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A STUDY OF Q-CONTIGUOUS FUNCTION RELATIONS

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Abstract. In 1812, Gauss obtained fifteen contiguous functions relations. Later on, 1847, Henie gave their q-analogue. Recently, good progress has been done in finding more contiguous functions relations by employing results due to Gauss. In 1999, Cho et al. have obtained 24 new and interesting contiguous functions relations with the help of Gauss's 15 contiguous relations. In fact, such type of 72 relations exists and therefore the rest 48 contiguous functions relations have very recently been obtained by Rakha et al..

Thus, the paper is in continuation of the paper [16] published in Computer & Mathematics with Applications **61** (2011), 620–629. In this paper, first we obtained 15 q-contiguous functions relations due to Henie by following a different method and then with the help of these 15 q-contiguous functions relations, we obtain 72 new and interesting q-contiguous functions relations. These q-contiguous functions relations have wide applications.

1. Introduction

The study of basic hypergeometric series (*q-series or q-hypergeometric series*) essentially started in (1748) when Euler [5], considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty}(1-q^{k+1})^{-1}$ as a generating function of $p(n)$, (*the number of partitions of a positive integer n into positive integers*).

One hundred years later, the basic hypergeometric series acquired an independent status when Heine [5], converted a simple observation that

$$\lim_{q\to 1}\frac{(1-q^a)}{(1-q)} = a \tag{1.1}$$

into a systematic theory of basic hypergeometric series parallel to the theory of Gauss hypergeometric series.

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The basic hypergeometric series have many significant applications in several areas of pure and applied mathematics including the theory of partitions, combinatorial identities, number theory, finite vector spaces, Lie theory, mathematical physics and statistics. q-integrals are the most common applications between q-series, Lie algebras and their root systems.

In 1812, Gauss [6], presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss (1813)) in which he considered the infinite series

$$(1.2)\quad 1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots,$$

as a function of a, b, c, z, where it is assumed that $c \neq 0, -1, -2, \ldots$, so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $\Re(c-a-b) > 0$, gave its (contiguous) recurrence relations, and derived his famous formula

$$(1.3)\qquad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \Re(c-a-b) > 0,$$

for the sum of his series when $z = 1$ and $\Re(c-a-b) > 0$.

Although Gauss used the notation $F(a, b, c, z)$ for his series, it is now customary to use $F(a, b; c; z)$ or either of the notations

$$(1.4)\qquad {}_2F_1(a, b; c; z) \text{ or } {}_2F_1\left[\begin{array}{c} a, b \\ c \end{array}; z\right],$$

for his series (and for its sum when it converges), because these notations separate the numerator parameters a, b from the denominator parameter c and the variable z.

Two hypergeometric functions with the same argument z are said to be *contiguous* if their parameters a, b and c differ by integers. Gauss derived analogous relations between ${}_2F_1[a, b; c; z]$ and any two contiguous hypergeometrics in which a parameter has been changed by ± 1. Rainville [14] generalized this to cases with more parameters.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series, they can be used to evaluate a hypergeometric function that is contiguous to a hypergeometric series which can be satisfactorily evaluated. Contiguous relations are also used to make a correspondence between Lie algebras and special functions. The correspondence yields formulas of special functions [13].

Gauss [6] defined as contiguous to ${}_2F_1(a, b; c; z)$ or simply $F(a, b; c; z)$ each of the six functions obtained by increasing or decreasing one of the parameters by unity. He also proved that between F and any two of its contiguous functions, there exists a linear relation with coefficients at most linear and obtained his fifteen interesting and useful results [8].

Thirty three years after Gauss's paper, Heine [7], introduced the series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\cdots, \tag{1.5}$$

where it is assumed that $c \neq 0, -1, -2, \ldots$. This series converges absolutely for $|z| < 1$ when $|q| < 1$ and it tends (at least term-wise) to Gauss' series as $q \to 1$, because of (1.1).

The series (1.5) is usually called Heine's series or, in view of the base q, the basic hypergeometric series or q-hypergeometric series.

Analogous to Gauss notation, Heine [5], used the notation $\Phi(a, b, c, q, z)$ for his series. It is now customary to define the basic hypergeometric series by

$$\Phi(a,b;c;q,z) \quad \text{or} \quad {}_2\Phi_1(a,b;c;q,z)$$

$$\text{or} \quad {}_2\Phi_1\left[\begin{matrix} a,b \\ c \end{matrix} ;q,z\right]=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n, \tag{1.6}$$

where

$$(a;q)_n=\begin{cases} 1, & n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\ldots, \end{cases} \tag{1.7}$$

is the q-shifted factorial and it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \ldots$.

Some other notations that have been used for the product $(a; q)_n$ are $(a)_{q,n}$, $[a]_n$ and even $(a)_n$ when (1.7) is not used and the base is not displayed.

Generalizing Heine's series, we shall define an ${}_r\Phi_s$ basic hypergeometric series by

$$\begin{aligned} &{}_r\Phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z) \\ &\equiv {}_r\Phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix} ;q,z\right] \\ &=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s + 1$. For more detail, see [5].

We also define

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k) \tag{1.8}$$

for $|q| < 1$.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \tag{1.9}$$

$$(a_1,a_2,\ldots,a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty, \tag{1.10}$$

$$(1-a)(aq)_n=(a)_{n+1}=(a)_n(1-aq^n), \tag{1.11}$$

(1.12) $$\left(\frac{a}{q}\right)_n = \left(1-\frac{a}{q}\right)(a)_{n-1} = \frac{(q-a)}{q}\cdot\frac{(a)_n}{(1-aq^{n-1})}.$$

Recently, Wei et al. [17] re-established fifteen interesting three-term relations for the ${}_2\phi_1$ series by the method of comparing coefficients. In such relations, their limiting cases recover Gauss' fifteen contiguous relations for ${}_2F_1$ series.

The paper is organized as follows. In Section 2, fifteen q-contiguous function relations due to Henie [7] will be derived by another method. In Section 3, we establish 24 q-contiguous function relations and in Section 4 as an application of our results, we obtained further 48 q-contiguous relations. In Section 5, special cases of our results (2.13)–(2.27), (3.28)–(3.51) and (4.52)–(4.99) are given. Finally, as an application we obtain 18 q-summation formulas in closed forms which will be given in Section 6. The results derived in this paper are simple, interesting, easily established and may be useful.

2. Heine q-contiguous functions relations

In this section, the following 15 q-contiguous functions relations due to Henie [7] will be derived by another method. These are

(2.13) $$(b-a)\phi = b(1-a)\phi(aq) - a(1-b)\phi(bq)$$

(2.14) $$(c-aq)\phi = c(1-a)\phi(aq) - a(q-c)\phi\left(\tfrac{c}{q}\right)$$

(2.15) $$\{c^2(1-a)+a^2z(c-b)\}\phi = c(1-a)(c-abz)\phi(aq) - \tfrac{a(c-a)(c-b)}{(1-c)}z\phi(cq)$$

(2.16) $$(1-c)(cq-abz)\phi = cq(1-c)\phi\left(\tfrac{a}{q}\right) + a(c-b)z\phi(cq)$$

(2.17) $$(1-c)(cq-abz)\phi = cq(1-c)\phi\left(\tfrac{b}{q}\right) + b(c-a)z\phi(cq)$$

(2.18) $$\{c(1-a)+q(c-a)+az(a-b)\}\phi = q(c-a)\phi\left(\tfrac{a}{q}\right) + (1-a)(c-abz)\phi(aq)$$

(2.19) $$\{bc(1-a)+aq(c-b)\}\phi = aq(c-b)\phi\left(\tfrac{b}{q}\right) + b(1-a)(c-abz)\phi(aq)$$

(2.20) $$\{ac(1-b)+bq(c-a)\}\phi = bq(c-b)\phi\left(\tfrac{a}{q}\right) + a(1-b)(c-abz)\phi(bq)$$

(2.21) $$(a-b)(cq-abz)\phi = aq(c-b)\phi\left(\tfrac{b}{q}\right) - bq(c-a)\phi\left(\tfrac{a}{q}\right)$$

(2.22) $$\{c^2(q-a)+a^2z(c-bq)\}\phi = qc(c-a)\phi\left(\tfrac{a}{q}\right) + a(q-c)(c-abz)\phi\left(\tfrac{c}{q}\right)$$

$$\{c(1-b)+q(c-b)+bz(b-a)\}\phi = q(c-b)\phi\left(\tfrac{b}{q}\right) + (1-b)(c-abz)\phi(bq) \tag{2.23}$$

$$\{c^2(1-b)+b^2z(c-a)\}\phi = c(1-b)(c-abz)\phi(bq) - \tfrac{b(c-b)(c-a)}{(1-c)}z\phi(cq) \tag{2.24}$$

$$(c-bq)\phi = c(1-b)\phi(bq) - b(q-c)\phi\left(\tfrac{c}{q}\right) \tag{2.25}$$

$$\{c^2(q-b)+b^2z(c-aq)\}\phi = qc(c-b)\phi\left(\tfrac{b}{q}\right) + b(q-c)(c-abz)\phi\left(\tfrac{c}{q}\right) \tag{2.26}$$

$$\{c(q-c)+(ac+bc-ab-abq)z\}\phi = (q-c)(c-abc)\phi\left(\tfrac{c}{q}\right) - \tfrac{(c-b)(c-a)}{(1-c)}z\phi(cq) \tag{2.27}$$

2.1. Derivations of (2.13)–(2.27)

The derivations of the 15 q-contiguous functions relations due to Heine (2.13)–(2.27) are straightforward, by expressing ϕ on the right-hand-side of these relations as a series and then simplifying using the identities (1.9) - (1.12). So we will derive only three of these relations, and the rest can be derived on similar lines.

