Commun. Korean Math. Soc. **31** (2016), No. 1, pp. 53–64
http://dx.doi.org/10.4134/CKMS.2016.31.1.053

GENERALIZATION OF THE FEJÉR–HADAMARD'S INEQUALITY FOR CONVEX FUNCTION ON COORDINATES

Ghulam Farid and Atiq Ur Rehman

Abstract. In this paper, we give generalization of the Fejér–Hadamard inequality by using definition of convex functions on n-coordinates. Results given in [8, 12] are particular cases of results given here.

1. Introduction

Convex functions are important and provide a base to build literature of mathematical inequalities. A function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

where $\lambda \in [0,1]$, $x, y \in I$.

A bundle of inequalities in literature, are due to convex functions or functions related to convex functions see [4, 9, 15]. A classical inequality for convex functions is Hadamard inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \to \mathbb{R}$ is a convex function $a, b \in I$, $a < b$ (see [17, p. 137]).

In many areas of analysis, application of the Hadamard inequality appear for different classes of functions (see [1, 3, 6, 10, 18] for convex functions). Some useful mappings connected to this inequality are also defined by many authors, for example, see [2, 5, 10, 14]. In recent years, the concept of convexity has been extended and generalized in various directions. In this regards, very novel and innovative techniques are used by different authors (see, [11, 16]).

Received March 13, 2015.
2010 *Mathematics Subject Classification.* 26A51, 26D15, 65D30.
Key words and phrases. convex functions, Hadamard inequality, convex functions on coordinates.

In 1906, Fejér (see [13] and [17, p. 138]) established the following weighted generalization of the Hadamard inequality. The inequalities

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

hold for every convex function $f : I \to \mathbb{R}$, $a, b \in I$, and $g : [a, b] \to \mathbb{R}^+$ is symmetric about $(a + b)/2$.

In [8] S. S. Dragomir gave Hadamard inequality for rectangle in plane by defining convex functions on coordinates.

Definition 1.1. Let $\Delta^2 := [a, b] \times [c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta^2 \to \mathbb{R}$ will be called convex on coordinates if the partial mapping $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$ and $f_x : [c, d] \to \mathbb{R}$, $f_x(v) := f(x, v)$ are convex, where defined for all $y \in [c, d]$ and $x \in [a, b]$.

Theorem 1.2. *Let $f : \Delta^2 \to \mathbb{R}$ be a convex mapping on coordinates in Δ^2. Also let $g_1 : [a, b] \to \mathbb{R}^+$ and $g_2 : [c, d] \to \mathbb{R}^+$ be two integrable and symmetric functions about $(a+b)/2$ and $(c+d)/2$ respectively. Then one has the following inequalities*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)g_1(x)dx + \frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)g_2(y)dy\right]\\
&\le \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)\,dydx\\
&\le \frac{1}{4}\left[\frac{1}{G_1}\int_a^b g_1(x)f(x,c)dx + \frac{1}{G_1}\int_a^b g_1(x)f(x,d)dx\right.\\
&\quad \left. + \frac{1}{G_2}\int_c^d g_2(y)f(a,y)dy + \frac{1}{G_2}\int_c^d g_2(y)f(b,y)dy\right]\\
&\le \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right],
\end{aligned} \tag{3}$$

where

$$G_1 = \int_a^b g_1(x)dx \text{ and } G_2 = \int_c^d g_2(y)dy.$$

There in [12] some mappings connected to above inequality are also considered and their properties are discussed.

In [12] authors extended the definition of convex functions on coordinates to n-coordinates and gave the Hadamard's inequality for n-coordinates and related results. In this paper we give Fejér–Hadamard's inequality for convex functions on coordinates and show that results proved in [8, 12] are particular case of results in this paper.

