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A NOTE ON THE MODIFIED k-FIBONACCI-LIKE SEQUENCE

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Abstract. The Fibonacci sequence is a sequence of numbers that has been studied for hundreds of years. In this paper, we introduce the modified k-Fibonacci-like sequence and prove Binet's formula for this sequence and then use it to introduce and prove the Catalan, Cassini, and d'Ocagne identities for the modified k-Fibonacci-like sequence. Also, the ordinary generating function of this sequence is stated.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 \tag{1}$$

with $F_0 = 0$ and $F_1 = 1$.

Many authors have studied the Fibonacci sequence, some of whom introduced new sequences related to it as well as proving many identities for them.

Falcón and Plaza [6] introduced the k-Fibonacci sequence and proved some related identities. For any real number k, the k-Fibonacci sequence $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined by the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad \text{for } n \geq 2 \tag{2}$$

with $F_{k,0} = 0$ and $F_{k,1} = 1$.

Edson and Yayenie [4] introduced the generalized Fibonacci sequence and proved some related identities. For any two nonzero real numbers a and b, the generalized Fibonacci sequence $\{q_n\}_{n=0}^{\infty}$ is defined by the recurrence relation

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even,} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad (n \geq 2), \tag{3}$$

with $q_0 = 0$ and $q_1 = 1$.

Edson, Lewis and Yayenie [3] introduced the k-periodic Fibonacci sequence and proved some related identities. For any k-tuple $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k$, the

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k-periodic Fibonacci sequence $\{q_n\}$ is defined by the recurrence relation

$$q_n = \begin{cases} x_1 q_{n-1} + q_{n-2}, & \text{if } n \equiv 2 \pmod{k} \\ x_2 q_{n-1} + q_{n-2}, & \text{if } n \equiv 3 \pmod{k} \\ \vdots & \\ x_{k-1} q_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod{k} \\ x_k q_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod{k} \end{cases} \quad (n \geq 2) \tag{4}$$

with $q_0 = 0$ and $q_1 = 1$.

Some authors introduced various Fibonacci-like sequences:

Singh, Sikhwal and Bhatnagar [11] introduced the Fibonacci-like sequence and proved several related identities. The Fibonacci-like sequence $\{S_n\}$ is defined by the recurrence relation

$$S_n = S_{n-1} + S_{n-2} \quad \text{for } n \geq 2 \tag{5}$$

with $S_0 = 2$ and $S_1 = 2$.

Badshah, Teeth and Dar [1] introduced a generalized Fibonacci-like sequence and proved some related identities. Their generalized Fibonacci-like sequence $\{M_n\}$ is defined by the recurrence relation

$$M_n = M_{n-1} + M_{n-2} \quad \text{for } n \geq 2 \tag{6}$$

with $M_0 = 2m$ and $M_1 = 1 + m$, m being a fixed positive integer.

Harne, Singh and Pal [7] introduced another generalized Fibonacci-like sequence and proved some related identities. Their generalized Fibonacci-like sequence $\{D_n\}$ is defined by the recurrence relation

$$D_n = D_{n-1} + D_{n-2} \quad \text{for } n \geq 2 \tag{7}$$

with $D_0 = 2$ and $D_1 = 1 + m$, m being a fixed positive integer.

Finally, Panwar, Rathore and Chawla [9] introduced the k-Fibonacci-like sequence and proved some related identities. For any positive real number k, the k-Fibonacci-like sequence $\{S_{k,n}\}$ is defined by the recurrence relation

$$S_{k,n} = kS_{k,n-1} + S_{k,n-2} \quad \text{for } n \geq 2 \tag{8}$$

with $S_{k,0} = 2$ and $S_{k,1} = 2k$.

Furthermore, the above authors utilize Binet's formula, a well-known tool for proving various identities. Thus, when authors study a new sequence related to the Fibonacci sequence, they always introduce Binet's formula for each sequence. As you know, most of identities for each sequence is that the left-hand side of equation is expressed using itself. For example, the Catalan identity of the k-Fibonacci sequence is

$$F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,r}^2,$$

where $F_{k,n}$ is the k-Fibonacci sequence.

In this paper, we introduce the modified k-Fibonacci-like sequence and prove Binet's formula for the modified k-Fibonacci-like sequence. And then using it, we prove the Catalan, Cassini, and d'Ocagne identities for this sequence. Moreover, we introduce the special sums of the modified k-Fibonacci-like sequence and prove them using Binet's formula.

In Section 2, we will introduce the modified k-Fibonacci-like sequence and related facts.

In Section 3, we will introduce and prove Binet's formula of the modified k-Fibonacci-like sequence.

In Section 4, we will introduce the Catalan, Cassini, and d'Ocagne identities for the modified k-Fibonacci-like sequence. In these identities, we can find that the left-hand side of equation is expressed using the k-Fibonacci sequence. Moreover, we will prove them using Binet's formula. We will also introduce and prove the sums of the modified k-Fibonacci-like sequence.

In Section 5, we will find the generating function of the modified k-Fibonacci-like sequence.

In Section 6, we will generalize the modified k-Fibonacci-like sequence for $S_{k,0} = S_{k,1} = a$, where a is an integer.

2. Preliminaries

In this section, we review basic definitions and introduce relevant facts.

Definition (The k-Fibonacci sequence [6])**.** For any positive real number k, the k-Fibonacci sequence $\{F_{k,n}\}$ is defined by the recurrence relation

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \quad \text{for } n \geq 2$$

with $F_{k,0} = 0$ and $F_{k,1} = 1$.