Derivation of (2.15): In order to derive (2.15), it is sufficient to show that

$$c^2(1-a)\phi = c(1-a)(c-abz)\phi(aq) - \frac{a(c-a)(c-b)}{(1-c)}z\phi(cq) - a^2z(c-b)\phi.$$

Now, we start with the right-hand side of the above equation

$$= c(1-a)(c-abz)\phi(aq) - \frac{a(c-a)(c-b)}{(1-c)}z\phi(cq) - a^2z(c-b)\phi.$$

Expressing ϕ as a series, we have

$$= c^2(1-a)\sum_{n=0}^{\infty}\frac{(aq)_n(b)_n}{(q)_n(c)_n}z^n - az\left\{bc(1-a)\sum_{n=0}^{\infty}\frac{(aq)_n(b)_n}{(q)_n\,(c)_n}z^n + \frac{(c-a)(c-b)}{(1-c)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n\,(cq)_n}z^n + a(c-b)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n\,(c)_n}z^n\right\}.$$

Using the identities (1.9)–(1.12), we have

$$= c^2\sum_{n=0}^{\infty}\frac{(a)_n(1-aq^n)(b)_n}{(q)_n(c)_n}z^n - az\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n\,(c)_n}z^n\left\{bc(1-aq^n) + \frac{(c-a)(c-b)}{(1-cq^n)} + a(c-b)\right\}.$$

After simplification, we get

$$= c^2 \sum_{n=0}^{\infty} \frac{(a)_n (1-aq^n)(b)_n}{(q)_n (c)_n} z^n - az \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n\, (c)_n} z^n \frac{c^2(1-aq^n)(1-bq^n)}{(1-cq^n)}$$

$$= c^2 \left[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n \left\{ (1-aq^n) - a(1-q^n) \right\} \right]$$

$$= (1-a) c^2 \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n.$$

Finally summing up the series, we get

$$= c^2 (1-a) \phi$$

which is the left-hand side. □

Derivation of (2.21): In order to derive (2.21), it is sufficient to show that

$$cq(a-b)\phi = aq(c-b)\phi\left(\frac{b}{q}\right) - bq(c-a)\phi\left(\frac{a}{q}\right) + abz(a-b)\phi.$$

Now, we start with the right-hand side of the above equation

$$= aq(c-b)\phi\left(\frac{b}{q}\right) - bq(c-a)\phi\left(\frac{a}{q}\right) + abz(a-b)\phi.$$

Expressing ϕ as a series, we have

$$= aq(c-b) \sum_{n=0}^{\infty} \frac{(a)_n \left(\frac{b}{q}\right)_n}{(q)_n (c)_n} z^n - bq(c-a) \sum_{n=0}^{\infty} \frac{\left(\frac{a}{q}\right)_n (b)_n}{(q)_n (c)_n} z^n$$

$$+ abz(a-b) \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(q)_n (c)_n} z^n.$$

Proceeding as before, we have

$$= a(c-b)(q-b) \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(q)_n (c)_n (1-bq^{n-1})} z^n$$

$$- b(c-a)(q-a) \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(q)_n (c)_n (1-aq^{n-1})} z^n$$

$$+ ab(a-b) \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(q)_n (c)_n} z^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} z^n \left\{ \frac{a(c-b)(q-b)}{(1-bq^{n-1})} - \frac{b(c-a)(q-a)}{(1-aq^{n-1})} + \frac{ab(a-b)(1-q^n)(1-cq^{n-1})}{(1-aq^{n-1})(1-bq^{n-1})} \right\}.$$

After little simplification, we get

$$= (a-b) \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(q)_n (c)_n} z^n \left\{ \frac{qc - cq^n(a+b) + abcq^{2n-1}}{(1-aq^{n-1})(1-bq^{n-1})} \right\}$$

$$= qc(a-b)\phi$$

which is the left-hand side. □

Derivation of (2.27):

In order to derive (2.27), it is sufficient to show that

$$c(q-c)\phi = (q-c)(c-abc)\phi\left(\tfrac{c}{q}\right) - \frac{(c-b)(c-a)}{(1-c)} z\phi(cq) - (ac+bc-ab-abq)z\phi.$$

Now, we start with the right-hand side of the above equation

$$= (q-c)(c-abc)\phi$$
$$- z\left[ab(q-c)\phi\left(\tfrac{c}{q}\right) + \frac{(c-b)(c-a)}{(1-c)}\phi(cq) + (ac+bc-abq)\phi\right].$$

Expressing ϕ as a series, we have

$$= c(q-c)\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n\left(\frac{c}{q}\right)_n}z^n$$
$$- z\left[ab(q-c)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n\left(\frac{c}{q}\right)_n}z^n + \frac{(c-b)(c-a)}{(1-c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n(cq)_n}z^n\right.$$
$$\left. +(ac+bc-ab-abq)\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n(c)_n}z^n\right].$$

Proceeding as before, we get

$$= cq\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(1-cq^{n-1})}{(q)_n\,(c)_n}z^n$$
$$-\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n(c)_n}z^{n+1}\left[abq(1-cq^{n-1}) + \frac{(c-b)(c-a)}{(1-cq^{n-1})}\right.$$
$$\left. +(ac+bc-ab-abq)\right].$$

After little simplification, we get

$$= cq\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(1-cq^{n-1})}{(q)_n\,(c)_n}z^n - c^2\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n(c)_n}z^{n+1}\frac{(1-aq^n)(1-bq^n)}{(1-cq^n)}$$
$$= c(q-c)\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(q)_n\,(c)_n}z^n$$

Finally, summing up the series, we get

$$= c(q-c)\phi$$

which is the left-hand side. □

Remarks. For a note on these contiguous functions relations, see a paper by Kim et al. [11].

3. New results

In this section, the following 24 results will be established with the help of the results given in Section 2. These are

$$b(q-a)\phi = (bq-a)\phi\left(\tfrac{a}{q}\right) + a(1-b)\phi\left(\tfrac{a}{q}, bq\right) \tag{3.28}$$

$$a(q-b)\phi = (aq-b)\phi\left(\tfrac{b}{q}\right) + b(1-a)\phi\left(aq, \tfrac{b}{q}\right) \tag{3.29}$$

$$c(q-a)\phi = q(c-a)\phi\left(\tfrac{a}{q}\right) + a(q-c)\phi\left(\tfrac{a}{q}, \tfrac{c}{q}\right) \tag{3.30}$$

$$a(1-c)\phi = (a-c)\phi(cq) + c(1-a)\phi(aq, cq) \tag{3.31}$$

$$\begin{aligned} c(q-a)(cq-abz)\phi = {} & \left\{qc^2(q-a) + a^2z(c-b)\right\}\phi\left(\tfrac{a}{q}\right) \\ & + \frac{a(c-b)(cq-a)}{(1-c)}z\phi\left(\tfrac{a}{q}, cq\right) \end{aligned} \tag{3.32}$$

$$\begin{aligned} \frac{aq(c-bq)(c-aq)}{(q-c)}z\phi = {} & \left\{(bq-c)a^2qz - c^2(1-a)\right\}\phi\left(\tfrac{c}{q}\right) \\ & + c(1-a)(c-abqz)\phi\left(aq, \tfrac{c}{q}\right) \end{aligned} \tag{3.33}$$

$$c(1-c)\phi = (1-c)(c-abz)\phi(aq) + a(b-c)z\phi(aq, cq) \tag{3.34}$$

$$a(bq-c)z\phi = (q-c)(abz-c)\phi\left(\tfrac{c}{q}\right) + c(q-c)\phi\left(\tfrac{a}{q}, \tfrac{c}{q}\right) \tag{3.35}$$

$$c(1-c)\phi = (c-abz)\phi(bq) + b(a-c)z\phi(bq, cq) \tag{3.36}$$

$$b(aq-c)z\phi = (q-c)(abz-c)\phi\left(\tfrac{c}{q}\right) + c(q-c)\phi\left(\tfrac{b}{q}, cq\right) \tag{3.37}$$

$$\begin{aligned} b(q-a)(cq-abz)\phi = {} & \left\{aq^2(c-b) + bq(q-a)\right\}\phi\left(\tfrac{a}{q}\right) \\ & + aq^2(b-c)z\phi\left(\tfrac{a}{q}, \tfrac{b}{q}\right) \end{aligned} \tag{3.38}$$

$$\begin{aligned} a(c-bq)\phi = {} & \{a(c-bq) + bc(1-a)\}\phi(bq) \\ & + b(1-a)(abz-c)\phi(aq, bq) \end{aligned} \tag{3.39}$$

$$\begin{aligned} b(c-aq)\phi = {} & \{a(c-bq) + ac(1-b)\}\phi(aq) \\ & + a(1-a)(abqz-c)\phi(aq, bq) \end{aligned} \tag{3.40}$$

$$\begin{aligned} a(q-b)(cq-abz)\phi = {} & \left\{bq^2(c-q) + acq(q-b)\right\}\phi\left(\tfrac{b}{q}\right) \\ & + bq^2(a-c)z\phi\left(\tfrac{a}{q}, \tfrac{b}{q}\right) \end{aligned} \tag{3.41}$$

$$b(c-aq)\phi = (b-aq)(c-abz)\phi\,(aq) + aq(c-b)\phi\left(aq, \tfrac{b}{q}\right) \tag{3.42}$$

$$a(c-bq)\phi = (a-bq)(c-abz)\phi\,(bq) + bq(c-a)\phi\left(\tfrac{a}{q}, bq\right) \tag{3.43}$$

$$\begin{aligned} c(ca-q)\phi = &\ \left\{c^2(1-a) + a^2qz(c-bq)\right\}\phi(aq) \\ &+ a(c-q)(c-abqz)\phi\left(aq, \tfrac{c}{q}\right) \end{aligned} \tag{3.44}$$

$$\begin{aligned} a(1-c)(cq-abz)\phi = &\ \left\{c^2q(q-a) + a^2z(c-b)\right\}\phi(cq) \\ &+ cq(a-cq)\phi\left(\tfrac{a}{q}, cq\right) \end{aligned} \tag{3.45}$$

