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HESITANT FUZZY SEMIGROUPS WITH TWO FRONTIERS

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Abstract. The notion of hesitant fuzzy semigroups with two frontiers is introduced, and related properties are investigated. Relations between a hesitant fuzzy semigroups with a frontier and a hesitant fuzzy semigroups with two frontiers are discussed. It is shown that the hesitant intersection of two hesitant fuzzy semigroups with two frontiers is a hesitant fuzzy semigroup with two frontiers. We provide an example to show that the hesitant union of two hesitant fuzzy semigroups with two frontiers may not be a hesitant fuzzy semigroup with two frontiers.

1. Introduction

A fuzzy set was introduced by Zadeh [14] in 1965. Since then, several extensions have been developed, such as intuitionistic fuzzy set, type-2 fuzzy set, type-n fuzzy set, fuzzy multiset and hesitant fuzzy sets. Intuitionistic fuzzy set has three main parts, that is, membership function, non-membership function and hesitancy function. As a useful generalization of the fuzzy set, Torra [7] introduced the hesitant fuzzy set which permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [7] and [8]). The hesitant fuzzy set permits the membership having a set of possible values, and provides a more accurate representation of hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making (see [6], [8], [9], [10], [11], [12], [13], and [15]). Jun et al. applied the notion of hesitant fuzzy sets to semigroups, MTL-algebras and EQ-algebras (see [1], [2], [3], [4] and [5]).

In this paper, we introduce the notion of hesitant fuzzy semigroups with two frontiers, and investigate several properties. We consider relations between a hesitant fuzzy semigroups with a frontier and a hesitant fuzzy semigroups with two frontiers. We show that the hesitant intersection of two hesitant fuzzy semigroups with two frontiers is a hesitant fuzzy semigroup with two frontiers. We provide an example to show that the hesitant union of two hesitant fuzzy

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semigroups with two frontiers may not be a hesitant fuzzy semigroup with two frontiers.

2. Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A nonempty subset A of S is called a subsemigroup of S if $AA \subseteq A$, that is, $ab \in A$ for all $a, b \in A$,

Torra [7] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. We display the basic notions on hesitant fuzzy sets. For more details, we refer to references.

Let S be a reference set. Then we define hesitant fuzzy set on S in terms of a function $\mathcal{G}$ that when applied to X returns a subset of $[0, 1]$.

For a hesitant fuzzy set $\mathcal{G}$ on S, we use the notations $\mathcal{G}_x := \mathcal{G}(x)$, $\mathcal{G}_x^y := \mathcal{G}(x) \cap \mathcal{G}(y)$, $\mathcal{G}_x(\lambda) := \mathcal{G}(x) \cap \lambda$, $\mathcal{G}_x^y(\lambda) := \mathcal{G}(x) \cap \mathcal{G}(y) \cap \lambda$, $\Sigma\{\tau, \lambda\} := \tau \cup \lambda$, $\Sigma\{\tau, \lambda\}(\mu) := (\tau \cup \lambda) \cap \mu$ for all $x, y \in S$ and $\lambda, \mu, \tau \in \mathscr{P}([0, 1])$. It is clear that $\mathcal{G}_x^y = \mathcal{G}_y^x$, $\mathcal{G}_x^y(\lambda) \subseteq \mathcal{G}_x(\lambda)$ and

$$\mathcal{G}_x = \mathcal{G}_y \;\Leftrightarrow\; \mathcal{G}_x \subseteq \mathcal{G}_y,\ \mathcal{G}_y \subseteq \mathcal{G}_x$$

for all $x, y \in S$.

For a hesitant fuzzy set $\mathcal{G}$ on S and a subset λ of $[0, 1]$, the set

$$S(\mathcal{G}; \lambda) := \{x \in S \mid \lambda \subseteq \mathcal{G}_x\}$$

is called the hesitant level set of $\mathcal{G}$.