A few k-Fibonacci numbers are

$$F_{k,2} = k,\ F_{k,3} = k^2 + 1,\ F_{k,4} = k^3 + 2k,\ F_{k,5} = k^4 + 3k^2 + 1,\ \ldots .$$

Theorem 2.1 (Binet's formula for the k-Fibonacci sequence [6])**.** *The nth k-Fibonacci number is given by*

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{9}$$

where α, β are the roots of the characteristic equation $x^2 - kx - 1 = 0$, and $\alpha > \beta$.

Definition (The k-Lucas sequence [5])**.** For any positive real number k, the k-Lucas sequence $\{L_{k,n}\}$ is defined by the recurrence relation

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2} \quad \text{for } n \geq 2$$

with $L_{k,0} = 2$ and $L_{k,1} = k$.

A few k-Lucas numbers are

$$L_{k,2} = k^2 + 2,\ L_{k,3} = k^3 + 3k,\ L_{k,4} = k^4 + 4k^2 + 2,\ \ldots .$$

Theorem 2.2 (Binet's formula for the k-Lucas sequence [5])**.** *The k-Lucas numbers are given by the formula*

$$L_{k,n} = \alpha^n + \beta^n, \tag{10}$$

where α, β are the roots of the characteristic equation $x^2 - kx - 1 = 0$, and $\alpha > \beta$.

Now, we will introduce the new sequence, called the modified k-Fibonacci-like sequence. This sequence contains features both of the k-Fibonacci sequence and the Fibonacci-like sequence.

Definition (The modified k-Fibonacci-like sequence)**.** For any positive real number k, the *modified k-Fibonacci-like sequence* $\{M_{k,n}\}$ is defined by the recurrence relation

$$M_{k,n} = kM_{k,n-1} + M_{k,n-2} \quad \text{for } n \geq 2$$

with $M_{k,0} = 2$ and $M_{k,1} = 2$.

The first few modified k-Fibonacci-like numbers are

$$\begin{aligned}
M_{k,2} &= 2k + 2, \\
M_{k,3} &= 2k^2 + 2k + 2, \\
M_{k,4} &= 2k^3 + 2k^2 + 4k + 2, \\
M_{k,5} &= 2k^4 + 2k^3 + 6k^2 + 4k + 2, \\
M_{k,6} &= 2k^5 + 2k^4 + 8k^3 + 6k^2 + 6k + 2.
\end{aligned}$$

Example 2.3 (Case $k = 1$)**.** *We obtain*

$$M_{1,0} = 2, M_{1,1} = 2, \text{ and } M_{1,n} = M_{1,n-1} + M_{1,n-2} \quad \text{for } n \geq 2:$$

$$\{M_{1,n}\}_{n\in\mathbb{N}} = \{2, 2, 4, 6, 10, 16, \ldots\}.$$

This sequence is the Fibonacci-like sequence in [11]*.*

Example 2.4 (Case $k = 2$)**.** *We obtain*

$$M_{2,0} = 2, M_{2,1} = 2, \text{ and } M_{2,n} = 2M_{2,n-1} + M_{2,n-2} \quad \text{for } n \geq 2:$$

$$\{M_{2,n}\}_{n\in\mathbb{N}} = \{2, 2, 6, 14, 34, 82, \ldots\}.$$

Example 2.5 (Case $k = \frac{1}{2}$)**.** *We obtain*

$$M_{\frac{1}{2},0} = 2, M_{\frac{1}{2},1} = 2, \text{ and } M_{\frac{1}{2},n} = \frac{1}{2}M_{\frac{1}{2},n-1} + M_{\frac{1}{2},n-2} \quad \text{for } n \geq 2:$$

$$\{M_{\frac{1}{2},n}\}_{n\in\mathbb{N}} = \left\{2, 2, 3, \frac{7}{2}, \frac{19}{4}, \frac{47}{8}, \ldots\right\}.$$

3. Binet's formula of the modified k-Fibonacci-like sequence

Binet's formulae are well known in the study of sequences like Fibonacci sequence [1, 2, 3, 4, 6, 7, 8, 10, 11, 12]. In this section, we introduce and prove Binet's formula for the modified k-Fibonacci-like sequence. Binet's formula of the modified k-Fibonacci-like sequence allows us to express the modified k-Fibonacci-like sequence in terms of the roots α and β of the characteristic equation, $x^2 - kx - 1 = 0$ $(\alpha > \beta)$. The roots of the characteristic equation are $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$. Note that $\beta < 0 < \alpha$, $\alpha + \beta = k$ and $\alpha\beta = -1$.

In this paper, Binet's formula for the modified k-Fibonacci-like sequence is the following:

Theorem 3.1 (Binet's formula)**.** *The nth modified k-Fibonacci-like number $M_{k,n}$ is given by*

$$M_{k,n} = 2\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right). \tag{11}$$

Proof. The general term of the modified k-Fibonacci-like sequence may be expressed in the form, $M_{k,n} = C_1\alpha^n + C_2\beta^n$ for some coefficients C_1 and C_2.

(1) $M_{k,0} = C_1 + C_2 = 2$,
(2) $M_{k,1} = C_1\alpha + C_2\beta = 2$.