$$\begin{aligned} c(q-b)(cq-abz)\phi = &\ \left\{c^2q(q-b) + b^2z(c-a)\right\}\phi\left(\tfrac{b}{q}\right) \\ &+ \frac{b(c-a)(cq-b)}{(1-c)}z\phi\left(\tfrac{b}{q}, cq\right) \end{aligned} \tag{3.46}$$

$$\begin{aligned} \frac{bq(c-aq)(c-bq)}{(q-c)}z\phi = &\ \left\{b^2qz(aq-c) - c^2(1-b)\right\}\phi\left(\tfrac{c}{q}\right) \\ &+ c(1-b)(c-abzq)\phi\left(bq, \tfrac{c}{q}\right) \end{aligned} \tag{3.47}$$

$$c(q-b)\phi = q(c-b)\phi\left(\tfrac{b}{q}\right) + b(q-c)\phi\left(bq, \tfrac{c}{q}\right) \tag{3.48}$$

$$b(1-c)\phi = (b-c)\phi\,(cq) + c(1-b)\phi\,(bq, cq) \tag{3.49}$$

$$\begin{aligned} c(c-bq)\phi = &\ \left\{c^2(1-b) + b^2qz(c-aq)\right\}\phi(bq) \\ &+ b(c-q)(c-abqz)\phi\left(bq, \tfrac{c}{q}\right) \end{aligned} \tag{3.50}$$

$$\begin{aligned} b(1-c)(cq-abz)\phi = &\ \left\{c^2q(q-b) + b^2z(c-a)\right\}\phi(cq) \\ &+ cq(b-cq)\phi\left(\tfrac{b}{q}, cq\right) \end{aligned} \tag{3.51}$$

3.1. Derivations of results (3.28) to (3.51)

The derivations of our new q-contiguous functions relations (3.28)–(3.51) are quite straightforward by algebraic manipulations. For example, if wish to derive the result (3.28), then in equation (2.13), replace a by $\frac{a}{q}$ and then multiply by q, we get

$$q\left(b-\frac{a}{q}\right)\phi\left(\frac{a}{q}\right) = bq\left(1-\frac{a}{q}\right)\phi - a(1-b)\phi\left(\frac{a}{q}, bq\right)$$

after rearrangement of the terms, we easily get (3.28). In a similar manner, other results can be easily obtained. The scheme is outlined in Table-1 including that of (3.28).

TABLE 1. Derivations of (3.28)–(3.51)

Derivation of	**In equation**	**Action**
(3.1)	(2.1)	Replace a by a/q and multiply by q
(3.2)	(2.1)	Replace b by b/q and multiply by q
(3.3)	(2.2)	Replace a by a/q and multiply by q
(3.4)	(2.2)	Replace c by cq and divide by q
(3.5)	(2.3)	Replace a by a/q and multiply by q^2
(3.6)	(2.3)	Replace c by c/q and multiply by q^2
(3.7)	(2.4)	Replace a by aq and divide by q
(3.8)	(2.4)	Replace c by c/q and multiply by q
(3.9)	(2.5)	Replace b by bq and multiply by q
(3.10)	(2.5)	Replace c by c/q and multiply by q
(3.11)	(2.7)	Replace a by a/q and multiply by q^2
(3.12)	(2.7)	Replace b by bq and divide by q
(3.13)	(2.8)	Replace a by aq and divide by q
(3.14)	(2.8)	Replace b by b/q and multiply by q^2
(3.15)	(2.9)	Replace a by aq and divide by q
(3.16)	(2.9)	Replace b by bq and divide by q
(3.17)	(2.10)	Replace a by aq and divide by q
(3.18)	(2.10)	Replace c by c/q and multiply by q
(3.19)	(2.12)	Replace b by b/q and divide by q
(3.20)	(2.12)	Replace c by c/q and divide by q^2
(3.21)	(2.13)	Replace b by b/q and divide by q
(3.22)	(2.13)	Replace c by cq and divide by q
(3.23)	(2.14)	Replace b by bq and divide by q
(3.24)	(2.14)	Replace c by cq and divide by q

4. Applications

In this section we shall obtain 48 more contiguous functions relations with the help of the results obtained in Sections 2 and 3. These are

$$\begin{aligned}(4.52)\qquad &(1-c)\left\{c(1-b)+bz(b-a)\right\}\phi \\ &= (1-b)(1-c)(c-abz)\phi(bq)+b(1-a)(b-c)z\phi(aq,cq)\end{aligned}$$

$$\begin{aligned}(4.53)\qquad &(1-c)\left\{c(1-a)+az(a-b)\right\}\phi \\ &= (1-a)(1-c)(c-abz)\phi(aq)+a(1-b)(a-c)z\phi(bq,cq)\end{aligned}$$

$$\begin{aligned}(4.54)\qquad &\left\{c(1-b)(b-q)-q(1-a)(b^2-c)+bz(b-a)(b-aq)\right\}\phi \\ &= (1-b)(b-aq)(c-abz)\phi(bq)+bq(1-a)(c-b)\phi\left(aq,\tfrac{b}{q}\right)\end{aligned}$$

$$\begin{aligned}\{c(1-a)(a-q)-q(1-b)(a^2-c)+az(a-b)(a-bq)\}\,\phi \\ =(1-a)(a-bq)(c-abz)\phi(aq)+aq(1-b)(c-a)\phi\left(\tfrac{a}{q},bq\right)\end{aligned} \tag{4.55}$$

$$\begin{aligned}&(c-bq)\left\{c^2(1-a)-aqz(b-a)\right\}\phi \\ &=(1-b)\left\{c^2(1-a)+a^2qz(c-bq)\right\}\phi(bq) \\ &\quad+b(1-a)(c-a)(c-abqz)\phi\left(aq,\tfrac{c}{q}\right)\end{aligned} \tag{4.56}$$

$$\begin{aligned}&(c-aq)\left\{c^2(1-b)-bqz(a-b)\right\}\phi \\ &=(1-a)\left\{c^2(1-b)+b^2qz(c-aq)\right\}\phi(aq) \\ &\quad+a(1-b)(c-q)(c-abqz)\phi\left(bq,\tfrac{c}{q}\right)\end{aligned} \tag{4.57}$$

$$\begin{aligned}(1-c)\left\{c^2(1-a)-(c-aq)(c-abz)\right\}\phi \\ =a(q-c)(1-c)(c-abz)\phi\left(\tfrac{c}{q}\right)+ac(b-c)(1-a)\phi(aq,cq)\end{aligned} \tag{4.58}$$

$$\{aq-c+az(b-c)\}\,\phi=(1-a)(abz-c)\phi(aq)+ac(q-c)\phi\left(\tfrac{a}{q},\tfrac{c}{q}\right) \tag{4.59}$$

$$(aq-c)\phi=(1-a)(abz-c)\phi(aq)+a(q-c)\phi\left(\tfrac{b}{q},\tfrac{c}{q}\right) \tag{4.60}$$

$$\begin{aligned}(c-aq)(bq-c)\phi=(q-c)\left\{b(c-aq)+ac(1-b)\right\}\phi\left(\tfrac{c}{q}\right) \\ +c(1-a)(1-b)(abqz-c)\phi\left(aq,bq\right)\end{aligned} \tag{4.61}$$

$$\begin{aligned}(c-aq)\left\{c(q-b)+bz(b-aq)\right\}\phi \\ =(q-c)(b-aq)(c-abz)\phi\left(\tfrac{c}{q}\right)+cq(1-a)(c-b)\phi\left(aq,\tfrac{b}{q}\right)\end{aligned} \tag{4.62}$$

$$\begin{aligned}&\left[c^3(1-a)(b-1)+bz(c-aq)\left\{c-b+bc(1-a)\right\}\right]\phi \\ &=(1-c)^{-1}(c-a)(c-b)\left\{b(c-aq)+ac(1-b)\right\}z\phi(cq) \\ &\quad+c(1-a)(1-b)(c-abz)(abqz-c)\phi(aq,bq)\end{aligned} \tag{4.63}$$

$$\begin{aligned}&\left\{qc(1-a)-az(b-aq)\right\}\phi \\ &=(1-c)^{-1}(c-a)(b-aq)z\phi(cq)+qc(1-a)\phi\left(aq,\tfrac{b}{q}\right)\end{aligned} \tag{4.64}$$

$$\begin{aligned}&\left[c(1-a)(c-aq)(c-abz)-\left\{c^2(1-a)+a^2z(c-b)\right\}\right. \\ &\quad\left.\left\{c^2(1-a)+a^2zq(c-bq)\right\}\right]\phi \\ &=a(1-c)^{-1}(c-a)(c-b)\left\{c^2(1-a)+a^2qz(c-bq)\right\}z\phi(cq) \\ &\quad+c(1-a)(c-q)(c-abz)(c-abqz)\phi\left(aq,\tfrac{c}{q}\right)\end{aligned} \tag{4.65}$$

$$\begin{aligned}(4.66)\qquad &\big[a^2(c-a)(c-b)(cq-abz)z\\ &+\left\{c^2(1-a)+a^2z(c-b)\right\}\left\{c^2(q-a)+a^2(c-b)\right\}\big]\,\phi\\ &=c(1-a)(c-abz)\left\{c^2(q-a)+a^2z(c-b)\right\}\phi(aq)\\ &+acq(1-c)^{-1}(c-a)(c-b)(a-cq)z\phi\left(\tfrac{a}{q},cq\right)\end{aligned}$$

$$\begin{aligned}(4.67)\qquad &[ab(c-a)(c-b)(cq-abz)\\ &+\left\{c^2(1-a)+a^2z(c-b)\right\}\left\{c^2(q-b)+b^2z(c-a)\right\}]\,\phi\\ &=c(1-a)(c-abz)\left\{c^2(q-b)+b^2z(c-a)\right\}\phi(aq)\\ &+acq(1-c)^{-1}(c-a)(c-b)(b-cq)z\phi\left(\tfrac{b}{q},cq\right)\end{aligned}$$