Let $\mathcal{G}$ and $\mathcal{H}$ be two hesitant fuzzy sets on S. The hesitant union $\mathcal{G} \sqcup \mathcal{H}$ and hesitant intersection $\mathcal{G} \sqcap \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ are defined to be hesitant fuzzy sets on S as follows:

$$\mathcal{G} \sqcup \mathcal{H} : S \to \mathscr{P}([0, 1]),\ x \mapsto \mathcal{G}_x \cup \mathcal{H}_x \tag{1}$$

and

$$\mathcal{G} \sqcap \mathcal{H} : S \to \mathscr{P}([0, 1]),\ x \mapsto \mathcal{G}_x \cap \mathcal{H}_x, \tag{2}$$

respectively.

3. Hesitant fuzzy semigroups with two frontiers

In what follows, we take a semigroup S as a reference set unless otherwise specified.

Definition 3.1 ([1])**.** A hesitant fuzzy set $\mathcal{G}$ on a semigroup S is called a hesitant fuzzy semigroup on S if $\mathcal{G}_x^y \subseteq \mathcal{G}_{xy}$ for all $x, y \in S$.

Definition 3.2 ([5])**.** Let $\lambda \in \mathscr{P}([0,1])$ with $\lambda \neq \emptyset$. If a hesitant fuzzy set $\mathcal{G}$ on S satisfies $\mathcal{G}_x^y(\lambda) \subseteq \mathcal{G}_{xy}$ for all $x, y \in S$, then we say that λ is a frontier of $\mathcal{G}$ and $\mathcal{G}$ is a hesitant fuzzy semigroup with a frontier λ (briefly, λ-hesitant fuzzy semigroup) on S.

Definition 3.3. Let $\lambda, \mu \in \mathscr{P}([0,1])$ with $\lambda \neq [0,1]$ and $\mu \neq \emptyset$. A hesitant fuzzy set $\mathcal{G}$ on S is called a hesitant fuzzy semigroup with two frontiers λ and μ (briefly, (λ, μ)-hesitant fuzzy semigroup) of S if $\mathcal{G}_x^y(\mu) \subseteq \Sigma\{\mathcal{G}_{xy}, \lambda\}$ for all $x, y \in S$.

Example 3.4. Let $S = \{a, b, c\}$ be a semigroup with the Cayley table which is appeared in Table 1.

TABLE 1. Cayley table for the multiplication

$\cdot$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Let $\mathcal{G}$ be a hesitant fuzzy set on S defined as follows:

$$\mathcal{G} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} (0,1) & \text{if } x = a, \\ (0.2, 0.5) \cup (0.6, 0.8] & \text{if } x = b, \\ [0.3, 0.5) & \text{if } x = c. \end{cases}$$

It is routine to verify that $\mathcal{G}$ is a (λ, μ)-hesitant fuzzy semigroup on S with $\lambda = (0.3, 0.5)$ and $\mu = [0.3, 0.5)$.

Example 3.5. Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table which is appeared in Table 2.

TABLE 2. Cayley table for the multiplication

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	a	d

Let $\mathcal{G}$ be a hesitant fuzzy set on S defined as follows:

$$\mathcal{G} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.1, 0.2, 0.3, 0.6\} & \text{if } x = a, \\ (0,1) & \text{if } x = b, \\ \{0.6, 0.7, 0.8\} & \text{if } x \in \{c, d\}. \end{cases}$$

Then $\mathcal{G}$ is a (λ, μ)-hesitant fuzzy semigroup on S with $\lambda = \{0.1, 0.2, 0.3, 0.8\}$ and $\mu = \{0.3, 0.6, 0.8\}$.

For any $\lambda, \mu \in \mathscr{P}([0,1])$ with $\lambda \neq [0,1]$ and $\mu \neq \emptyset$, the following assertions are clearly valid.

(1) Every μ-hesitant fuzzy semigroup on S is both a (λ, μ)-hesitant fuzzy semigroup on S and a $\lambda \cap \mu$-hesitant fuzzy semigroup on S .
(2) Every λ-hesitant fuzzy semigroup on S is a $\lambda \cap \mu$-hesitant fuzzy semigroup on S.
(3) Every $\lambda \cup \mu$-hesitant fuzzy semigroup on S is both a λ-hesitant fuzzy semigroup and a μ-hesitant fuzzy semigroup on S.
(4) Every $\lambda \cup \mu$-hesitant fuzzy semigroup on S is a $\lambda \cap \mu$-hesitant fuzzy semigroup on S.
(5) Every $\lambda \cup \mu$-hesitant fuzzy semigroup on S is a (λ, μ)-hesitant fuzzy semigroup on S.