Then

$$C_1 = \frac{2 - 2\beta}{\alpha - \beta},\ C_2 = \frac{2\alpha - 2}{\alpha - \beta}.$$

Therefore,

$$M_{k,n} = \frac{2 - 2\beta}{\alpha - \beta}\alpha^n + \frac{2\alpha - 2}{\alpha - \beta}\beta^n = 2\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right).$$

□

Note that

$$M_{k,2} = 2\left(\frac{\alpha^2 - \beta^2}{\alpha - \beta} + \frac{\alpha^1 - \beta^1}{\alpha - \beta}\right) = 2k + 2,$$
$$M_{k,3} = 2\left(\frac{\alpha^3 - \beta^3}{\alpha - \beta} + \frac{\alpha^2 - \beta^2}{\alpha - \beta}\right) = 2k^2 + 2k + 2,$$
$$M_{k,4} = 2\left(\frac{\alpha^4 - \beta^4}{\alpha - \beta} + \frac{\alpha^3 - \beta^3}{\alpha - \beta}\right) = 2k^3 + 2k^2 + 4k + 2.$$

Note that $M_{k,n} = 2(F_{k,n} + F_{k,n-1})$, where $F_{k,n}$ is the k-Fibonacci sequence.

- If $k = 1$,

$$M_{1,n} = 2(F_{1,n} + F_{1,n-1}) = 2F_{1,n+1} = 2\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

- If $k = 2$,

$$\begin{aligned} M_{2,n} &= 2(F_{2,n} + F_{2,n-1}) \\ &= (2F_{2,n} + F_{2,n-1}) + F_{2,n-1} \\ &= F_{2,n+1} + F_{2,n-1} \\ &= L_{2,n}, \text{where } L_{2,n} \text{ is the 2-Lucas sequence.} \end{aligned}$$

Also, we can express the nth k-Fibonacci number $F_{k,n}$ as follows:

Theorem 3.2.

$$F_{k,n} = \frac{1}{2}\sum_{i=0}^{n-1} M_{k,n-i}(-1)^i. \tag{12}$$

Proof. By the note of Binet's formula,

$$F_{k,n} = \frac{M_{k,n}}{2} - F_{k,n-1}.$$

Then, inductively and using the fact that $F_{k,0} = 0$, we have

$$\begin{aligned} F_{k,n} &= \frac{M_{k,n}}{2} - F_{k,n-1} \\ &= \frac{M_{k,n}}{2} - \left(\frac{M_{k,n-1}}{2} - F_{k,n-2}\right) \\ &= \frac{M_{k,n}}{2} - \frac{M_{k,n-1}}{2} + F_{k,n-2} \\ &= \frac{M_{k,n}}{2} - \frac{M_{k,n-1}}{2} + \left(\frac{M_{k,n-2}}{2} - F_{k,n-3}\right) \\ &= \cdots \\ &= \frac{1}{2}\left(M_{k,n} - M_{k,n-1} + M_{k,n-2} - M_{k,n-3} + \cdots + (-1)^{n-1}M_{k,1}\right) \\ &\quad + (-1)^n F_{k,0} \\ &= \frac{1}{2}\sum_{i=0}^{n-1} M_{k,n-i}(-1)^i. \end{aligned}$$

□

4. Identities of the modified k-Fibonacci-like sequence

Many authors who study sequences like Fibonacci sequence have introduced special identities, such as the Catalan, Cassini, and d'Ocagne identities [2, 4, 6, 10, 12]. They have then proved them using Binet's formula for each identity.

In this section, we also introduce the Catalan, Cassini, and d'Ocagne identities for the modified k-Fibonacci-like sequence, and prove them using Binet's formula stated in the previous section. In addition, we introduce and prove the sums of the modified k-Fibonacci-like sequence using Binet's formula.

4.1. The Catalan identity

Many authors have expressed the Catalan identity as the following:

- $F_{n-r}F_{n+r} - F_n^2 = (-1)^{n+1-r}F_r^2$, where F_n is the Fibonacci sequence [2].
- $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,r}^2$, where $F_{k,n}$ is the k-Fibonacci sequence [2, 6].
- $a^{\xi(n-r)}b^{1-\xi(n-r)}q_{n-r}q_{n+r} - a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a^{\xi(r)}b^{1-\xi(r)}(-1)^{n+1-r}q_r^2$, where q_n is the generalized Fibonacci sequence [4].

As we can see, the left-hand side of each equation is expressed using itself.

However, in the modified k-Fibonacci-like sequence, $M_{k,n-r}M_{k,n+r} - M_{k,n}^2$ is expressed using the k-Fibonacci sequence $F_{k,r}$.

Theorem 4.1 (the Catalan identity). *For any two nonnegative integers n and r with $n \geq r$, we have*

$$M_{k,n-r}M_{k,n+r} - M_{k,n}^2 = 4k(-1)^{n-r}F_{k,r}^2, \tag{13}$$

where $F_{k,r}$ is the k-Fibonacci sequence.

Proof. Note that $\alpha + \beta = k$, $\alpha\beta = -1$. By Binet's formula, we have