$$\begin{aligned}(4.68)\qquad \left\{(1-b)+z(bq-a)\right\}\phi &=(1-c)^{-1}(b-c)(bq-a)z\phi(cq)\\ &+cq(1-b)\phi\left(\tfrac{a}{q},bq\right)\end{aligned}$$

$$\begin{aligned}(4.69)\qquad \left\{c(q-c)+z(a-c)\right\}\phi &=(1-c)^{-1}(c-a)(c-b)z\phi(cq)\\ &+c(c-q)\phi\left(\tfrac{a}{q},\tfrac{c}{q}\right)\end{aligned}$$

$$\begin{aligned}(4.70)\qquad \left\{a(a-c)+az(b-a)\right\}\phi &=q(a-c)\phi\left(\tfrac{a}{q}\right)\\ &+a(1-c)^{-1}(1-a)(b-c)z\phi(aq,cq)\end{aligned}$$

$$(4.71)\qquad q(cq-abz)\phi=(1-c)^{-1}\left\{aq(c-b)+bc(q-a)\right\}z\phi(cq)+cq^2\phi\left(\tfrac{a}{q},\tfrac{b}{q}\right)$$

$$(4.72)\qquad q\phi=q\left(\tfrac{a}{q}\right)+a(1-c)^{-1}(1-b)z\phi(bq,cq)$$

$$\begin{aligned}(4.73)\qquad &(cq-abz)\left\{cq(q-b)+bz(b-a)\right\}\phi\\ &=q\left\{c^2q(q-b)+b^2z(c-a)\right\}\phi\left(\tfrac{a}{q}\right)\\ &-aq(1-c)^{-1}(c-b)(b-cq)z\phi\left(\tfrac{b}{q},cq\right)\end{aligned}$$

$$(4.74)\qquad q\phi=q\left(\tfrac{b}{q}\right)+b(1-c)^{-1}(1-a)z\phi(aq,cq)$$

$$\begin{aligned}(4.75)\qquad &(cq-abz)\left\{cq(q-a)+az(a-b)\right\}\phi\\ &=q\left\{c^2q(q-a)+a^2z(c-b)\right\}\phi\left(\tfrac{b}{q}\right)\\ &-bq(1-c)^{-1}(c-a)(a-cq)z\phi\left(\tfrac{a}{q},cq\right)\end{aligned}$$

$$(4.76)\qquad \left\{c(c-q)-az(c-b)\right\}\phi=(1-c)^{-1}(c-a)(c-b)z\phi\,(cq)$$

$$- c(q-c)\phi\left(\tfrac{b}{q},\tfrac{c}{q}\right)$$

(4.77) $$\{q(b-c)-bz(b-a)\}\,\phi = q(b-c)\phi\left(\tfrac{b}{q}\right) - b(1-c)^{-1}(c-b)\phi\left(\tfrac{b}{q},\tfrac{c}{q}\right)$$

(4.78) $$\begin{aligned}&[\{c(1-a)+q(c-a)\}\{aq(c-b)+bc(q-a)\}-bcq(q-a)(c-a)\\&+az\left\{aq(c-b)(a-b)+bc(q-a)(a-b)+b^2(q-a)(c-a)\right\}\big]\,\phi\\&=(1-a)(c-abz)\{aq(c-b)+bc(q-a)\}\,\phi(aq)\\&-aq^2(c-a)(b-c)\phi\left(\tfrac{a}{q},\tfrac{b}{q}\right)\end{aligned}$$

(4.79) $$\begin{aligned}&\big[ac^2(1-a)(1-b)+bq(c-a)(c-aq)+acq(c-a)(1-b)\\&-az\left\{b^2(1-a)(c-aq)+b(a-b)(c-aq)+ac(1-b)\right\}\big]\,\phi\\&=q(a-c)\{b(c-aq)+ac(1-b)\}\,\phi\left(\tfrac{a}{q}\right)\\&+a(1-a)(1-b)(abqz-c)\phi(aq,bq)\end{aligned}$$

(4.80) $$\begin{aligned}&[\{c(1-a)+q(c-a)+az(a-b)\}(b-aq)-b(1-a)(c-aq)]\,\phi\\&=q(c-a)(b-aq)\phi\left(\tfrac{a}{q}\right)-aq(1-a)(c-b)\phi\left(aq,\tfrac{b}{q}\right)\end{aligned}$$

(4.81) $$\begin{aligned}&\big[\{c(1-a)+q(c-a)+az(a-b)\}\left\{c^2(1-a)+a^2qz(c-bq)\right\}\\&-c(1-a)(c-abz)(c-aq)]\,\phi\\&=q(c-a)\left\{c^2(1-a)+a^2qz(c-bq)\right\}\phi\left(\tfrac{a}{q}\right)\\&-a(1-a)(c-q)(c-abz)(c-abqz)\phi\left(aq,\tfrac{c}{q}\right)\end{aligned}$$

(4.82) $$\begin{aligned}&\big[b(c-aq)\left\{a(c-b)+b^2z(1-a)\right\}\\&+c(1-b)\{aq(c-b)+bc(1-a)\}]\,\phi\\&=q(c-b)\{b(c-aq)+a(1-b)\}\,\phi\left(\tfrac{b}{q}\right)\\&-b(1-a)(1-b)(c-abz)(abqz-c)\phi(aq,bq)\end{aligned}$$

(4.83) $$\begin{aligned}&\big[c^3(1-a)(q-b)\\&+z\left\{b^2(1-a)(c-aq)+a^2q^2(c-bq)(c-b)+abcq(c-bq)(1-a)\right\}\big]\,\phi\\&=q(c-b)\left\{c^2(1-a)+a^2qz(c-bq)\right\}\phi\left(\tfrac{b}{q}\right)\\&-a(1-a)(c-q)(c-abz)(c-abqz)\phi\left(aq,\tfrac{c}{q}\right)\end{aligned}$$

(4.84) $$\{bc(1-q)+a(c-b)\}\,\phi=(1-b)(c-abz)\phi\left(\tfrac{b}{q}\right)-ab(q-c)\phi\left(\tfrac{a}{q},\tfrac{c}{q}\right)$$

(4.85) $$\big[qc^3(1-b)(q-a)$$

$$+z\left\{abq(c-a)(c-b)+b^2cq(c-a)(q-a)+a^2c(c-b)\right\}\big]\,\phi$$
$$=(1-b)(c-abz)\phi(bq)-bq(c-b)(c-a)(q-a)(1-c)^{-1}z\phi\left(\tfrac{a}{q},\tfrac{c}{q}\right)$$

(4.86)
$$\big[\{bq(c-a)+ac(1-b)\}\left\{aq^2(c-b)+bqc(q-a)\right\}$$
$$-b^2q(c-a)(q-a)(cq-abz)\big]\,\phi$$
$$=a(1-b)(c-abz)\left\{aq^2(c-b)+bqc(q-a)\right\}\phi(bq)$$
$$-\,abq^3(b-c)(c-a)\phi\left(\tfrac{a}{q},\tfrac{b}{q}\right)$$

(4.87)
$$\big[c(1-b)\left\{bcq(c-a)+ac^2(1-b)-a(c-bq)\right\}$$
$$+bz\left\{b^2q^2(c-a)(c-aq)+abcq(1-b)(c-aq)+a^2(1-b)(c-bq)\right\}\big]\,\phi$$
$$=bq(c-a)\left\{c^2(1-b)+b^2qz(c-aq)\right\}\phi\left(\tfrac{a}{q}\right)$$
$$-\,ab(1-b)(c-q)(c-abz)(c-abqz)\phi\left(bq,\tfrac{c}{q}\right)$$

(4.88)
$$\left\{b^2q^2-b^2qc+ab^2q+bcq^2-acq+bcq-bz(a-b)(bq-b)\right\}\phi$$
$$=q(c-b)(bq-a)\phi\left(\tfrac{b}{q}\right)+bq(1-b)(c-a)\phi\left(\tfrac{a}{q},bq\right)$$

(4.89)
$$\{c(q-b)-bz(a-b)\}\,\phi=q(c-b)\phi\left(\tfrac{b}{q}\right)+b(q-c)\phi\left(\tfrac{a}{q},\tfrac{c}{q}\right)$$

(4.90)
$$\{c(q-a)+az(a-b)\}\,\phi=q(c-a)\phi\left(\tfrac{a}{q}\right)+a(q-c)\phi\left(\tfrac{b}{q},\tfrac{c}{q}\right)$$

(4.91)
$$(c-bq)\left\{c(a-q)+az(bq-a)\right\}\phi$$
$$=(q-c)(bq-a)(c-abz)\phi\left(\tfrac{c}{q}\right)-qc(1-b)(c-a)\phi\left(\tfrac{a}{q},bq\right)$$

(4.92)
$$\big[\left\{c^2(q-a)+a^2z(c-bq)\right\}\left\{qc^2(q-a)+a^2z(c-b)\right\}$$
$$-c^2q(c-a)(q-a)(cq-abz)\big]\,\phi$$
$$=a(q-c)(c-abz)\left\{qc^2(q-a)+a^2z(c-b)\right\}\phi\left(\tfrac{c}{q}\right)$$
$$-\,aqc(1-c)^{-1}(c-a)(1-b)(q-a)\phi\left(\tfrac{a}{q},cq\right)$$

(4.93)
$$\big[\left\{c^2(q-a)+a^2z(c-bq)\right\}\left\{aq^2(c-b)+bqc(q-a)\right\}$$
$$-bqc(q-a)(c-q)(cq-abz)\big]\,\phi$$
$$=a(q-c)(c-abz)\left\{aq^2(c-b)+bqc(q-a)\right\}\phi\left(\tfrac{c}{q}\right)$$
$$-\,aq^3c(b-c)(c-a)\phi\left(\tfrac{a}{q},\tfrac{b}{q}\right)$$

