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ON A COMPUTATION OF PLURIGENUS OF A CANONICAL THREEFOLD

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Abstract. For a canonical threefold X, it is known that p_n does not vanish for a sufficiently large n, where $p_n = h^0(X, \mathcal{O}_X(nK_X))$. We have shown that p_n does not vanish for at least one n in $\{6, 8, 10\}$. Assuming an additional condition $p_2 \geq 1$ or $p_3 \geq 1$, we have shown that $p_{12} \geq 2$ and $p_n \geq 2$ for $n \geq 14$ with one possible exceptional case. We have also found some inequalities between $\chi(\mathcal{O}_X)$ and K_X^3.

Throughout this paper X is assumed to be a projective threefold with only canonical singularities and an ample canonical divisor K_X over the complex number field $\mathbb{C}$, i.e., a canonical threefold.

It is well known that $H^0(X, \mathcal{O}_X(mK_X))$ does not vanish and generates a birational map for a sufficiently large m. If there exists a positive integer n such that $h^0(X, \mathcal{O}_X(nK_X)) \geq 2$, then by using Kollár's technique we can find the integer m which generates a birational map (see Kollár [4]).

A. R. Fletcher showed $h^0(X, \mathcal{O}_X(12K_X)) \geq 1$ and $h^0(X, \mathcal{O}_X(24K_X)) \geq 2$ when $\chi(\mathcal{O}_X) = 1$ in Fletcher [3]. Shin [6] improved the above results. J. A. Chen and M. Chen showed that $h^0(X, \mathcal{O}_X(nK_X)) \geq 1$ for every integer $n \geq 27$ and that $h^0(X, \mathcal{O}_X(24K_X)) \geq 2$ and $h^0(X, \mathcal{O}_X(n_0K_X)) \geq 2$ for some integer $n_0 \leq 18$ (see Chen and Chen [1, 2]).

Plurigenus p_n of canonical threefolds were extensively studied by J. A. Chen and M. Chen (see Chen and Chen [1, 2]). They inspect linear combinations of p_n and baskets of singularities. In this paper, we study also linear combinations of p_n. But our approach is slightly different and includes less complex calculations. To find special linear combinations of p_n, our strategy is searching linear combinations which satisfy the following (1) or (2):

(1) linear combinations of p_n are non-positive at every point in $(0, \frac{1}{2}]$.
(2) linear combinations of p_n can be expressed as pure linear forms a_ib+d_ir of singularity type $\frac{b}{r}$ on some partition of $(0, \frac{1}{2}]$,

where $a_i,\ d_i$ are integers.

From (1), we may obtain information of p_n in linear combinations.

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From (2), we may compute p_n using special singularity types.

Finally, with above information we may construct a system of linear equations of numbers of singularities.

In this paper, we have introduced techniques to compute p_n and shown the following theorems:

Theorem A (=Theorem 2)**.** *$p_n \geq 1$ for at least one n in $\{6, 8, 10\}$.*

Theorem B (=Theorem 3)**.** *Suppose that $p_2 \geq 1$ or $p_3 \geq 1$. Then*

(1) $p_{12} \geq 2$.

(2) *$p_n \geq 2$ for $n \geq 14$ with a possible exceptional case which must satisfy:*

i) *$p_2 \geq 1$, $p_3 = p_5 = p_7 = p_9 = 0$ and $p_{15} \leq 1$.*

ii) *$p_n \geq 2$ for an even integer n $(n \geq 6)$.*

iii) $K_X^3 \leq \frac{1}{12}\chi(\mathcal{O}_X) - \frac{1}{12}p_2$.

Furthermore, we have obtained the following table:

case	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	$p_{n(\geq 14)}$
$p_2 \geq 1^*$	≥ 1	?	≥ 2	?	≥ 2	≥ 1	≥ 2	≥ 1	≥ 2	≥ 2	≥ 2
$p_3 \geq 1$	?	?	≥ 1	?	≥ 1	≥ 1	?	≥ 1	≥ 2	?	≥ 2

The symbol ? means that it is not known or can be computed with mild additional conditions. The symbol $*$ means that there is one possible exceptional case which is described in Theorem 3.