2. Main results

For $n \geq 2$, let a_i, b_i $(i = 1, 2, \ldots, n)$ be real numbers such that $a_i < b_i$ for $i = 1, 2, \ldots, n$. We consider an n-dimensional interval Δ^n defined as $\Delta^n = \prod_{i=1}^{n} [a_i, b_i]$. In [12] the definition of a convex function on n-coordinates is given as follows:

Definition 2.1. Let $(x_1, \ldots, x_n) \in \Delta^n$. A mapping $f : \Delta^n \to \mathbb{R}$ is called convex on n-coordinates if the functions $f^i_{x_n}$, where $f^i_{x_n}(t) := f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$, are convex on $[a_i, b_i]$ for $i = 1, 2, \ldots, n$.

Recall that a mapping $f : \Delta^n \to \mathbb{R}$ is convex in Δ^n if for $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \Delta^n$ and $\alpha \in [0, 1]$, the following inequality holds:

$$f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y}).$$

It can be seen that every convex mapping $f : \Delta^n \to \mathbb{R}$ is convex on the n-coordinates but converse is not true.

Let $f : \Delta^n \to \mathbb{R}$ be convex in Δ^n. Consider $f^i_{x_n} : [a_i, b_i] \to \mathbb{R}$, defined by

$$f^i_{x_n}(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n), \quad t \in [a_i, b_i].$$

Now for $x, y \in [a_i, b_i]$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
&f^i_{x_n}(\alpha x + (1 - \alpha) y) \\
&= f(x_1, \ldots, x_{i-1}, \alpha x + (1 - \alpha) y, x_{i+1}, \ldots, x_n) \\
&= f(\alpha x_1 + (1 - \alpha) x_1, \ldots, \alpha x + (1 - \alpha) y, \ldots, \alpha x_n + (1 - \alpha) x_n) \\
&\leq \alpha f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) + (1 - \alpha) f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \\
&= \alpha f^i_{x_n}(x) + (1 - \alpha) f^i_{x_n}(y),
\end{aligned}$$

which implies $f^i_{x_n}$ is convex on $[a_i, b_i]$, that is, f is convex on n-coordinates. For converse we give the following counter example:

Example 2.2. Let us consider a mapping $f : [0, 1]^n \to \mathbb{R}$ defined as

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n.$$

It is convex on n-coordinates as follows:

$$\begin{aligned}
&f^i_{x_n}(\alpha x + (1 - \alpha) y) \\
&= x_1 \cdots x_{i-1} \cdot (\alpha x + (1 - \alpha) y) \cdot x_{i+1} \cdots x_n \\
&= \alpha(x_1 \cdots x_{i-1} \cdot x \cdot x_{i+1} \cdots x_n) + (1 - \alpha)(x_1 \cdots x_{i-1} \cdot y \cdot x_{i+1} \cdots x_n) \\
&= \alpha f^i_{x_n}(x) + (1 - \alpha) f^i_{x_n}(y).
\end{aligned}$$

But for $\mathbf{x} = (1, 1, \ldots, 1, 0), \mathbf{y} = (0, 1, 1, \ldots, 1) \in [0, 1]^n$, we have

$$\begin{aligned}
f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) &= f(\alpha, 1, 1, \ldots, 1 - \alpha) \\
&= \alpha(1 - \alpha)
\end{aligned}$$

and

$$\alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y}) = \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0.$$

This gives

$$f(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) > \alpha f(\mathbf{x})+(1-\alpha)f(\mathbf{y}) \quad \text{for all } \alpha\in(0,1),$$

that is, f is not convex on $[0,1]^n$.

It is interesting to note that if $f:\Delta^n\to\mathbb{R}$ is a convex mapping on n-coordinates, then $f^i_{x_n}:[a_i,b_i]\to\mathbb{R}$ is a convex function on $[a_i,b_i]$ for each $i=1,2,\dots,n$. Also if $g_i:[a_i,b_i]\to\mathbb{R}$ is a symmetric function about $\frac{a_i+b_i}{2}$, then from Fejér–Hadamard's inequality, we have

$$f^i_{x_n}\left(\frac{a_i+b_i}{2}\right)\le\frac{1}{G_i}\int_{a_i}^{b_i}f^i_{x_n}(x_i)g(x_i)dx_i,\quad i=1,2,\dots,n,$$

where

$$G_i=\int_{a_i}^{b_i}g_i(x_i)dx_i.$$

This gives us

$$\sum_{k=1}^{n}f^k_{x_n}\left(\frac{a_k+b_k}{2}\right)\le\sum_{k=1}^{n}\frac{1}{G_k}\int_{a_k}^{b_k}f^k_{x_n}(x_k)g_k(x_k)dx_k. \tag{4}$$