(4.94)
$$\big[\{c(1-b)+q(c-b)+bz(b-a)\}\left\{c^2q(q-b)+b^2z(c-a)\right\}$$

$$\begin{aligned}&-cq(c-b)(q-b)(cq-abz)\big]\,\phi\\&=(1-b)(c-abz)\left\{c^2q(q-b)+b^2z(c-a)\right\}\phi(bq)\\&\quad-bq(1-c)^{-1}(c-a)(c-b)(cq-b)\phi\left(\tfrac{b}{q},cq\right)\end{aligned}$$

(4.95) $$\{c-bq+bz(b-a)\}\,\phi=(1-b)(c-abz)\phi(bq)-b(q-c)\phi\left(\tfrac{b}{q},\tfrac{c}{q}\right)$$

(4.96) $$\begin{aligned}&\big[\{c(1-b)+q(c-b)+bz(b-a)\}\left\{c^2(1-b)+b^2qz(c-aq)\right\}\\&-c(1-b)(c-bq)(c-abz)\big]\,\phi\\&=q(c-a)\left\{c^2(1-b)+b^2qz(c-aq)\right\}\phi\left(\tfrac{b}{q}\right)\\&\quad-b(1-b)(c-q)(c-abz)(c-abqz)\phi\left(bq,\tfrac{c}{q}\right)\end{aligned}$$

(4.97) $$\begin{aligned}&\big[c^2(1-b)(c-bq)(c-abz)\\&-\left\{c^2(1-b)+b^2z(c-a)\right\}\left\{c^2(1-b)+b^2qz(c-aq)\right\}\big]\,\phi\\&=b(1-c)^{-1}(c-a)(c-b)\left\{c^2(1-b)+b^2qz(c-aq)\right\}\phi\left(\tfrac{b}{q}\right)\\&\quad+bc(1-b)(c-q)(c-abz)(c-abqz)\phi\left(bq,\tfrac{c}{q}\right)\end{aligned}$$

(4.98) $$\begin{aligned}\{c(q-c)+az(c-bq)\}\,\phi&=(q-c)(c-abz)\phi\left(\tfrac{c}{q}\right)\\&\quad+c(1-c)^{-1}(1-b)(a-c)z\phi(bq,cq)\end{aligned}$$

(4.99) $$\begin{aligned}&\big[\left\{c^2(q-b)+b^2z(c-aq)\right\}\left\{c^2q(q-b)+b^2q(c-a)\right\}\\&-qc^2(q-b)(c-b)(cq-abz)\big]\,\phi\\&=b(q-c)(c-abz)\left\{c^2q(q-b)+b^2q(c-a)\right\}\phi\left(\tfrac{c}{q}\right)\\&\quad-bcq(1-c)^{-1}(c-a)(c-b)(cq-b)\phi\left(\tfrac{b}{q},cq\right)\end{aligned}$$

4.1. Derivation of results (4.52)–(4.99)

The derivations of these new contiguous function relations (4.52)–(4.99) are quite straight forward. For example, by algebraic manipulation, if we wish to derive (4.52), we consider Henie's q-contiguous relation (2.13) and the result (3.34) and eliminate $\phi(aq)$, we easily arrive (4.52). Similarly other results can also be obtained. The scheme is outlined in Table 2 and Table 3.

5. Special cases

(i) In (2.13)–(2.27), if we take $q \to 1$, we get the corresponding contiguous functions relations in hypergeometric series, due to Gauss given in [15].

TABLE 2. Derivations of (4.52)–(4.86)

S. No.	If we take Henie's relation	and relation	if we eliminate	we get the result
1.	(2.1)	(3.7)	$\phi\,(aq)$	(4.1)
2.	(2.1)	(3.9)	$\phi\,(bq)$	(4.2)
3.	(2.1)	(3.15)	$\phi\,(aq)$	(4.3)
4.	(2.1)	(3.16)	$\phi\,(bq)$	(4.4)
5.	(2.1)	(3.17)	$\phi\,(aq)$	(4.5)
6.	(2.1)	(3.29)	$\phi\,(bq)$	(4.6)
7.	(2.2)	(3.7)	$\phi(aq)$	(4.7)
8.	(2.2)	(3.8)	$\phi(\frac{c}{q})$	(4.8)
9.	(2.2)	(3.10)	$\phi(\frac{c}{q})$	(4.9)
10.	(2.2)	(3.13)	$\phi(aq)$	(4.10)
11.	(2.2)	(3.15)	$\phi(aq)$	(4.11)
12.	(2.3)	(3.13)	$\phi(aq)$	(4.12)
13.	(2.3)	(3.15)	$\phi(aq)$	(4.13)
14.	(2.3)	(3.17)	$\phi(aq)$	(4.14)
15..	(2.3)	(3.18)	$\phi(cq)$	(4.15)
16.	(2.3)	(3.24)	$\phi(cq)$	(4.16)
17.	(2.4)	(3.1)	$\phi(\frac{a}{q})$	(4.17)
18.	(2.4)	(3.3)	$\phi(\frac{a}{q})$	(4.18)
19.	(2.4)	(3.4)	$\phi(cq)$	(4.19)
20.	(2.4)	(3.11)	$\phi(\frac{a}{q})$	(4.20)
21.	(2.4)	(3.22)	$\phi(cq)$	(4.21)
22.	(2.4)	(3.24)	$\phi(cq)$	(4.22)
23.	(2.5)	(3.4)	$\phi(cq)$	(4.23)
24.	(2.5)	(3.18)	$\phi(cq)$	(4.24)
25.	(2.5)	(3.21)	$\phi\left(\frac{b}{q}\right)$	(4.25)
26.	(2.5)	(3.22)	$\phi\,(cq)$	(4.26)
27.	(2.6)	(3.11)	$\phi\left(\frac{a}{q}\right)$	(4.27)
28.	(2.6)	(3.13)	$\phi\,(aq)$	(4.28)
29.	(2.6)	(3.15)	$\phi\,(aq)$	(4.29)
30.	(2.6)	(3.17)	$\phi\,(aq)$	(4.30)
31.	(2.7)	(3.13)	$\phi\,(aq)$	(4.31)
32.	(2.7)	(3.17)	$\phi\,(aq)$	(4.32)
33.	(2.8)	(3.3)	$\phi\left(\frac{a}{q}\right)$	(4.33)
34.	(2.8)	(3.5)	$\phi\left(\frac{a}{q}\right)$	(4.34)
35.	(2.8)	(3.11)	$\phi\left(\frac{a}{q}\right)$	(4.35)

(ii) In (3.28)–(3.51), if we take $q \to 1$, then after little simplification, we get the corresponding contiguous function relations in hypergeometric series obtained by Cho et al. [3].

(iii) In (4.52)–(4.99), if we take $q \to 1$, then after little simplification, we get the corresponding contiguous function relations in hypergeometric series obtained very recently by Rakha et al. [16].

TABLE 3. Derivations of (4.87)–(4.99)

S. No.	If we take Henie's relation	and relation	if we eliminate	we get the result
36.	(2.8)	(3.23)	$\phi\,(bq)$	(4.36)
37.	(2.9)	(3.1)	$\phi\left(\frac{a}{q}\right)$	(4.37)
38.	(2.9)	(3.3)	$\phi\left(\frac{a}{q}\right)$	(4.38)
39.	(2.9)	(3.21)	$\phi\left(\frac{b}{q}\right)$	(4.39)
40.	(2.10)	(3.1)	$\phi\left(\frac{a}{q}\right)$	(4.40)
41.	(2.10)	(3.5)	$\phi\left(\frac{a}{q}\right)$	(4.41)
42.	(2.10)	(3.11)	$\phi\left(\frac{a}{q}\right)$	(4.42)
43.	(2.11)	(3.19)	$\phi\left(\frac{b}{q}\right)$	(4.43)
44.	(2.11)	(3.21)	$\phi\left(\frac{b}{q}\right)$	(4.44)
45.	(2.11)	(3.23)	$\phi\,(bq)$	(4.45)
46.	(2.12)	(3.23)	$\phi\,(bq)$	(4.46)
47.	(2.13)	(3.9)	$\phi\,(bq)$	(4.47)
48.	(2.14)	(3.18)	$\phi\left(\frac{b}{q}\right)$	(4.48)

6. Applications

Our main aim in this section is to obtain a closed form for the following summation formulas for the q-series

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ \frac{aq^i}{b}, & \end{matrix} ; \; q, -\frac{q}{b}\right], \; i = 2, 3, 4, 5, 6, $$

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ \frac{a}{bq^i}, & \end{matrix} ; \; q, -\frac{q}{b}\right], \; i = 0, 1, 2, 3, 4, $$

$$ {}_2\phi_1\left[\begin{matrix} a, & bq^i, \\ \frac{aq}{b}, & \end{matrix} ; \; q, -\frac{q}{b}\right], \; i = 1, 2, 3, 4, $$

and

$$ {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^i}, \\ \frac{aq}{b}, & \end{matrix} ; \; q, -\frac{q}{b}\right], \; i = 1, 2, 3, 4. $$

6.1. Main results

In this section, the results to be established are

(6.100)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ & & ; \quad q, -\frac{q}{b} \\ \frac{aq^2}{b}, & \end{matrix}\right] $$

$$ = \frac{(-q;q)_\infty}{\left(1-\frac{q}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} \left\{\left(aq;q^2\right)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty - \frac{q}{b}\left(a;q^2\right)_\infty \left(\frac{aq^3}{b^2};q^2\right)_\infty\right\} $$