Theorem 3.1. *Let λ and μ be elements of $\mathscr{P}([0,1])$ such that $\lambda \neq [0,1]$, $\mu \neq \emptyset$, $\lambda \cap \mu \neq \emptyset$, $\lambda \nsubseteq \mu$ and $\mu \nsubseteq \lambda$. If $\mathcal{G}$ is both a $\lambda \cap \mu$-hesitant fuzzy semigroup of S and a (λ, μ)-hesitant fuzzy semigroup of S, then it is a μ-hesitant fuzzy semigroup of S.*

Proof. Let $\mathcal{G}$ be both a (λ, μ)-hesitant fuzzy semigroup of S and a $\lambda \cap \mu$-hesitant fuzzy semigroup of S. For any $x, y \in S$, we have $\mathcal{G}_{xy} \supseteq \mathcal{G}_x^y(\lambda \cap \mu)$ and

$$\mathcal{G}_{xy} \supseteq \Sigma\{\mathcal{G}_{xy}, \lambda\}(\lambda^c) \supseteq \mathcal{G}_x^y(\mu \cap \lambda^c).$$

Hence $\mathcal{G}_{xy} \supseteq \Sigma\{\mathcal{G}_x^y(\lambda \cap \mu), \mathcal{G}_x^y(\mu \cap \lambda^c)\} = G_x^y(\mu)$. Therefore $\mathcal{G}$ is a μ-hesitant fuzzy semigroup on S. □

We provide conditions for a (λ, μ)-hesitant fuzzy semigroup to be a μ-hesitant fuzzy semigroup.

Theorem 3.2. *For any $\lambda, \mu \in \mathscr{P}([0,1])$ with $\lambda \neq [0,1]$ and $\mu \neq \emptyset$, if λ and μ are disjoint, then every (λ, μ)-hesitant fuzzy semigroup on S is a μ-hesitant fuzzy semigroup on S.*

Proof. Let $\mathcal{G}$ be a (λ, μ)-hesitant fuzzy semigroup on S. Then $\mathcal{G}_x^y(\mu) \subseteq \Sigma\{\mathcal{G}_{xy}, \lambda\}$ for all $x, y \in S$. If $n \in \mathcal{G}_x^y(\mu)$, then $n \in \Sigma\{\mathcal{G}_{xy}, \lambda\}$ and $n \in \mu$. Since λ and μ are disjoint, it follows that $n \notin \lambda$ and that $n \in \mathcal{G}_{xy}$. Hence $\mathcal{G}_x^y(\mu) \subseteq \mathcal{G}_{xy}$ for all $x, y \in S$. Therefore $\mathcal{G}$ is a μ-hesitant fuzzy semigroup on S. □

If λ and μ are not disjoint in Theorem 3.2, then Theorem 3.2 is false as seen in the following example.

Example 3.6. Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table which is appeared in Table 3.
Let $\mathcal{G}$ be a hesitant fuzzy set on S defined as follows:

$$\mathcal{G} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.1, 0.2, 0.3\} & \text{if } x = a, \\ \{0.4, 0.5\} & \text{if } x = b, \\ \{0.2, 0.3, 0.4\} & \text{if } x \in \{c, d\}. \end{cases}$$

TABLE 3. Cayley table for the multiplication

$\cdot$	a	b	c	d
a	b	b	a	b
b	b	b	b	b
c	a	b	c	b
d	b	b	d	b

Let $\lambda = \{0.1, 0.2, 0.3\}$ and $\mu = \{0.1, 0.4\}$. Then $\mathcal{G}$ is a (λ, μ)-hesitant fuzzy semigroup on S. But it is not a μ-hesitant fuzzy semigroup on S since

$$\mathcal{G}_{aa} = \mathcal{G}_b = \{0.4, 0.5\} \not\supseteq \{0.1\} = \mathcal{G}_a^a(\mu).$$