$$\begin{aligned}
&M_{k,n-r}M_{k,n+r} - M_{k,n}^2 \\
&= \frac{2^2}{(\alpha-\beta)^2}\big[(\alpha^{n-r} - \beta^{n-r} + \alpha^{n-r-1} - \beta^{n-r-1}) \\
&\quad \times(\alpha^{n+r} - \beta^{n+r} + \alpha^{n+r-1} - \beta^{n+r-1}) - (\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1})^2\big] \\
&= \frac{2^2}{(\alpha-\beta)^2}\left[\left(\alpha^{n-r}\left(1+\tfrac{1}{\alpha}\right) - \beta^{n-r}\left(1+\tfrac{1}{\beta}\right)\right)\left(\alpha^{n+r}\left(1+\tfrac{1}{\alpha}\right) - \beta^{n+r}\left(1+\tfrac{1}{\beta}\right)\right)\right. \\
&\quad \left. - \left(\alpha^n\left(1+\tfrac{1}{\alpha}\right) - \beta^n\left(1+\tfrac{1}{\beta}\right)\right)^2\right] \\
&= \frac{2^2}{(\alpha-\beta)^2}\left[\alpha^{2n}\left(1+\tfrac{1}{\alpha}\right)^2 - \left(\alpha^{n-r}\beta^{n+r} + \alpha^{n+r}\beta^{n-r}\right)\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right)\right. \\
&\quad \left. +\beta^{2n}\left(1+\tfrac{1}{\beta}\right)^2 - \left(\alpha^{2n}\left(1+\tfrac{1}{\alpha}\right)^2 - 2\alpha^n\beta^n\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right) + \beta^{2n}\left(1+\tfrac{1}{\beta}\right)^2\right)\right] \\
&= \frac{2^2}{(\alpha-\beta)^2}\left(-\alpha^{n-r}\beta^{n+r} - \alpha^{n+r}\beta^{n-r} + 2\alpha^n\beta^n\right)\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right) \\
&= \frac{-2^2(\alpha\beta)^n}{(\alpha-\beta)^2}\left[\left(\tfrac{\beta}{\alpha}\right)^r + \left(\tfrac{\alpha}{\beta}\right)^r - 2\right]\left(1 + \tfrac{1}{\alpha} + \tfrac{1}{\beta} + \tfrac{1}{\alpha\beta}\right) \\
&= \frac{4(-1)^{n+1}}{(\alpha-\beta)^2}\left[\tfrac{\alpha^{2r} - 2\alpha^r\beta^r + \beta^{2r}}{(\alpha\beta)^r}\right]\left(\tfrac{1}{\alpha} + \tfrac{1}{\beta}\right) \\
&= \frac{4(-1)^{n-r+1}}{(\alpha-\beta)^2}(\alpha^r - \beta^r)^2\tfrac{\alpha+\beta}{\alpha\beta}
\end{aligned}$$

$$= 4k(-1)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2$$
$$= 4k(-1)^{n-r}F_{k,r}^2. \qquad \square$$

Remark 4.2. We can express the left-hand side using itself. Using Theorem 3.2,

$$M_{k,n-r}M_{k,n+r} - M_{k,n}^2 = k(-1)^{n-r}\left(\sum_{i=0}^{r-1} M_{k,r-i}(-1)^i\right)^2. \tag{14}$$

4.2. The Cassini identity

The Cassini identity is a special case of the Catalan identity with $r = 1$. Many authors have expressed the Cassini identity as the following:

- $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, where F_n is the Fibonacci sequence [2].
- $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$, where $F_{k,n}$ is the k-Fibonacci sequence [2, 6].
- $a^{\xi(n-1)}b^{1-\xi(n-1)}q_{n-1}q_{n+1} - a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a(-1)^n$, where q_n is the generalized Fibonacci sequence [4].

Similarly, the Cassini identity for the modified k-Fibonacci-like sequence is the following:

Theorem 4.3 (the Cassini identity)**.** *For any nonnegative integer n, we have*

$$M_{k,n-1}M_{k,n+1} - M_{k,n}^2 = 4k(-1)^{n-1}. \tag{15}$$

Proof. Taking $r = 1$ in the Catalan identity, we get

$$M_{k,n-1}M_{k,n+1} - M_{k,n}^2 = 4k(-1)^{n-1}F_{k,1}^2 = 4k(-1)^{n-1},$$

since $F_{k,1} = 1$. $\square$

4.3. The d'Ocagne identity

The d'Ocagne identity is similar to the Catalan identity. Many authors have expressed the d'Ocagne identity as the following:

- $F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}$, where F_n is the Fibonacci sequence [2].
- $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^nF_{k,m-n}$, where $F_{k,n}$ is the k-Fibonacci sequence [2, 6].
- $$a^{\xi(mn+m)}b^{\xi(mn+n)}q_mq_{n+1} - a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m+1}q_n$$
  $$= (-1)^na^{\xi(m-n)}q_{m-n},$$

  where q_n is the generalized Fibonacci sequence [4].

Once more, the left-hand side of each equation is expressed using itself.

However, in the modified k-Fibonacci-like sequence, the d'Ocagne identity is expressed using the k-Fibonacci sequence similar to the Catalan identity.

Theorem 4.4 (the d'Ocagne identity)**.** *For any two nonnegative integers m and n with $m \geq n$, we have*

$$M_{k,m}M_{k,n+1} - M_{k,m+1}M_{k,n} = 4k(-1)^{n-1}F_{k,m-n}, \tag{16}$$

where $F_{k,n}$ is the k-Fibonacci number.

Proof. We check each term on left-hand side using Binet's formula, and then we prove the theorem.

(i) First, we check $M_{k,m}M_{k,n+1}$.

$$\begin{aligned}
&M_{k,m}M_{k,n+1}\\
&= \frac{2^2}{(\alpha-\beta)^2}(\alpha^m - \beta^m + \alpha^{m-1} - \beta^{m-1})(\alpha^{n+1} - \beta^{n+1} + \alpha^n - \beta^n)\\
&= \frac{2^2}{(\alpha-\beta)^2}\left(\alpha^m\left(1+\tfrac{1}{\alpha}\right) - \beta^m\left(1+\tfrac{1}{\beta}\right)\right)\left(\alpha^{n+1}\left(1+\tfrac{1}{\alpha}\right) - \beta^{n+1}\left(1+\tfrac{1}{\beta}\right)\right)\\
&= \frac{2^2}{(\alpha-\beta)^2}\left[\alpha^{m+n+1}\left(1+\tfrac{1}{\alpha}\right)^2 - (\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m)\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right)\right.\\
&\quad \left. +\beta^{m+n+1}\left(1+\tfrac{1}{\beta}\right)^2\right].
\end{aligned}$$