(6.101)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ & & ; \quad q, -\frac{q}{b} \\ \frac{aq^3}{b}, & \end{matrix}\right] $$

$$ = \frac{(-q;q)_\infty}{\left(1-\frac{q^2}{b}\right)\left(1-\frac{q}{b}\right)\left(\frac{aq^3}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} \left\{\left(aq;q^2\right)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty \right. $$

$$ \left. -\frac{q}{b}(1+q)\left(a;q^2\right)_\infty \left(\frac{aq^3}{b^2};q^2\right)_\infty + \frac{q^3}{b^2}(1-a)(aq;q^2)_\infty \left(\frac{aq^4}{b^2};q^2\right)_\infty\right\} $$

(6.102)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ & & ; \quad q, -\frac{q}{b} \\ \frac{aq^4}{b}, & \end{matrix}\right] $$

$$ = \frac{(-q;q)_\infty}{\left(1-\frac{q^3}{b}\right)\left(1-\frac{q^2}{b}\right)\left(1-\frac{q}{b}\right)\left(\frac{aq^4}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} \left\{\left(aq;q^2\right)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty \right. $$

$$ - \frac{q}{b}(1+q+q^2)\left(a;q^2\right)_\infty \left(\frac{aq^3}{b^2};q^2\right)_\infty $$

$$ + \frac{q^3}{b^2}(1-a)(1+q+q^2)(aq;q^2)_\infty \left(\frac{aq^4}{b^2};q^2\right)_\infty $$

$$ \left. -\frac{q^6}{b^3}(1-aq)\left(a;q^2\right)_\infty \left(\frac{aq^5}{b^2};q^2\right)_\infty\right\} $$

(6.103)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, \\ & & ; \quad q, -\frac{q}{b} \\ \frac{aq^5}{b}, & \end{matrix}\right] $$

$$ = \frac{(-q;q)_\infty}{\left(1-\frac{q^4}{b}\right)\left(1-\frac{q^3}{b}\right)\left(1-\frac{q^2}{b}\right)\left(1-\frac{q}{b}\right)\left(\frac{aq^5}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} \left\{\left(aq;q^2\right)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty \right. $$

$$ - \frac{q}{b}(1+q+q^2+q^3)\left(a;q^2\right)_\infty \left(\frac{aq^3}{b^2};q^2\right)_\infty $$

$$ + \frac{q^3}{b^2}(1+q+2q^2+q^3+q^4)(1-a)(aq;q^2)_\infty \left(\frac{aq^4}{b^2};q^2\right)_\infty $$

$$ -\frac{q^6}{b^3}(1+q+q^2+q^3)(1-aq)\left(a;q^2\right)_\infty \left(\frac{aq^5}{b^2};q^2\right)_\infty $$

$$ \left. +\frac{q^{10}}{b^4}(1-a)(1-aq^2)\left(aq;q^2\right)_\infty \left(\frac{aq^6}{b^2};q^2\right)_\infty\right\} $$

(6.104)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ; \quad q, -\frac{q}{b} \\ \frac{aq^6}{b}, & & \end{matrix}\right] $$

$$ = \frac{(-q;q)_\infty}{\left(1-\frac{q^5}{b}\right)\left(1-\frac{q^4}{b}\right)\left(1-\frac{q^3}{b}\right)\left(1-\frac{q^2}{b}\right)\left(1-\frac{q}{b}\right)\left(\frac{aq^6}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} \Big\{ \left(aq;q^2\right)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty $$
$$ - \frac{q}{b}(1+q+q^2+q^3+q^4)\left(a;q^2\right)_\infty \left(\frac{aq^3}{b^2};q^2\right)_\infty $$
$$ + \frac{q^3}{b^2}(1-a)(1+q+2q^2+2q^3+2q^4+q^5+q^6)(aq;q^2)_\infty \left(\frac{aq^4}{b^2};q^2\right)_\infty $$
$$ - \frac{q^6}{b^3}(1+q+2q^2+2q^3+2q^4+q^5+q^6)\,(1-aq)\left(a;q^2\right)_\infty \left(\frac{aq^5}{b^2};q^2\right)_\infty $$
$$ + \frac{q^{10}}{b^4}(1-a)(1-aq^2)(1+q+q^2+q^3+q^4)\left(aq;q^2\right)_\infty \left(\frac{aq^6}{b^2};q^2\right)_\infty $$
$$ - \frac{q^{15}}{b^5}(1-aq)(1-aq^3)\left(aq;q^2\right)_\infty \left(\frac{aq^7}{b^2};q^2\right)_\infty \Big\} $$

(6.105)
$$ {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ; \quad q, -\frac{q}{b} \\ \frac{a}{b}, & & \end{matrix}\right] $$
$$ = \frac{(-q;q)_\infty}{\left(\frac{a}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty} \left\{ \left(aq;q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty + \frac{1}{b}(a;q^2)_\infty \left(\frac{aq}{b^2};q^2\right)_\infty \right\} $$

(6.106)
$$ {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ; \quad q, -\frac{q}{b} \\ \frac{a}{bq}, & & \end{matrix}\right] $$
$$ = \frac{(-q;q)_\infty}{\left(\frac{a}{bq};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty} \Big\{ \left(1-\frac{a}{b^2q}\right)\left(aq;q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty $$
$$ + \frac{1}{bq}(1+q)(a;q^2)_\infty \left(\frac{a}{b^2q};q^2\right)_\infty + \frac{1}{b^2q}\left(\frac{a}{q};q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty \Big\} $$

(6.107)

$$ {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ; \quad q, -\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] $$
$$ = \frac{(-q;q)_\infty}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty} \Big\{ \left(1-\frac{a}{b^2q}\right)\left(aq;q^2\right)_\infty \left(\frac{a}{b^2q^2};q^2\right)_\infty $$
$$ + \frac{1}{bq^2}(1+q+q^2)\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty \left(\frac{a}{b^2q};q^2\right)_\infty $$
$$ + \frac{1}{b^2q^3}\left(1+q+q^2\right)\left(\frac{a}{q};q^2\right)_\infty \left(\frac{a}{b^2q^2};q^2\right)_\infty + \frac{1}{b^3q^3}\left(\frac{a}{q^2};q^2\right)_\infty \left(\frac{a}{b^2q};q^2\right)_\infty \Big\} $$

(6.108)
$$ {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ; \quad q, -\frac{q}{b} \\ \frac{a}{bq^3}, & & \end{matrix}\right] $$

$$= \frac{(-q;q)_\infty}{\left(\frac{a}{bq^3};q\right)_\infty \left(-\frac{1}{bq^3};q\right)_\infty} \Big\{ \left(1-\frac{a}{b^2q^3}\right)\left(1-\frac{a}{b^2q}\right)\left(aq;q^2\right)_\infty \left(\frac{a}{b^2q^2};q^2\right)_\infty$$
$$+\frac{1}{bq^3}(1+q+q^2+q^3)\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty$$
$$+\frac{1}{b^2q^5}(1+q+2q^2+q^3+q^4)\left(1-\frac{a}{b^2q^3}\right)(\tfrac{a}{q};q^2)_\infty \left(\frac{a}{b^2q^2};q^2\right)_\infty$$
$$+\frac{1}{b^3q^6}\left(1+q+q^2+q^3\right)\left(\frac{a}{q^2};q^2\right)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty$$
$$+\frac{1}{b^4q^6}\left(\frac{a}{q^3};q^2\right)_\infty \left(\frac{a}{b^2q^2};q^2\right)_\infty \Big\}$$

(6.109)
$$ {}_2\phi_1 \left[\begin{matrix} a, & b, \\ \frac{a}{bq^4}, & \end{matrix} ; \quad q, -\frac{q}{b} \right]$$
$$= \frac{(-q;q)_\infty}{\left(\frac{a}{bq^4};q\right)_\infty \left(-\frac{1}{bq^4};q\right)_\infty} \Big\{ \left(1-\frac{a}{b^2q^3}\right)\left(1-\frac{a}{b^2q}\right)\left(aq;q^2\right)_\infty \left(\frac{a}{b^2q^4};q^2\right)_\infty$$
$$+\frac{1}{bq^4}(1+q+q^2+q^3+q^4)\left(1-\frac{a}{b^2q^4}\right)\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty$$
$$+\frac{1}{b^2q^7}(1+q+2q^2+2q^3+2q^4+q^5+q^6)\left(1-\frac{a}{b^2q^3}\right)(\tfrac{a}{q};q^2)_\infty \left(\frac{a}{b^2q^4};q^2\right)_\infty$$
$$+\frac{1}{b^3q^9}(1+q+2q^2+2q^3+2q^4+q^5+q^6)\left(1-\frac{a}{b^2q^3}\right)\left(\frac{a}{q^2};q^2\right)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty$$
$$+\frac{1}{b^4q^{10}}\left(1+q+q^2+q^3+q^4\right)\left(\frac{a}{q^3};q^2\right)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty$$
$$+\frac{1}{b^5q^{10}}\left(\frac{a}{q^4};q^2\right)_\infty \left(\frac{a}{b^2q^3};q^2\right)_\infty \Big\}$$

(6.110)
$$ {}_2\phi_1 \left[\begin{matrix} a, & bq, \\ \frac{aq}{b}, & \end{matrix} ; \quad q, -\frac{q}{b} \right]$$
$$= \frac{b(-q;q)_\infty}{a(1-b)\left(\frac{aq}{b};q\right)_\infty \left(-\frac{1}{b};q\right)_\infty} \left\{ \left(a;q^2\right)_\infty \left(\frac{aq}{b^2};q^2\right)_\infty - \left(aq;q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty \right\}$$

(6.111)
$$ {}_2\phi_1 \left[\begin{matrix} a, & bq^2, \\ \frac{aq}{b}, & \end{matrix} ; \quad q, -\frac{q}{b} \right]$$
$$= \frac{b^2(-q;q)_\infty}{a^2(1-b)(1-bq)\left(\frac{aq}{b};q\right)_\infty \left(-\frac{1}{bq};q\right)_\infty} \Big[\left(aq;q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty \left\{ q\left(1-\frac{a}{b^2q^2}\right) + (1-a) \right\}$$
$$- \left(a;q^2\right)_\infty \left(\frac{aq}{b^2};q^2\right)_\infty \left\{ q\left(1-\frac{a}{b^2q^2}\right) + \left(1-\frac{a}{b^2}\right) \right\} \Big]$$