Theorem 2.3. *Let* $(x_1,\dots,x_n)\in\Delta^n$ *and* $f:\Delta^n\to\mathbb{R}$ *be a convex mapping on n-coordinates. Also, let* $g_i:[a_i,b_i]\to\mathbb{R}$ *be an integrable and symmetric function about* $\frac{a_i+b_i}{2}$ *for each* $i=1,\dots,n$*. Then we have*

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{G_k}\int_{a_k}^{b_k}f^{k+1}_{x_n}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)g_k(x_k)dx_k\\
&\le\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_k+1}^{b_k+1}f(\mathbf{x})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\
&\le\frac{1}{2}\sum_{k=1}^{n}\left[\frac{1}{G_k}\int_{a_k}^{b_k}\left(f^{k+1}_{x_n}(a_{k+1})+f^{k+1}_{x_n}(a_{k+1})\right)g_k(x_k)dx_k\right],
\end{aligned}\tag{5}$$

where

$$G_k=\int_{a_k}^{b_k}g_k(x_k)dx_k,$$

and with $n+1\mapsto 1$*. These inequalities are sharp.*

Proof. By applying the Fejér–Hadamard's inequality for convex function $f^{k+1}_{x_n}$ on interval $[a_{k+1},b_{k+1}]$ we have

$$\begin{aligned}
f^{k+1}_{x_n}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)G_{k+1}&\le\int_{a_{k+1}}^{b_{k+1}}f^{k+1}_{x_n}(x_{k+1})g_{k+1}(x_{k+1})dx_{k+1}\\
&\le\left(\frac{f^{k+1}_{x_n}(a_{k+1})+f^{k+1}_{x_n}(b_{k+1})}{2}\right)G_{k+1}.
\end{aligned}\tag{6}$$

Multiplying (6) by $g_k(x_k)$ we have

$$f_{x_n}^{k+1}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)g_k(x_k)G_{k+1} \leq \int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}$$
$$\leq \left(\frac{f_{x_n}^{k+1}(a_{k+1})+f_{x_n}^{k+1}(b_{k+1})}{2}\right)g_k(x_k)G_{k+1}.$$

Now by integrating on $[a_k, b_k]$ we get

$$G_{k+1}\int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)g_k(x_k)dx_k$$
$$\leq \int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k$$
$$\leq G_{k+1}\int_{a_{k+1}}^{b_{k+1}}\left(\frac{f_{x_n}^{k+1}(a_{k+1})+f_{x_n}^{k+1}(b_{k+1})}{2}\right)g_k(x_k)dx_k.$$

As $G_k > 0, G_{k+1} > 0$, then divide by G_kG_{k+1} we get

$$\frac{1}{G_k}\int_{a_k}^{b_k} f_{x_n}^{k+1}\left(\frac{a_{k+1}+b_{k+1}}{2}\right)g_k(x_k)dx_k$$
$$\leq \frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f_{x_n}^{k+1}(x_{k+1})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k$$
$$\leq \frac{1}{G_k}\int_{a_{k+1}}^{b_{k+1}}\left(\frac{f_{x_n}^{k+1}(a_{k+1})+f_{x_n}^{k+1}(b_{k+1})}{2}\right)g_k(x_k)dx_k.$$

Taking summation from 1 to n we get (5).