Theorem 3.3. *The hesitant intersection of two (λ, μ)-hesitant fuzzy semigroups on S is a (λ, μ)-hesitant fuzzy semigroup on S.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ be (λ, μ)-hesitant fuzzy semigroups on S. For any $x, y \in S$, we have

$$\begin{aligned} \Sigma\{(\mathcal{G} \sqcap \mathcal{H})_{xy}, \lambda\} &= \Sigma\{\mathcal{G}_{xy} \cap \mathcal{H}_{xy}, \lambda\} \\ &= \Sigma\{\mathcal{G}_{xy}, \lambda\} \, (\Sigma\{\mathcal{H}_{xy}, \lambda\}) \\ &\supseteq \mathcal{G}_x^y(\mu) \cap \mathcal{H}_x^y(\mu) \\ &= (\mathcal{G} \sqcap \mathcal{H})_x^y(\mu). \end{aligned}$$

Hence $\mathcal{G} \sqcap \mathcal{H}$ is a (λ, μ)-hesitant fuzzy semigroup on S. □

The converse of Theorem 3.3 is not true in general as seen in the following example.

Example 3.7. Let $S = \{a, b, c\}$ be a semigroup with the Cayley table which is appeared in Table 4.

TABLE 4. Cayley table for the multiplication

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	b

Let $\mathcal{G}$ and $\mathcal{H}$ be hesitant fuzzy sets on S given by

$$\mathcal{G} : S \to \mathscr{P}([0,1]), \; x \mapsto \begin{cases} \{0.2, 0.3, 0.4\} & \text{if } x = a, \\ (0, 1] & \text{if } x = b, \\ \{0.1\} & \text{if } x = c, \end{cases}$$

and

$$\mathcal{H} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.2, 0.3, 0.5\} & \text{if } x = a, \\ \{0.1, 0.2, 0.3, 0.5\} & \text{if } x = b, \\ \{0.1, 0.4\} & \text{if } x = c, \end{cases}$$

respectively. The hesitant intersection $\mathcal{G} \sqcap \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ is given as follows:

$$\mathcal{G} \sqcap \mathcal{H} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.2, 0.3\} & \text{if } x = a, \\ \{0.1, 0.2, 0.3, 0.5\} & \text{if } x = b, \\ \{0.1\} & \text{if } x = c. \end{cases}$$

It is easy to verify that $\mathcal{G}$ and $\mathcal{G} \sqcap \mathcal{H}$ are (λ, μ)-hesitant fuzzy semigroups on S. But, if we take $\lambda := \{0.1\}$ and $\mu := \{0.1, 0.4, 0.5\}$, then $\mathcal{H}$ is not a (λ, μ)-hesitant fuzzy semigroup on S since $\Sigma\{\mathcal{H}_{cc}, \lambda\} = \Sigma\{\mathcal{H}_b, \lambda\} = \{0.1, 0.2, 0.3, 0.5\} \not\supseteq \{0.1, 0.4\} = \mathcal{H}_c^c(\mu)$.

The following example shows that the hesitant union of two (λ, μ)-hesitant fuzzy semigroups on S is not a (λ, μ)-hesitant fuzzy semigroup on S in general.

Example 3.8. Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table which is appeared in Table 5.

TABLE 5. Cayley table for the multiplication

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Let $\mathcal{G}$ and $\mathcal{H}$ be hesitant fuzzy sets on S defined as follows:

$$\mathcal{G} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.2, 0.3\} & \text{if } x = a, \\ \{0.3, 0.4\} & \text{if } x = b, \\ (0, 1] & \text{if } x = c, \\ \{0.2, 0.4, 0.5\} & \text{if } x = d, \end{cases}$$

and

$$\mathcal{H} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.2, 0.3\} & \text{if } x = a, \\ (0, 1) & \text{if } x = b, \\ \{0.2, 0.4\} & \text{if } x = c, \\ \{0.2, 0.3, 0.5\} & \text{if } x = d, \end{cases}$$

respectively. Then $\mathcal{G}$ and $\mathcal{H}$ are (λ, μ)-hesitant fuzzy semigroups on S with $\lambda = \{0.4\}$ and $\mu = \{0.1, 0.2, 0.3, 0.4\}$. But

$$\begin{aligned} \Sigma\{(\mathcal{G} \sqcup \mathcal{H})_{bc}, \lambda\} &= \Sigma\{(\mathcal{G} \sqcup \mathcal{H})_a, \lambda\} = \{0.2, 0.3, 0.4\} \\ &\not\supseteq \{0.1, 0.2, 0.3, 0.4\} = (\mathcal{G} \sqcup \mathcal{H})_b^c(\mu). \end{aligned}$$