(6.112)
$$ {}_2\phi_1 \left[\begin{matrix} a, & bq^3, \\ \frac{aq}{b}, & \end{matrix} ; \quad q, -\frac{q}{b} \right]$$

$$= \frac{b^3(-q;q)_\infty}{a^3(1-b)(1-bq)(1-bq^2)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}$$
$$\left\{\left(a;q^2\right)_\infty\left(\tfrac{aq}{b^2};q^2\right)_\infty X_1-\left(aq;q^2\right)_\infty\left(\tfrac{aq^2}{b^2};q^2\right)_\infty Y_1\right\},$$

where

$$X_1 = q^2\left(1-\frac{a}{b^2q^4}\right)\left\{q\left(1-\frac{a}{b^2q^2}\right)+\left(1-\frac{a}{b^2}\right)\right\}$$
$$+\left(1-\frac{a}{b^2q}\right)\left\{q\left(1-\frac{a}{b^2q}\right)+(1-aq)\right\}$$

and

$$Y_1 = q^2\left(1-\frac{a}{b^2}\right)\left(1-\frac{a}{b^2q^4}\right)\left\{q\left(1-\frac{a}{b^2q}\right)+(1-a)\right\}$$
$$+\left(1-\frac{a}{b^2q}\right)(1-a)\left\{q\left(1-\frac{a}{b^2q}\right)+\left(1-\frac{aq}{b^2}\right)\right\}$$

(6.113)
$$ {}_2\phi_1\left[\begin{matrix} a, & bq^4, & \\ & & ;\quad q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$
$$= \frac{b^4(-q;q)_\infty}{a^4(1-b)(1-bq)(1-bq^2)(1-bq^3)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{bq^3};q\right)_\infty}$$
$$\left\{\left(aq;q^2\right)_\infty\left(\tfrac{aq^2}{b^2};q^2\right)_\infty X_2-\left(a;q^2\right)_\infty\left(\tfrac{aq^3}{b^2};q^2\right)_\infty Y_2\right\},$$

where

$$X_2 = \left(1-\frac{a}{b^2q^2}\right)(X_1)_{aq}+q^3\left(1-\frac{a}{b^2q^6}\right)Y_1$$

and

$$Y_2 = q^3\left(1-\frac{a}{b^2q^6}\right)\left(1-\frac{aq}{b}\right)X_1+q^3\left(1-\frac{a}{b^2q^6}\right)(Y_1)_{aq}$$

also, $(X_1)_{aq}$ and $(Y_1)_{aq}$ can be obtained from X_1 and Y_1 by changing a to aq.

(6.114)
$$ {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q}, & \\ & & ;\quad q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$
$$= \frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(aq;q^2\right)_\infty\left(\tfrac{aq^2}{b^2};q^2\right)_\infty+\left(a;q^2\right)_\infty\left(\tfrac{aq^3}{b^2};q^2\right)_\infty\right\}$$

(6.115)
$$ {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^2}, & \\ & & ;\quad q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$
$$= \frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left[\left(aq;q^2\right)_\infty\left(\tfrac{aq^4}{b^2};q^2\right)_\infty\left\{\left(1-\tfrac{aq^2}{b^2}\right)+\tfrac{(1-a)}{q}\right\}\right.$$
$$\left.+\ \tfrac{1}{q}(1+q)\left(a;q^2\right)_\infty\left(\tfrac{aq^3}{b^2};q^2\right)_\infty\right]$$

(6.116)

$$
{}_2\phi_1\left[\begin{array}{cc} a, & \frac{b}{q^3}, \\ & ; \quad q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{array}\right]
$$

$$
= \frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty \left(-\frac{q}{b};q\right)_\infty} \left\{ \left(aq;q^2\right)_\infty \left(\frac{aq^4}{b^2};q^2\right)_\infty X_3 + \left(a;q^2\right)_\infty \left(\frac{aq^5}{b^2};q^2\right)_\infty Y_3 \right\},
$$

where

$$
X_3 = \left\{ \left(1 - \frac{aq^2}{b^2}\right) + \frac{(1-a)}{q} \right\} + \frac{1}{q^3}(1+q)(1-a)
$$

and

$$
Y_3 = \frac{1}{q}(1+q)\left(1 - \frac{aq^3}{b^2}\right) + \frac{1}{q^2}\left\{ \left(1 - \frac{aq^3}{b^2}\right) + \frac{(1-aq)}{q} \right\}.
$$

(6.117)

$$
{}_2\phi_1\left[\begin{array}{cc} a, & \frac{b}{q^4}, \\ & ; \quad q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{array}\right]
$$

$$
= \frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty \left(-\frac{q}{b};q\right)_\infty} \left\{ \left(aq;q^2\right)_\infty \left(\frac{aq^6}{b^2};q^2\right)_\infty X_4 + \left(a;q^2\right)_\infty \left(\frac{aq^5}{b^2};q^2\right)_\infty Y_4 \right\},
$$

where

$$
X_4 = \left(1 - \frac{aq^4}{b^2}\right) X_3 + \frac{1}{q^3}(Y_3)_{aq}
$$

and

$$
Y_4 = Y_3 + \frac{1}{q^3}(X_3)_{aq}
$$

also, $(X_3)_{aq}$ and $(Y_3)_{aq}$ can be obtained from X_3 and Y_3 by changing a to aq.

6.2. Derivations

In order to start the derivations of (6.100)–(6.117), the contiguous functions relations (3.4), (3.8), (4.1) and (4.23) together the Bailey-Dhum's summation formula [2, 4], viz

$$
{}_2\phi_1\left[\begin{array}{cc} a, & b, \\ & ; \quad q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{array}\right] = \frac{\left(aq;q^2\right)_\infty (-q;q)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty \left(-\frac{q}{b};q\right)_\infty} \tag{6.118}
$$

will be required in our investigations.

The derivations of (6.100)–(6.117) are quite straightforward. So we shall derive only (6.100), (6.107), (6.111) and (6.115) and the rest can be derived on similar lines.

- In order to derive the result (6.100), we use the result (3.31) in the form

$$
(a-c)\phi(cq) = a(1-c)\phi - c(1-a)\phi(aq,cq). \tag{6.119}
$$

In (6.119), if we select

$$\phi = {}_2\phi_1 \left[\begin{matrix} a, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix} \right]$$

then after simplification, we get

$$(6.120) \quad {}_2\phi_1 \left[\begin{matrix} a, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix} \right]$$
$$= \frac{(b-aq)}{(b-q)}\, {}_2\phi_1 \left[\begin{matrix} a, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix} \right] - \frac{q(1-a)}{(b-q)}\, {}_2\phi_1 \left[\begin{matrix} aq, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix} \right].$$

Now, it is easy to see that, first ${}_2\phi_1$ on the right-hand-side of (6.120) can be evaluated with the help of the Bailey-Dhum's summation formula (6.118) and second ${}_2\phi_1$ can also be evaluated with the help of Bailey-Dhum's summation formula (6.118) by simply changing a by aq, we get

$$(6.121) \quad {}_2\phi_1 \left[\begin{matrix} a, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix} \right]$$
$$= \frac{(b-aq)\left(aq;q^2\right)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{(b-q)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} - \frac{q(1-a)\left(aq^2;q^2\right)_\infty(-q;q)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty}{(b-q)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}.$$

Noting that

$$(6.122) \quad (1-a)(aq^2;q^2)_\infty = (a;q^2),$$

and

$$(6.123) \quad \frac{(b-aq)}{\left(\frac{aq}{b};q\right)_\infty(q-b)} = \frac{1}{\left(\frac{aq^2}{b};q\right)_\infty\left(1-\frac{q}{b}\right)}$$

we get

$$(6.124) \quad {}_2\phi_1 \left[\begin{matrix} a, & b, & \\ & & ; q, -\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix} \right]$$
$$= \frac{\left(aq;q^2\right)_\infty(-q;q)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(1-\frac{q}{b}\right)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty} - \frac{q}{b\left(1-\frac{q}{b}\right)}\frac{\left(a;q^2\right)_\infty(-q;q)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty}{\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty},$$

which on simplification gives

(6.125)

$$
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix}\right]
= \frac{(-q;q)_\infty}{\left(1-\frac{q}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(aq,q^2\right)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty-\frac{q}{b}\left(a;q^2\right)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right\}.
$$

This completes the derivation of (6.100). □

- In order to derive (6.107), we use (3.35) and select

$$
\phi = {}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq}, & & \end{matrix}\right]
$$

then after little simplification, we

$$
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] \tag{6.126}
$$
$$
= \frac{q^2(b^2q^2-a)}{(bq^2-a)(1+bq^2)}\,{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq}, & & \end{matrix}\right]
+ \frac{(bq^2-a)}{(bq^2-a)(1+bq^2)}\,{}_2\phi_1\left[\begin{matrix} \frac{a}{q}, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right].
$$

Now, the first ${}_2\phi_1$ on the right-hand side of (6.126) can be evaluated with the help of the q-contiguous Kummer's formula (6.106) and the second ${}_2\phi_1$ can also be evaluated with the q-contiguous Kummer's formula (6.106) by simply changing a by $\frac{a}{q}$, we get

$$
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] \tag{6.127}
$$
$$
= \frac{q^2(b^2q^2-a)}{(bq^2-a)(1+bq^2)}\,\frac{(-q;q)_\infty}{\left(\frac{a}{bq};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}\left\{\left(1-\frac{a}{b^2q}\right)(aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\right.
$$
$$
\left.+\frac{1}{bq}(1+q)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)+\frac{1}{b^2q}\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\right\}
$$
$$
+\frac{1}{(1+bq^2)}\,\frac{(-q;q)_\infty}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}\left\{\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\right.
$$
$$
\left.+\frac{1}{bq}(1+q)\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2q^2};q^2\right)+\frac{1}{b^2q}\left(\frac{a}{q^2};q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\right\}.
$$