If we consider $f(x_1,\ldots,x_n) = x_1\cdots x_n$, then inequalities in (5) become equality, which shows these are sharp. □

Theorem 2.4. *Let $(x_1,\ldots,x_n) \in \Delta^n$ and $f : \Delta^n \to \mathbb{R}$ be a convex mapping on n-coordinates. Also, let $g_i : [a_i, b_i] \to \mathbb{R}$ be an integrable and symmetric function about $\frac{a_i+b_i}{2}$ for each $i = 1,\ldots,n$. Then we have*

$$\sum_{k=1}^{n}\frac{1}{G_k}\int_{a_k}^{b_k}\left(f_{a_n}^k(x_k)+f_{b_n}^k(x_k)\right)dx_k \tag{7}$$
$$\leq \frac{n}{2}\left(f(\boldsymbol{a})+f(\boldsymbol{b})\right)+\frac{1}{2}\sum_{k=1}^{n}\left(f_{a_n}^k(b_k)+f_{b_n}^k(a_k)\right),$$

where $\boldsymbol{a} = (a_1, a_2,\ldots,a_n)$ and $\boldsymbol{b} = (b_1, b_2,\ldots,b_n)$. The above inequality is sharp.

Proof. As $f : \Delta^n \to \mathbb{R}$ is a convex mapping on n-coordinates, therefore $f_{x_n}^i : [a_i, b_i] \to \mathbb{R}$ is convex on $[a_i, b_i]$ for each $i = 1, 2, 3,\ldots,n$. From Fejér inequality for each $i = 1, 2, 3, \ldots, n$ we have,

$$\frac{1}{G_i}\int_{a_i}^{b_i} f_{a_n}^i(x_i)dx_i g_i(x_i) \le \frac{f(\mathbf{a}) + f_{a_n}^i(b_i)}{2} \tag{8}$$

and

$$\frac{1}{G_i}\int_{a_i}^{b_i} f_{b_n}^i(x_i)dx_i g_i(x_i) \le \frac{f_{b_n}^i(a_i) + f(\mathbf{b})}{2}. \tag{9}$$

Adding (8) and (9) we get,

$$\begin{aligned}\frac{1}{G_i}\int_{a_i}^{b_i} &\left(f_{a_n}^i(x_i) + f_{b_n}^i(x_i)\right) g_i(x_i)dx_i \\ &\le \frac{1}{2}\left(f(\mathbf{a}) + f(\mathbf{b})\right) + \frac{1}{2}\left(f_{a_n}^i(b_i) + f_{b_n}^i(a_i)\right),\end{aligned} \tag{10}$$

where $i = 1, 2, \ldots, n$. Taking sum from 1 to n we get (7).

If we consider $f(x_1, \ldots, x_n) = x_1 \cdots x_n$, then inequalities in (7) become equality, which shows these are sharp. □

A special case of inequalities (4), (5), and (7) is stated in the following, which is main result of [12, Theorem 1].

Corollary 2.5. *Let $\Delta^2 = [a, b] \times [c, d]$ and $f : \Delta^2 \to \mathbb{R}$ be a convex mapping on 2-coordinates. Also, let $g_i : [a_i, b_i] \to \mathbb{R}$ be an integrable and symmetric function about $\frac{a_i+b_i}{2}$ for each $i = 1, 2$. Then* (3) *is valid.*

Proof. By putting $n = 2$ in Theorem 2.3 and Theorem 2.4, and taking $a_1 = a$, $b_1 = b$, $a_2 = c$, and $b_2 = d$, we get the required result. □

Remark 2.6. Further if we put $g_1(x) = 1$ and $g_2(x) = 1$, then we get main result of [8, Theorem 1].

3. Associated mappings

In this section we are interested to associate some mappings with the generalized Fejér–Hadamard inequality for a convex mapping on n-coordinates.

For $n \ge 2$, let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and A_i denotes arithmetic means of numbers a_i and b_i, that is,

$$A_i = A(a_i, b_i) = \frac{a_i + b_i}{2}.$$

Also for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta^n := \prod_{i=1}^n [a_i, b_i]$ and $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in [0, 1]^n$, we consider s_i be a point on a segment between x_i and A_i, that is,

$$s_i = t_i x_i + (1 - t_i)A_i.$$

For the mapping $f : \Delta^n \to \mathbb{R}$ defined in previous section, we can associate a mapping $\widehat{H} : [0;1]^n \to \mathbb{R}$ given by

$$\widehat{H}(\mathbf{t}) = \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(\mathbf{s}) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k,$$

where $\mathbf{s} = (s_1, s_2, \dots, s_n)$.
Consider $\widehat{H}^i_{t_n} : [0,1] \to \mathbb{R}$, defined by

$$\widehat{H}^i_{t_n}(t) = \widehat{H}(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n).$$