(ii) Second, we check $M_{k,m+1}M_{k,n}$.

$$\begin{aligned}
&M_{k,m+1}M_{k,n}\\
&= \frac{2^2}{(\alpha-\beta)^2}(\alpha^{m+1} - \beta^{m+1} + \alpha^m - \beta^m)(\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1})\\
&= \frac{2^2}{(\alpha-\beta)^2}\left[\left(\alpha^{m+1}\left(1+\tfrac{1}{\alpha}\right) - \beta^{m+1}\left(1+\tfrac{1}{\beta}\right)\right)\left(\alpha^n\left(1+\tfrac{1}{\alpha}\right) - \beta^n\left(1+\tfrac{1}{\beta}\right)\right)\right]\\
&= \frac{2^2}{(\alpha-\beta)^2}\left[\alpha^{m+n+1}\left(1+\tfrac{1}{\alpha}\right)^2 - (\alpha^{m+1}\beta^n + \alpha^n\beta^{m+1})\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right)\right.\\
&\quad \left. +\beta^{m+n+1}\left(1+\tfrac{1}{\beta}\right)^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&M_{k,m}M_{k,n+1} - M_{k,m+1}M_{k,n}\\
&= \frac{2^2}{(\alpha-\beta)^2}\left[(\alpha^{m+1}\beta^n + \alpha^n\beta^{m+1}) - (\alpha^m\beta^{n+1} + \alpha^{n+1}\beta^m)\right]\left(1+\tfrac{1}{\alpha}\right)\left(1+\tfrac{1}{\beta}\right)\\
&= \frac{2^2(\alpha\beta)^n}{(\alpha-\beta)^2}\left[(\alpha^{m-n+1} + \beta^{m-n+1}) - (\alpha^{m-n}\beta + \alpha\beta^{m-n})\right]\left(1+\tfrac{1}{\alpha}+\tfrac{1}{\beta}+\tfrac{1}{\alpha\beta}\right)\\
&= \frac{2^2(-1)^n}{(\alpha-\beta)^2}\left[(\alpha^{m-n+1} - \alpha\beta^{m-n}) + (\beta^{m-n+1} - \alpha^{m-n}\beta)\right]\left(\tfrac{1}{\alpha}+\tfrac{1}{\beta}\right)\\
&= \frac{2^2(-1)^n}{(\alpha-\beta)^2}\left[\alpha(\alpha^{m-n} - \beta^{m-n}) - \beta(\alpha^{m-n} - \beta^{m-n})\right]\left(\tfrac{\alpha+\beta}{-1}\right)
\end{aligned}$$

$$= \frac{2^2(-1)^{n-1}}{(\alpha-\beta)^2}(\alpha^{m-n}-\beta^{m-n})(\alpha-\beta)(\alpha+\beta)$$
$$= 2^2k(-1)^{n-1}\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}$$
$$= 4k(-1)^{n-1}F_{k,m-n}. \qquad \square$$

Remark 4.5. Similarly, we can express the left-hand side using itself. Using Theorem 3.2,

$$M_{k,m}M_{k,n+1} - M_{k,m+1}M_{k,n} = 2k\sum_{i=0}^{m-n-1} M_{k,m-n-i}(-1)^{n+i-1}. \tag{17}$$

4.4. The sums of the modified *k*-Fibonacci-like sequence

Binet's formula allows us to express the sum of the first n terms of the modified k-Fibonacci-like sequence.

Theorem 4.6. *The sum of first n terms of the modified k-Fibonacci-like sequence* $\{M_{k,n}\}$ *is*

$$M_{k,1} + M_{k,2} + \cdots + M_{k,n} = \sum_{i=1}^{n} M_{k,i} = \frac{1}{k}(M_{k,n+1} + M_{k,n}) - \frac{4}{k}. \tag{18}$$

Proof. Note that $\alpha+\beta = k$, $\alpha\beta = -1$ and $(\alpha-1)(\beta-1) = -k$.

By Binet's formula, we have

$$\sum_{i=1}^{n} M_{k,i} = \frac{2}{\alpha-\beta}\sum_{i=1}^{n}(\alpha^i - \beta^i + \alpha^{i-1} - \beta^{i-1}).$$

By summing up the geometric partial sums, we have

$$\sum_{i=1}^{n} M_{k,i} = \frac{2}{\alpha-\beta}\left[\frac{\alpha(\alpha^n-1)}{\alpha-1} - \frac{\beta(\beta^n-1)}{\beta-1} + \frac{\alpha^n-1}{\alpha-1} - \frac{\beta^n-1}{\beta-1}\right].$$

Then, using the above note, we have

$$\sum_{i=1}^{n} M_{k,i}$$
$$= \frac{2}{(\alpha-\beta)(\alpha-1)(\beta-1)}[\alpha(\alpha^n-1)(\beta-1) - \beta(\beta^n-1)(\alpha-1)$$
$$+(\alpha^n-1)(\beta-1) - (\beta^n-1)(\alpha-1)]$$
$$= \frac{2}{-k(\alpha-\beta)}\left[(\alpha^{n+1}\beta - \alpha^{n+1} - \alpha\beta + \alpha) - (\alpha\beta^{n+1} - \beta^{n+1} - \alpha\beta + \beta)\right.$$
$$\left.+(\alpha^n\beta - \alpha^n - \beta + 1) - (\alpha\beta^n - \beta^n - \alpha + 1)\right]$$
$$= \frac{2}{-k(\alpha-\beta)}\left[(\alpha\beta)(\alpha^n-\beta^n) - (\alpha^{n+1}-\beta^{n+1}) + (\alpha-\beta)\right.$$
$$\left.+(\alpha\beta)(\alpha^{n-1}-\beta^{n-1}) - (\alpha^n-\beta^n) + (\alpha-\beta)\right]$$