Hence the hesitant union $\mathcal{G} \sqcup \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ is not a (λ, μ)-hesitant fuzzy semigroup on S.

Let λ and μ be elements of $\mathscr{P}([0,1])$ such that $\lambda \neq [0,1]$, $\mu \neq \emptyset$ and $\mu \nsubseteq \lambda$. For a hesitant fuzzy set $\mathcal{G}$ on S and fixed $w \in S$, we consider a set

$$S^w_{(\lambda,\mu)} := \{x \in S \mid \Sigma\{\mathcal{G}_x, \lambda\} \supseteq \mathcal{G}_w(\mu)\}$$

which is called a $(\lambda, \mu)^w$-set with respect to w under $\mathcal{G}$.

In the following example, we know that the $(\lambda, \mu)^w$-set $S^w_{(\lambda,\mu)}$ is a subsemigroup of S for some $w = a \in S$, but not a subsemigroup of S for some $w = b \in S$.

Example 3.9. Let $S = \{a, b, c, d\}$ be a semigroup with the Cayley table which is appeared in Table 6.

TABLE 6. Cayley table for the multiplication

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	d	d

Let $\mathcal{G}$ be a hesitant fuzzy set on S defined as follows:

$$\mathcal{G} : S \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \{0.1, 0.2, 0.3, 0.4, 0.5\} & \text{if } x = a, \\ \{0.2, 0.4, 0.6, 0.8, 1\} & \text{if } x = b, \\ \{0.2, 0.8, 1\} & \text{if } x = c, \\ \{0.5, 0.7, 0.8, 1\} & \text{if } x = d. \end{cases}$$

Given $\lambda := \{0.3, 0.4, 0.8\}$ and $\mu := \{0.2, 0.3, 0.4, 0.8, 1\}$, $\mathcal{G}$ is not a (λ, μ)-hesitant fuzzy semigroup on S since

$$\Sigma\{\mathcal{G}_{db}, \lambda\} = \{0.1, 0.2, 0.3, 0.4, 0.5\} \cup \{0.3, 0.4, 0.8\} \not\supseteq \{0.8, 1\} = \mathcal{G}^b_d(\mu).$$

We know that the set $S^a_{(\lambda,\mu)} = \{a, b, c\}$ is a subsemigroup of S. But $S^b_{(\lambda,\mu)} = S^c_{(\lambda,\mu)} = \{b, c\}$ and $S^d_{(\lambda,\mu)} = \{b, c, d\}$ are not subsemigroups of S.

Definition 3.10. Let λ and μ be elements of $\mathscr{P}([0,1])$ such that $\lambda \neq [0,1]$, $\mu \neq \emptyset$ and $\mu \nsubseteq \lambda$. Given a hesitant fuzzy set $\mathcal{G}$ on S, if the $(\lambda, \mu)^w$-set $S^w_{(\lambda,\mu)}$ is a subsemigroup of S for all $w \in S$, then we say that $S^w_{(\lambda,\mu)}$ is a (λ, μ)-subsemigroup of S.

We provide conditions for the $(\lambda, \mu)^w$-set to be a subsemigroup for all $w \in S$.