We can rewrite $\widehat{H}^i_{t_n}(t)$ as follows:

$$\begin{aligned}
&\widehat{H}^i_{t_n}(t) = \widehat{H}(t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n) \\
&= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \dots, s_{i-1}, tx_i + (1-t)A_i, s_{i+1}, \dots, s_n) \\
&\quad \times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\
&= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f^i_{s_n}(tx_i + (1-t)A_i) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\
&= \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f^i_{s_n}(\widehat{t}) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k,
\end{aligned}$$

where $\widehat{t} = tx_i + (1-t)A_i$. We will use this notation throughout the paper. We also need a following lemma given by Levin and Stečkin in [17, p. 200] to get desired results.

Lemma 3.1. *Let f be convex on $[a,b]$ and g be symmetric about $(a+b)/2$ and nonincreasing function on $[a,(a+b)/2]$. Then*

$$\int_a^b f(x)g(x)dx \ge \frac{1}{b-a} \int_a^b f(x)dx \int_a^b g(x)dx.$$

Theorem 3.2. *Let $f : \Delta^n \to \mathbb{R}$ be a convex mapping on n-coordinates on Δ^n. Then the mapping $\widehat{H}$ is convex on n-coordinates on $[0,1]^n$. We also have*

$$\widehat{H}(\boldsymbol{t}) \ge \sum_{k=1}^{n} f(s_1, \dots, s_{i-1}, A_k, A_{k+1}, \dots, s_n) \tag{11}$$

and

(12)
$$\begin{aligned}
\widehat{H}(\boldsymbol{t}) \le \; &\sum_{k=1}^{n} \frac{t_k + t_{k+1}(1-t_k)}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \dots, s_{k-1}, x_k, x_{k+1}, \dots, s_n) \\
&\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k
\end{aligned}$$

$$+\sum_{k=1}^{n}(1-t_k)(1-t_{k+1})f(s_1,\dots,s_{k-1},A_k,A_{k+1},\dots,s_n),$$

with $n+1\mapsto 1$.

Proof. Let $u,v\in[0,1]$ and $\alpha,\beta\in[0,1]$ such that $\alpha+\beta=1$, then we have

$$\begin{aligned}&\widehat{H}^i_{t_n}(\alpha u+\beta v)\\ &=\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}}f^i_{s_n}(\widehat{\alpha u+\beta v})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k.\end{aligned}$$

Now

$$\begin{aligned}\widehat{\alpha u+\beta v}&=(\alpha u+\beta v)x_i+(1-\alpha u-\beta v)A_i\\ &=\alpha(ux_i+(1-u)A_i)+\beta(vx_i+(1-v)A_i)\\ &=\alpha\widehat{u}+\beta\widehat{v}.\end{aligned}$$

This gives us

$$\begin{aligned}&\widehat{H}^i_{t_n}(\alpha u+\beta v)\\ &=\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}}f^i_{s_n}(\alpha\widehat{u}+\beta\widehat{v})g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k.\end{aligned}$$

Since given that $f^i_{s_n}$ is convex, therefore we have

$$\begin{aligned}&\widehat{H}^i_{t_n}(\alpha u+\beta v)\\ &\le\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}}(\alpha f^i_{s_n}(\widehat{u})+\beta f^i_{s_n}(\widehat{v}))g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\ &=\alpha\widehat{H}^i_{s_n}(\widehat{u})+\beta\widehat{H}^i_{s_n}(\widehat{v}).\end{aligned}$$

Which implies $\widehat{H}^i_{t_n}$ is convex, that is, $\widehat{H}$ is convex on n-coordinates.