Noting that

$$\frac{q^2(b^2q^2-a)}{(bq^2-a)(1+bq^2)\left(\frac{a}{bq};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty} \doteq \frac{\left(1-\frac{a}{b^2q^2}\right)}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}, \tag{6.128}$$

we get

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] \qquad (6.129)\\
= \frac{(-q;q)_\infty\left(1-\frac{a}{b^2q^2}\right)}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}\Big\{\left(1-\frac{a}{b^2q}\right)(aq;q^2)_\infty\left(\frac{a}{b^2};q^2\right)_\infty \\
+\frac{1}{bq}(1+q)\left(a;q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty+\frac{1}{b^2q}\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\Big\} \\
+\frac{(-q;q)_\infty}{bq^2\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}\Big\{\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty \\
+\frac{1}{bq}(1+q)\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty+\frac{1}{b^2q}\left(\frac{a}{q^2};q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\Big\}.
\end{aligned}$$

Finally, using (6.122), we get

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] \qquad (6.130)\\
= \frac{(-q;q)_\infty}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}\Big\{\left(1-\frac{a}{b^2q}\right)(aq;q^2)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty \\
+\frac{1}{bq}(1+q)\left(a;q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty+\frac{1}{b^2q}\left(\frac{a}{b};q^2\right)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty\Big\} \\
+\frac{(-q;q)_\infty}{bq^2\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}\Big\{\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty \\
+\frac{1}{bq}(1+q)\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty+\frac{1}{b^2q}\left(\frac{a}{q^2};q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\Big\}
\end{aligned}$$

which after simplification gives

(6.131)

$$\begin{aligned}
{}_2\phi_1\left[\begin{matrix} a, & b, & \\ & & ;q,-\frac{q}{b} \\ \frac{a}{bq^2}, & & \end{matrix}\right] \\
= \frac{(-q;q)_\infty}{\left(\frac{a}{bq^2};q\right)_\infty\left(-\frac{1}{bq^2};q\right)_\infty}\Big\{\left(1-\frac{a}{b^2q}\right)(aq;q^2)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty \\
+\frac{1}{bq^2}(1+q+q^2)\left(1-\frac{a}{b^2q^2}\right)(a;q^2)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty \\
+\frac{1}{b^2q^3}(1+q+q^2)\left(\frac{a}{q};q^2\right)_\infty\left(\frac{a}{b^2q^2};q^2\right)_\infty+\frac{1}{b^3q^3}\left(\frac{a}{q^2};q^2\right)_\infty\left(\frac{a}{b^2q};q^2\right)_\infty\Big\}
\end{aligned}$$

which completes the derivation of (6.107). □

- In order to derive the result (6.111), we use the result (4.52) and select

$$\phi = {}_2\phi_1\left[\begin{matrix} a, & bq, \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{matrix}\right]$$

then after little simplification, we

$$(6.132) \qquad {}_2\phi_1\left[\begin{matrix} a, & bq^2, \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{matrix}\right]$$
$$= \frac{(a-b^2q^2)}{a(1-b^2q^2)}\, {}_2\phi_1\left[\begin{matrix} a, & bq, \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{matrix}\right]$$
$$- \frac{bq^2(1-a)(b^2-a)}{a(1-b^2q^2)(b-aq)}\, {}_2\phi_1\left[\begin{matrix} aq, & bq, \\ & & ; q, -\frac{q}{b} \\ \frac{aq^2}{b}, & \end{matrix}\right].$$

Now, it is easy to see that, first ${}_2\phi_1$ on the right-hand side of (6.132) can be evaluated with the help of the summation formula (6.110) and second ${}_2\phi_1$ can also be evaluated with the help of the formula (6.110) by simply changing a by aq, we get

(6.133)

$$ {}_2\phi_1\left[\begin{matrix} a, & bq^2, \\ & & ; q, -\frac{q}{b} \\ \frac{aq}{b}, & \end{matrix}\right]$$
$$= \frac{(a-b^2q^2)}{a(1-b^2q^2)} \frac{b(-q;q)_\infty}{a(1-b)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty} \left\{\left(a;q^2\right)_\infty\left(\frac{aq}{b^2};q\right)_\infty - \left(aq;q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\right\}$$
$$+ \frac{bq^2(1-a)(b^2-a)}{a(1-b^2q^2)(b-aq)} \frac{b(-q;q)_\infty}{aq(1-b)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty}$$
$$\left\{\left(aq;q^2\right)_\infty\left(\frac{aq^2}{b^2};q\right)_\infty - \left(aq^2;q^2\right)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty\right\}.$$

Noting that

$$\left(1-\frac{a}{b^2q^2}\right)\left(\frac{a}{b^2};q^2\right)_\infty = \left(\frac{a}{b^2q^2};q^2\right)_\infty,$$
$$\left(1+\frac{1}{bq}\right)\left(-\frac{1}{b};q\right)_\infty = \left(\frac{-1}{bq};q\right)_\infty,$$
$$\left(1-\frac{aq}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty = \left(\frac{aq}{b};q\right)_\infty$$

and

$$\left(1-\frac{a}{b^2}\right)\left(\frac{aq^2}{b^2};q^2\right)_\infty = \left(\frac{a}{b^2};q^2\right)_\infty$$

together with (6.122), we get

$$(6.134)\quad {}_2\phi_1\left[\begin{matrix} a, & bq^2, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$

$$= \frac{b^2(-q;q)_\infty}{a^2(1-bq)(1-b)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{b};q\right)_\infty}\left\{-q\left(1-\frac{a}{b^2q^2}\right)\left(a;q^2\right)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty\right.$$
$$+ q\left(aq;q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty + (1-a)\left(aq;q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty$$
$$\left.-\left(1-\frac{a}{b^2}\right)\left(a;q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\right\}$$

which after simplification

(6.135)

$${}_2\phi_1\left[\begin{matrix} a, & bq^2, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$

$$= \frac{b^2(-q;q)_\infty}{a^2(1-b)(1-bq)\left(\frac{aq}{b};q\right)_\infty\left(-\frac{1}{bq};q\right)_\infty}\left[\left(aq;q^2\right)_\infty\left(\frac{a}{b^2};q^2\right)_\infty\left\{q\left(1-\frac{a}{b^2q^2}\right)+(1-a)\right\}\right.$$
$$\left.-\left(a;q^2\right)_\infty\left(\frac{aq}{b^2};q^2\right)_\infty\left\{q\left(1-\frac{a}{b^2q^2}\right)+\left(1-\frac{a}{b^2}\right)\right\}\right].$$

This completes the derivation of (6.111). □

- In order to derive the result (6.115), we use the result (4.74) and select

$$\phi = {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$

then after little simplification, we

$$(6.136)\quad {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^2}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$

$$= {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right] + \frac{(1-a)}{q\left(1-\frac{aq}{b}\right)}\,{}_2\phi_1\left[\begin{matrix} aq, & \frac{b}{q}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq^2}{b}, & & \end{matrix}\right].$$

Now, it is easy to see that, first ${}_2\phi_1$ on the right-hand side of (6.136) can be evaluated with the help of the summation formula (6.114) and second ${}_2\phi_1$ can also be evaluated with the help of the summation formula (6.114) by simply changing a by aq, we get

(6.137)

$${}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^2}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$

$$= \frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(aq;q^2\right)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty + \left(a;q^2\right)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right\}$$

$$+\frac{(1-a)}{q\left(1-\frac{aq}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(aq^2;q^2\right)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty+\left(aq;q^2\right)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty\right\}.$$

Noting that,

$$\left(1-\frac{aq}{b}\right)\left(\frac{aq^2}{b};q\right)_\infty=\left(\frac{aq}{b};q\right)_\infty$$

$$\left(\frac{aq^2}{b};q^2\right)_\infty=\left(1-\frac{aq^2}{b}\right)\left(\frac{aq^4}{b};q^2\right)_\infty$$

we get

(6.138)

$$ {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^2}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$
$$=\frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(1-\frac{aq^2}{b^2}\right)\left(aq;q^2\right)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty+\left(a;q^2\right)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right\}$$
$$+\frac{(-q;q)_\infty}{q\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left\{\left(a;q^2\right)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty+(1-a)\left(aq;q^2\right)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty\right\}$$

which after simplification gives

$$ {}_2\phi_1\left[\begin{matrix} a, & \frac{b}{q^2}, & \\ & & ;q,-\frac{q}{b} \\ \frac{aq}{b}, & & \end{matrix}\right]$$
$$=\frac{(-q;q)_\infty}{\left(\frac{aq}{b};q\right)_\infty\left(-\frac{q}{b};q\right)_\infty}\left[\left(aq;q^2\right)_\infty\left(\frac{aq^4}{b^2};q^2\right)_\infty\left\{\left(1-\frac{aq^2}{b^2}\right)+\frac{(1-a)}{q}\right\}\right.$$
$$\left.+\frac{1}{q}(1+q)(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right].$$

This completes the derivation of (6.115). □

Similarly, other results can also be obtained

Remarks. (1) For q-Gauss second, Kummer and Bailey summation formulas, we refers the paper by Andrews [1].

(2) The results (6.100) and (6.101) were also obtained by Kim et al. [12] by following a different method.

(3) For q-contiguous Gauss's second summation formulas we prefer the paper by Kim and Rathie [10].

7. Concluding remark

In addition to 15 q-contiguous functions relations available in the literature, we have, in this paper, obtained 72 new and interesting q-contiguous functions relations. These relations have wide applications. Several new and interesting results by employing the q-contiguous functions relations given in the paper are under investigations and will form a part of subsequent paper in this direction.

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