{n-1}\frac{(1-np_{\nu_k})^2}{1+\max(1,np_{\nu_k})}\\
&= \frac{(n-1-n\sum_{k=1}^{n-1}p_{\nu_k})^2}{n(n-1+\max(n-1,n\sum_{k=1}^{n-1}p_{\nu_k}))} + \frac{(1-np_{\nu_n})^2}{n(1+\max(1,np_{\nu_n}))} \geq 0,
\end{aligned}$$

this completes the proof. □

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