To prove inequality (11), we consider

$$\begin{aligned}\widehat{H}(\mathbf{t})&=\sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}}f^i_{s_n}(s_i)g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\ &=\sum_{k=1}^{n}\frac{1}{G_{k+1}}\int_{a_{k+1}}^{b_{k+1}}\left[\frac{1}{G_k}\int_{a_k}^{b_k}f^i_{s_n}(s_i)g_k(x_k)g_{k+1}(x_{k+1})dx_k\right]dx_{k+1}.\end{aligned}$$

Since f is convex on the kth coordinate and $\frac{1}{G_k}\int_{a_k}^{b_k}g_k(x_k)dx_k=1$, we apply Jensen's inequality for integrals on kth coordinate to get

$$\begin{aligned}&H(\mathbf{t})\\ &\ge\sum_{k=1}^{n}\frac{1}{G_{k+1}}\int_{a_{k+1}}^{b_{k+1}}\left[f^k_{s_n}\left(\frac{1}{G_k}\int_{a_k}^{b_k}(s_i)g_k(x_k)dx_k\right)\right]g_{k+1}(x_{k+1})dx_{k+1}.\end{aligned}$$

Now it follows from Lemma 3.1, that

$$\widehat{H}(t) \geq \sum_{k=1}^{n} \frac{1}{G_{k+1}} \int_{a_{k+1}}^{b_{k+1}} f_{s_n}^{k}\left(\frac{a_k+b_k}{2}\right) g_{k+1}(x_{k+1})dx_{k+1}.$$

Now using convexity of f on $(k+1)$th coordinate. Again applying Jensen's inequality and Lemma 3.1 on $(k+1)$th coordinate we get inequality in (11).

Now to prove inequality (12), we first use convexity of f on kth coordinate, then on $(k+1)$th coordinate, we have

(13)

$$\begin{aligned}
&\widehat{H}(t)\\
&\leq \frac{t_k}{G_kG_{k+1}} \int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f_{s_n}^{k}(x_k)g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\
&\quad + \frac{1-t_k}{G_kG_{k+1}} \int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f_{s_n}^{k}\left(\frac{a_k+b_k}{2}\right) g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\
&\leq \frac{t_kt_{k+1}}{G_kG_{k+1}} \int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f(s_1,\dots,s_{i-1},x_k,x_{k+1},s_{k+2},\dots,s_n)\\
&\quad \times g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k\\
&\quad + \frac{t_k(1-t_{k+1})}{G_k}\int_{a_k}^{b_k} f(s_1,\dots,s_{i-1},x_k,\frac{a_{k+1}+b_{k+1}}{2},s_{k+2},\dots,s_n)g_k(x_k)dx_k\\
&\quad + \frac{t_{k+1}(1-t_k)}{G_{k+1}}\int_{a_{k+1}}^{b_{k+1}} f(s_1,\dots,s_{i-1},A_k,x_{k+1},s_{k+2},\dots,s_n)g_k(x_k)dx_k\\
&\quad + \frac{(1-t_k)(1-t_{k+1})}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_{k+1}}^{b_{k+1}} f(s_1,\dots,s_{k-1},A_k,A_{k+1},s_{k+2},\dots,s_n)\\
&\quad \times g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k.
\end{aligned}$$

Now by (4), we can have

(14)

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{G_{k+1}}\int_{a_{k+1}}^{b_{k+1}} f(s_1,\dots,A_k,x_{k+1},\dots,s_n)g_k(x_{k+1})dx_{k+1}\\
&\leq \sum_{k=1}^{n}\frac{1}{G_kG_{k+1}}\int_{a_k}^{b_k}\int_{a_k+1}^{b_k+1} f(s_1,\dots,x_k,x_{k+1},\dots,s_n)g_k(x_k)g_{k+1}(x_{k+1})dx_{k+1}dx_k
\end{aligned}$$

and from the first inequality in Theorem 2.3

(15)

$$\sum_{k=1}^{n}\frac{1}{G_k}\int_{a_k}^{b_k} f(s_1,\dots,x_k,A_{k+1},\dots,s_n)g_k(x_{k+1})dx_{k+1}$$

$$\leq \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k}\int_{a_k+1}^{b_k+1} f(s_1, \ldots, x_k, x_{k+1}, \ldots, s_n) g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k.$$

Now using the inequalities (14) and (15) in (13) we get (12). □

The particular case of above theorem is the following result, which is Theorem 2.4 given in [12].