$$= \frac{2}{-k(\alpha-\beta)}\left[-(\alpha^n-\beta^n)-(\alpha^{n+1}-\beta^{n+1})+(\alpha-\beta)-(\alpha^{n-1}-\beta^{n-1})\right.$$
$$\left.-(\alpha^n-\beta^n)+(\alpha-\beta)\right]$$
$$= \frac{2}{k(\alpha-\beta)}\left[(\alpha^{n+1}-\beta^{n+1}+\alpha^n-\beta^n)+(\alpha^n-\beta^n+\alpha^{n-1}-\beta^{n-1})\right.$$
$$\left.-2(\alpha-\beta)\right]$$
$$= \frac{1}{k}\left[2\frac{\alpha^{n+1}-\beta^{n+1}+\alpha^n-\beta^n}{\alpha-\beta}+2\frac{\alpha^n-\beta^n+\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}-4\right]$$
$$= \frac{1}{k}(M_{k,n+1}+M_{k,n})-\frac{4}{k}.$$
□

Note that from Theorem 4.6,

$$\sum_{i=0}^{n} M_{k,i} = \sum_{i=1}^{n} M_{k,i} + M_{k,0} = \frac{1}{k}(M_{k,n+1}+M_{k,n})-\frac{4}{k}+2.$$

Theorem 4.7. *The sum of the first n terms with odd indices is*

$$M_{k,1}+M_{k,3}+\cdots+M_{k,2n-1} = \sum_{i=1}^{n} M_{k,2i-1} = \frac{1}{k^2}(M_{k,2n+1}-M_{k,2n-1})-\frac{2}{k}. \tag{19}$$

Proof. Note that $(\alpha^2-1)(\beta^2-1)=-k^2$.
By Binet's formula, we have

$$\sum_{i=1}^{n} M_{k,2i-1} = \frac{2}{\alpha-\beta}\sum_{i=1}^{n}\left(\alpha^{2i-1}-\beta^{2i-1}+\alpha^{2i-2}-\beta^{2i-2}\right).$$

By summing up the geometric partial sums, we have

$$\sum_{i=1}^{n} M_{k,2i-1} = \frac{2}{\alpha-\beta}\left[\frac{\alpha(\alpha^{2n}-1)}{\alpha^2-1}-\frac{\beta(\beta^{2n}-1)}{\beta^2-1}+\frac{\alpha^{2n}-1}{\alpha^2-1}-\frac{\beta^{2n}-1}{\beta^2-1}\right].$$

Then, using the above note, we have

$$\sum_{i=1}^{n} M_{k,2i-1}$$
$$= \frac{-2}{k^2(\alpha-\beta)}\left[(\alpha^{2n+1}\beta^2-\alpha^{2n+1}-\alpha\beta^2+\alpha)-(\alpha^2\beta^{2n+1}-\beta^{2n+1}-\alpha^2\beta+\beta)\right.$$
$$\left.+(\alpha^{2n}\beta^2-\alpha^{2n}-\beta^2+1)-(\alpha^2\beta^{2n}-\beta^{2n}-\alpha^2+1)\right]$$
$$= \frac{-2}{k^2(\alpha-\beta)}\left[(\alpha\beta)^2(\alpha^{2n-1}-\beta^{2n-1})-(\alpha^{2n+1}-\beta^{2n+1})+(\alpha\beta)(\alpha-\beta)\right.$$
$$\left.+(\alpha-\beta)+(\alpha\beta)^2(\alpha^{2n-2}-\beta^{2n-2})-(\alpha^{2n}-\beta^{2n})+(\alpha^2-\beta^2)\right]$$
$$= \frac{-2}{k^2(\alpha-\beta)}\left[(\alpha^{2n-1}-\beta^{2n-1}+\alpha^{2n-2}-\beta^{2n-2})\right.$$
$$\left.-(\alpha^{2n+1}-\beta^{2n+1}+\alpha^{2n}-\beta^{2n})+(\alpha^2-\beta^2)\right]$$

$$= \frac{1}{k^2}(M_{k,2n+1} - M_{k,2n-1}) - \frac{2}{k}.$$ □

Theorem 4.8. *The sum of the first n terms with even indices is*

$$M_{k,2} + M_{k,4} + \cdots + M_{k,2n} = \sum_{i=1}^{n} M_{k,2i} = \frac{1}{k^2}(M_{k,2n+2} - M_{k,2n}) - \frac{2}{k}. \tag{20}$$

Proof. By Binet's formula, we have

$$\sum_{i=1}^{n} M_{k,2i} = \frac{2}{\alpha-\beta}\sum_{i=1}^{n}\left(\alpha^{2i} - \beta^{2i} + \alpha^{2i-1} - \beta^{2i-1}\right).$$

By summing up the geometric partial sums, we have

$$\sum_{i=1}^{n} M_{k,2i} = \frac{2}{\alpha-\beta}\left[\frac{\alpha^2(\alpha^{2n}-1)}{\alpha^2-1} - \frac{\beta^2(\beta^{2n}-1)}{\beta^2-1} + \frac{\alpha(\alpha^{2n}-1)}{\alpha^2-1} - \frac{\beta(\beta^{2n}-1)}{\beta^2-1}\right].$$