Theorem 3.4. *Let λ and μ be elements of $\mathscr{P}([0,1])$ such that $\lambda \neq [0,1]$, $\mu \neq \emptyset$ and $\mu \nsubseteq \lambda$. If $\mathcal{G}$ is a (λ, μ)-hesitant fuzzy semigroup on S, then the nonempty $(\lambda, \mu)^w$-set is a subsemigroup of S for all $w \in S$.*

Proof. Let $w \in S$ and assume that $S^w_{(\lambda,\mu)} \neq \emptyset$. Let $x, y \in S^w_{(\lambda,\mu)}$. Then $\Sigma\{\mathcal{G}_x, \lambda\} \supseteq \mathcal{G}_w(\mu)$ and $\Sigma\{\mathcal{G}_y, \lambda\} \supseteq \mathcal{G}_w(\mu)$. It follows that

$$\Sigma\{\mathcal{G}_{xy}, \lambda\} \supseteq \Sigma\{\mathcal{G}^y_x(\mu), \lambda\} = \Sigma\{\mathcal{G}_x, \lambda\} \left(\Sigma\{\mathcal{G}_y, \lambda\}(\mu \cup \lambda)\right) \supseteq \mathcal{G}_w(\mu).$$

Hence $xy \in S^w_{(\lambda,\mu)}$ and $S^w_{(\lambda,\mu)}$ is a subsemigroup of S. □

Theorem 3.5. *Let λ and μ be elements of $\mathscr{P}([0,1])$ such that $\lambda \neq [0,1]$, $\mu \neq \emptyset$ and $\lambda \subseteq \mu$. If $\mathcal{G}$ is a (λ,μ)-hesitant fuzzy semigroup on S, then the set*

$$S_a := \{x \in S \mid \Sigma\{\mathcal{G}_x(\mu), \lambda\} \supseteq \Sigma\{\mathcal{G}_a(\mu), \lambda\}\}$$

is a subsemigroup of S for all $a \in S$.

Proof. Note that $a \in S_a$ for all $a \in S$. Let $x, y \in S_a$. Then

$$\Sigma\{\mathcal{G}_x(\mu), \lambda\} \supseteq \Sigma\{\mathcal{G}_a(\mu), \lambda\} \text{ and } \Sigma\{\mathcal{G}_y(\mu), \lambda\} \supseteq \Sigma\{\mathcal{G}_a(\mu), \lambda\}.$$

Since $\lambda \subseteq \mu$, it follows that

$$\begin{aligned}\Sigma\{\mathcal{G}_a(\mu), \lambda\} &\subseteq \Sigma\{\mathcal{G}_x(\mu), \lambda\} \left(\Sigma\{\mathcal{G}_y(\mu), \lambda\}\right) \\ &= \Sigma\{\mathcal{G}^y_x(\mu), \lambda\} \\ &\subseteq \Sigma\{\Sigma\{\mathcal{G}_{xy}, \lambda\}(\mu), \lambda\} \\ &= \Sigma\{\mathcal{G}_{xy}(\mu), \lambda\}.\end{aligned}$$

Thus $xy \in S_a$, and S_a is a subsemigroup of S for all $a \in S$. □

Theorem 3.6. *If $\mathcal{G}$ is a (λ,μ)-hesitant fuzzy semigroup on S, then the hesitant level set $S(\mathcal{G};\gamma)$ of $\mathcal{G}$ is a subsemigroup of S for all $\gamma \in \mathscr{P}([0,1])$ with $\gamma \subseteq \mu \setminus \lambda$.*

Proof. Let γ be an element of $\mathscr{P}([0,1])$ such that $\gamma \subseteq \mu \setminus \lambda$. Assume that $\mathcal{G}$ is a (λ,μ)-hesitant fuzzy semigroup on S. Let $x, y \in S(\mathcal{G};\gamma)$. Then $\mathcal{G}_x \supseteq \gamma$ and $\mathcal{G}_y \supseteq \gamma$, which imply that $\Sigma\{\mathcal{G}_{xy}, \lambda\} \supseteq \mathcal{G}^y_x(\mu) \supseteq \gamma \cap \mu = \gamma$. Since $\lambda \cap \gamma = \emptyset$, it follows that $\mathcal{G}_{xy} \supseteq \gamma$, that is, $xy \in S(\mathcal{G};\gamma)$. Therefore $S(\mathcal{G};\gamma)$ is a subsemigroup of S for all $\gamma \in \mathscr{P}([0,1])$ with $\gamma \subseteq \mu \setminus \lambda$. □

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