Corollary 3.3. *Let $f : \Delta^2 \to \mathbb{R}$ be a convex function on 2-coordinates. Then the mapping $\widehat{H}$, defined as*

$$\widehat{H}(t,s) = \frac{1}{G_1 G_2} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) g_1(x) g_2(y) dy dx,$$

is convex on the coordinates on $[0,1]^2$. Further if g_1 is nonincreasing on $[a, (a+b)/2]$ and g_2 is nonincreasing on $[c, (c+d)/2]$, then

$$\inf_{(t,s)\in[0,1]^2} \widehat{H}(t,s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \widehat{H}(0,0)$$

and

$$\sup_{(t,s)\in[0,1]^2} \widehat{H}(t,s) = \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x,y) g_1(x) g_2(y) dy dx = \widehat{H}(1,1).$$

Proof. By putting $n = 2$ in Theorem 3.2, we get required result. □

Remark 3.4. Further if we take $g_1(x) = 1$ and $g_2(x) = 1$, then we get Theorem 2 in [8].

Theorem 3.5. *Let $f : \Delta^n \to \mathbb{R}$ be a convex mapping on Δ^n. Then the mapping $\widehat{H}$ is convex on $[0,1]^n$. Also the mapping $\widehat{h} : [0,1] \to \mathbb{R}$, defined by $\widehat{h}(t) = \widehat{H}(t, \ldots, t)$ is convex and one has the bounds*

$$\widehat{h}(\boldsymbol{t}) \geq \sum_{k=1}^{n} f(s_1, \ldots, s_{i-1}, A_k, A_{k+1}, \ldots, s_n) \tag{16}$$

and

$$\begin{aligned}\widehat{h}(\boldsymbol{t}) \leq & \sum_{k=1}^{n} \frac{t(2-t)}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f(s_1, \ldots, s_{k-1}, x_k, x_{k+1}, \ldots, s_n) \\ & \times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\ & + \sum_{k=1}^{n} (1-t)^2 f(s_1, \ldots, s_{k-1}, A_k, A_{k+1}, \ldots, s_n).\end{aligned} \tag{17}$$

Proof. Let $\mathbf{u}, \mathbf{v} \in [0,1]^n$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Then

$$\begin{aligned}\widehat{H}(\alpha\mathbf{u} + \beta\mathbf{v}) = & \sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f\left(\alpha \widehat{u_1 + \beta} v_1, \ldots, \alpha \widehat{u_n + \beta} v_n\right) \\ & \times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k.\end{aligned}$$

For each $i = 1, 2, \dots, n$, we have

$$\begin{aligned}\widehat{\alpha u_i + \beta v_i} &= (\alpha u_i + \beta v_i)x_i + (1 - \alpha u_i - \beta v_i)A_i \\ &= \alpha(u_i x_i + (1 - u_i)A_i) + \beta(v_i x_i + (1 - v_i)A_i) \\ &= \alpha\widehat{u}_i + \beta\widehat{v}_i.\end{aligned}$$

This gives

$$\begin{aligned}H(\alpha\mathbf{u} + \beta\mathbf{v}) = &\sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} f\left(\alpha\widehat{u}_1 + \beta\widehat{v}_1, \dots, \alpha\widehat{u}_n + \beta\widehat{v}_n\right) \\ &\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\ \leq &\sum_{k=1}^{n} \frac{1}{G_k G_{k+1}} \int_{a_k}^{b_k} \int_{a_{k+1}}^{b_{k+1}} (\alpha f(\widehat{u}_1, \dots, \widehat{u}_n) + \beta f(\widehat{v}_1, \dots, \widehat{v}_n)) \\ &\times g_k(x_k) g_{k+1}(x_{k+1}) dx_{k+1} dx_k \\ = &\,\alpha H(\mathbf{u}) + \beta H(\mathbf{v}).\end{aligned}$$

This shows that $\widehat{H}$ is convex on $[0,1]^n$. Similar to above, we can show that $\widehat{h}$ is convex on $[0,1]$ and using bounds of mapping $\widehat{H}$ in Theorem 3.2 we can get bounds of mapping $\widehat{h}$. □

The particular case of above theorem is the following result, which is Theorem 2.6 in [12].