Then, using the note in Theorem 4.7, we have

$$\begin{aligned}
&\sum_{i=1}^{n} M_{k,2i}\\
&= \frac{-2}{k^2(\alpha-\beta)}\big[(\alpha^{2n+2}\beta^2 - \alpha^{2n+2} - \alpha^2\beta^2 + \alpha^2)\\
&\quad -(\alpha^2\beta^{2n+2} - \beta^{2n+2} - \alpha^2\beta^2 + \beta^2) + (\alpha^{2n+1}\beta^2 - \alpha^{2n+1} - \alpha\beta^2 + \alpha)\\
&\quad -(\alpha^2\beta^{2n+1} - \beta^{2n+1} - \alpha^2\beta + \beta)\big]\\
&= \frac{-2}{k^2(\alpha-\beta)}\big[(\alpha\beta)^2(\alpha^{2n} - \beta^{2n}) - (\alpha^{2n+2} - \beta^{2n+2}) + (\alpha^2 - \beta^2)\\
&\quad +(\alpha\beta)^2(\alpha^{2n-1} - \beta^{2n-1}) - (\alpha^{2n+1} - \beta^{2n+1}) + (\alpha\beta)(\alpha-\beta) + (\alpha-\beta)\big]\\
&= \frac{2}{k^2(\alpha-\beta)}\big[(\alpha^{2n+2} - \beta^{2n+2} + \alpha^{2n+1} - \beta^{2n+1})\\
&\quad -(\alpha^{2n} - \beta^{2n} + \alpha^{2n-1} - \beta^{2n-1}) - (\alpha^2 - \beta^2)\big]\\
&= \frac{1}{k^2}(M_{k,2n+2} - M_{k,2n}) - \frac{2}{k}.
\end{aligned}$$ □

Remark 4.9. Note that we can check

$$\sum_{i=1}^{n} M_{k,2i} + \sum_{i=1}^{n} M_{k,2i-1} = \sum_{i=1}^{2n} M_{k,i}$$

using the above three theorems.

Theorem 4.10 (Sum involving binomial coefficients)**.** *For any integer $n \geq 0$,*

$$\sum_{i=0}^{n}\binom{n}{i}k^i M_{k,i} = M_{k,2n}. \tag{21}$$

Proof. By Binet's formula, we have

$$\sum_{i=0}^{n}\binom{n}{i}k^iM_{k,i}=\frac{2}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}k^i\left(\alpha^i-\beta^i+\alpha^{i-1}-\beta^{i-1}\right).$$

Then,

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}k^iM_{k,i}&=\frac{2}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}\left[(k\alpha)^i-(k\beta)^i+\frac{1}{\alpha}(k\alpha)^i-\frac{1}{\beta}(k\beta)^i\right]\\&=\frac{2}{\alpha-\beta}\left[\sum_{i=0}^{n}\binom{n}{i}(k\alpha)^i-\sum_{i=0}^{n}\binom{n}{i}(k\beta)^i+\frac{1}{\alpha}\sum_{i=0}^{n}\binom{n}{i}(k\alpha)^i\right.\\&\qquad\left.-\frac{1}{\beta}\sum_{i=0}^{n}\binom{n}{i}(k\beta)^i\right].\end{aligned}$$

By the binomial theorem, we have

$$\sum_{i=0}^{n}\binom{n}{i}k^iM_{k,i}=\frac{2}{\alpha-\beta}\left[(k\alpha+1)^n-(k\beta+1)^n+\frac{1}{\alpha}(k\alpha+1)^n-\frac{1}{\beta}(k\beta+1)^n\right].$$

Since $x^2=kx+1$, we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}k^iM_{k,i}&=\frac{2}{\alpha-\beta}\left[(\alpha^2)^n-(\beta^2)^n+\frac{1}{\alpha}(\alpha^2)^n-\frac{1}{\beta}(\beta^2)^n\right]\\&=\frac{2}{\alpha-\beta}(\alpha^{2n}-\beta^{2n}+\alpha^{2n-1}-\beta^{2n-1})\\&=M_{k,2n}.\end{aligned}$$

□

4.5. The limit of the quotient of two consecutive terms

In the modified k-Fibonacci-like sequence, the limit of the quotient of two consecutive terms is equal to the positive root α of the corresponding characteristic equation, $x^2-kx-1=0$.

Theorem 4.11.

$$\lim_{n\to\infty}\frac{M_{k,n+1}}{M_{k,n}}=\alpha. \tag{22}$$

Proof. By Binet's formula, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{M_{k,n+1}}{M_{k,n}}&=\lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}+\alpha^n-\beta^n}{\alpha^n-\beta^n+\alpha^{n-1}-\beta^{n-1}}\\&=\lim_{n\to\infty}\frac{1-\left(\frac{\beta}{\alpha}\right)^{n+1}+\frac{1}{\alpha}-\frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n}}{\frac{1}{\alpha}-\frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)^{n}+\frac{1}{\alpha^2}-\frac{1}{\alpha^2}\left(\frac{\beta}{\alpha}\right)^{n-1}},\end{aligned}$$

and taking into account that $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0$ since $|\beta| < \alpha$. Thus

$$\lim_{n\to\infty}\frac{M_{k,n+1}}{M_{k,n}} = \frac{1+\frac{1}{\alpha}}{\frac{1}{\alpha}+\frac{1}{\alpha^2}} = \alpha.$$