Corollary 3.6. *Suppose that $f : \Delta^2 \to \mathbb{R}$ is a convex mapping on 2-coordinates. Let $\widehat{h} : [0,1] \to \mathbb{R}$ be the mapping defined as $\widehat{h}(t) = \widehat{H}(t,t)$, then $\widehat{h}$ is convex on coordinates on Δ. Also if g_1 is nonincreasing on $[a, (a+b)/2]$ and g_2 is nonincreasing on $[c, (c+d)/2]$, then*

$$\inf_{t \in [0,1]} \widehat{h}(t) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \widehat{H}(0,0)$$

and

$$\sup_{t \in [0,1]} h_{g_1 g_2}(t) = \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x,y) g_1(x) g_2(y) dy dx = H_{g_1 g_2}(1,1).$$

Proof. By putting $n = 2$ in Theorem (3.5), we get (3.6). □

Remark 3.7. Further if we take $g_1(x) = 1$ and $g_2(x) = 1$, the we obtain Theorem 3 in [8].

References

[1] S. Abramovich, G. Farid, and J. Pečarić, *More about Jensen's inequality and Cauchy's means for superquadratic functions*, J. Math. Inequal. **7** (2013), no. 1, 11–14.

[2] S. Banić, *Mappings connected with Hermite-Hadamard inequalities for superquadratic functions*, J. Math. Inequal. **3** (2009), no. 4, 577–589.

[3] S. I. Butt, J. Pečarić, and A. U. Rehman, *Non-symmetric Stolarsky means*, J. Math. Inequal. **7** (2013), no. 2, 227–237.

[4] F. Chen and S. Wu, *Hermite-Hadamard type inequalities for harmonically s-convex functions*, Sci. World J. **2014** (2014), no. 7, Article ID 279158.
[5] S. S. Dragomir, *A mapping in connection to Hadamard's inequalities*, Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. **128** (1991), no. 2, 17–20.
[6] ———, *On Hadamard's inequalities for convex functions*, Math. Balkanica (N.S.) **6** (1992), no. 3, 215–222.
[7] ———, *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl. **167** (1992), no. 1, 49–56.
[8] ———, *On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math. **5** (2001), no. 4, 775–788.
[9] ———, *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Proyecciones **34** (2015), no. 4, 323–341.
[10] S. S. Dragomir and A. McAndrew, *Refinements of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math. **6** (2005), no. 5, Article 140, 6 pp.
[11] S. S. Dragomir, J. E. Pečarić, and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), no. 3, 335–341.
[12] G. Farid, M. Marwan, and A. U. Rehman, *Fejér-Hadamard inequality for convex functions on the coordinates in a rectangle from the plane*, Int. J. Analysis Appl. **10** (2016), no. 1, 40–47.
[13] L. Fejér, *Über die Fourierreihen. II*, Math. Naturwiss Anz Ungar. Akad. Wiss. **24** (1906), 369–390.
[14] D.-Y. Hwang, K.-L. Tseng, and G.-S. Yang, *Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane*, Taiwanese J. Math. **11** (2007), no. 1, 63–73.
[15] M. A. Noor, K. I. Noor, and M. U. Awan, *Integral inequalities for coordinated harmonically convex functions*, Complex Var. Elliptic Equ. **60** (2015), no. 6, 776–786.
[16] M. A. Noor, F. Qi, and M. U. Awan, *Some Hermite-Hadamard type inequalities for log-h-convex functions*, Analysis (Berlin) **33** (2013), no. 4, 367–375.
[17] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
[18] M. Tunç, *On some new inequalities for convex functions*, Turkish J. Math. **36** (2012), no. 2, 245–251.

Ghulam Farid
COMSATS Institute of Information Technology
Attock Campus
Pakistan
E-mail address: faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk

Atiq Ur Rehman
COMSATS Institute of Information Technology
Attock Campus
Pakistan
E-mail address: atiq@mathcity.org