□

5. The ordinary generating function of the modified *k*-Fibonacci-like sequence

In this section, the ordinary generating function for the modified k-Fibonacci-like sequence is presented. The modified k-Fibonacci-like sequence can be seen as the coefficients of the power series which is called the ordinary generating function of the modified k-Fibonacci-like sequence. Therefore, if $M_k(x)$ is the ordinary generating function, we can write

$$M_k(x) = \sum_{i=0}^{\infty} M_{k,i}x^i = M_{k,0} + M_{k,1}x + M_{k,2}x^2 + M_{k,3}x^3 + \cdots + M_{k,n}x^n + \cdots .$$

First, we can find the radius of convergence. By Theorem 4.11,

$$\lim_{n\to\infty}\left|\frac{M_{k,n+1}x^{n+1}}{M_{k,n}x^n}\right| = \alpha \mid x \mid .$$

Thus, the radius of convergence is $\frac{1}{\alpha}$. And then,

$$\begin{aligned}
M_k(x) &= M_{k,0} + xM_{k,1} + x^2M_{k,2} + x^3M_{k,3} + x^4M_{k,4} + x^5M_{k,5} + \cdots ,\\
kxM_k(x) &= kxM_{k,0} + kx^2M_{k,1} + kx^3M_{k,2} + kx^4M_{k,3} + kx^5M_{k,4} + \cdots ,\\
x^2M_k(x) &= x^2M_{k,0} + x^3M_{k,1} + x^4M_{k,2} + x^5M_{k,3} + x^6M_{k,4} + \cdots .
\end{aligned}$$

Since $M_{k,n+2} - kM_{k,n+1} - M_{k,n} = 0, M_{k,0} = 2$, and $M_{k,1} = 2$, we obtain

$$\begin{aligned}
(1-kx-x^2)M_k(x) &= M_{k,0} + (M_{k,1} - kM_{k,0})x + (M_{k,2} - kM_{k,1} - M_{k,0})x^2\\
&\quad + (M_{k,3} - kM_{k,2} - M_{k,1})x^3 + \cdots\\
&= M_{k,0} + (M_{k,1} - kM_{k,0})x\\
&= 2 + (2-2k)x.
\end{aligned}$$

Hence, the ordinary generating function of the modified k-Fibonacci-like sequence $\{M_{k,n}\}_{n=0}^{\infty}$ is

$$M_k(x) = \frac{2+2x(1-k)}{1-kx-x^2}. \tag{23}$$

6. Conclusions

In this paper, we have introduced and studied the modified k-Fibonacci-like sequence. Many identities, such as the Catalan, Cassini, and d'Ocagne identities for the modified k-Fibonacci-like sequence, are stated and in their proof we use Binet's formula. In addition, we have introduced the sums of the modified k-Fibonacci-like sequence and proved them. Moreover, we have found the ordinary generating function for the modified k-Fibonacci-like sequence.

Moreover, we can generalize the modified k-Fibonacci-like sequence for $N_{k,0} = N_{k,1} = a$, where a is an integer. In this case, the nth modified k-Fibonacci-like number $N_{k,n}$ is given by

$$N_{k,n} = a\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right).$$

The Catalan identity is

$$N_{k,n}^2 - N_{k,n+r}N_{k,n-r} = a^2k(-1)^{n-r}F_{k,r}^2.$$

The Cassini identity is

$$N_{k,n}^2 - N_{k,n+1}N_{k,n-1} = a^2k(-1)^{n-1}.$$

The d'Ocagne identity is

$$N_{k,m}N_{k,n+1} - N_{k,m+1}N_{k,n} = a^2k(-1)^nF_{k,m-n}.$$

The ordinary generating function is

$$N_k = \frac{a + ax(1-k)}{1 - kx - x^2}.$$

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References

[1] V. H. Badshah, M. S. Teeth, and M. M. Dar, *Generalized Fibonacci-like sequence and its properties*, Int. J. Contemp. Math. Sci. **7** (2012), no. 21-24, 1155–1164.

[2] P. Catarino, *On some identities for k-Fibonacci sequence*, Int. J. Contemp. Math. Sci. **9** (2014), no. 1, 37–42.

[3] M. Edson, S. Lewis, and O. Yayenie, *The k-periodic Fibonacci sequence and an extended Binet's formula*, Integers **11** (2011), A32, 12 pp.

[4] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's formula*, Integers **9** (2009), A48, 639–654.

[5] S. Falcón, *On the k-Lucas Numbers*, Int. J. Contemp. Math. Sci. **6** (2011), no. 21, 1039–1050.

[6] S. Falcón and Á. Plaza, *The k-Fibonacci sequence and the Pascal* 2*-triangle*, Chaos Solitons Fractals **33** (2007), no. 1, 38–49.

[7] S. Harne, B. Singh, and S. Pal, *Generalized Fibonacci-like sequence and Fibonacci sequence*, Int. J. Contemp. Math. Sci. **9** (2014), no. 5, 235–241.

[8] V. E. Hoggat, *Fibonacci and Lucas numbers*, Palo Alto, CA: Houghton; 1969.

[9] Y. K. Panwar, G. P. S. Rathore, and R. Chawla, *On the k-Fibonacci-like numbers*, Turkish J. Anal. Number Theory **2** (2014), no. 1, 9–12.

[10] Y. K. Panwar, B. Singh, and V. K. Gupta, *Generalized Fibonacci sequence and its properties*, Palest. J. Math. **3** (2014), no. 1, 141–147.

[11] B. Singh, O. Sikhwal, and S. Bhatnagar, *Fibonacci-like sequence and its properties*, Int. J. Contemp. Math. Sci. **5** (2010), no. 17-18, 859–868.

[12] O. Yayenie, *A note on generalized Fibonacci sequences*, Appl. Math. Comput. **217** (2011), no. 12, 5603–5611.

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