M. Reid and A. R. Fletcher described the formula for $\chi(\mathcal{O}_X(nK_X))$. Combining the formula for $\chi(\mathcal{O}_X(nK_X))$ with a vanishing theorem, it is possible to compute $h^0(X, \mathcal{O}_X(nK_X))$. The formula for $\chi(\mathcal{O}_X(nK_X))$ is as follows:

$$\chi(\mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_{Q \in \mathcal{B}} l(Q, n),$$

where the summation is over a basket $\mathcal{B}$ of singularities. Although singularities in a basket are not necessarily singularities in X, singularities in X make the contribution as if they were in a basket. For detailed explanations about a basket of singularities, see Reid [5] or Fletcher [3].

The exact formula for $l(Q, n)$ is described as follows:

$$l(Q, n) = \sum_{i=1}^{n-1} \frac{\overline{ib}(r - \overline{ib})}{2r},$$

where Q is a singularity of type $\frac{1}{r}(1, -1, b)$, r and b are relatively prime, and $\overline{ib}$ is the least residue of ib modulo r.

For the sake of simplicity, denote $\sum_{Q \in \mathcal{B}} l(Q, n)$ by $L(n)$. Switch two summations in $L(n)$ and denote $\sum_{Q \in \mathcal{B}} \frac{\overline{ib}(r-\overline{ib})}{2r}$ by l_i. Then we have

$$L(n) = \sum_{Q \in \mathcal{B}} l(Q, n) = \sum_{Q \in \mathcal{B}} \sum_{i=1}^{n-1} \frac{\overline{ib}(r - \overline{ib})}{2r} = \sum_{i=1}^{n-1} \sum_{Q \in \mathcal{B}} \frac{\overline{ib}(r - \overline{ib})}{2r} = \sum_{i=1}^{n-1} l_i.$$

Let's denote the singularity type $\frac{1}{r}(1,-1,b)$ by $\frac{b}{r}$ unless there is some confusion. Moreover, identify the singularity type $\frac{b}{r}$ with the rational number $\frac{b}{r}$ in the interval $(0,1]$. By identifying the type $\frac{b}{r}$ with the rational number $\frac{b}{r}$ in $(0,1]$, our situation is defined more effectively for the computation of $L(n)$.

The following proposition is a standard application of the Kawamata-Viehweg Vanishing Theorem.

Proposition 1. *For all $n \geq 2$,*

$$p_n \overset{def}{:=} h^0(X, \mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) + L(n).$$

Lemma 1. *Let Q be a point of type $\frac{b}{r}$. Let $k = \min\{b, r-b\}$. Then $\overline{ib}(r-\overline{ib}) = \overline{ik}(r-\overline{ik})$ for a positive integer i.*

Proof. If $k = r-b$, then $\overline{ik} \equiv \overline{ir-ib} \equiv \overline{-ib} \equiv r - \overline{ib} \mod r$. The graph of $x(r-x)$ yields $\overline{ib}(r-\overline{ib}) = \overline{ik}(r-\overline{ik})$. ☐

To compute p_n, by Lemma 1, it may be assumed that the basket of singularities consists of points related only to types $\frac{b}{r}$ ($\frac{b}{r} \leq \frac{1}{2}$) because $\frac{b}{r}$ and $\frac{k}{r}$ produce the same value for $\frac{\overline{ib}(r-\overline{ib})}{2r}$.

From now on, we are going to consider only the points $\frac{b}{r}$ in $(0,\frac{1}{2}]$ for a basket of singularities, where $(r,b)=1$.

Lemma 2. *Let $\mathcal{B} = \{\frac{b}{r}\}$ be a basket of singularities of X. Then*

(1) $\chi(\mathcal{O}_X) = \sum_{\mathcal{B}} \frac{b}{10} + \frac{-5p_2+p_3}{10}$.

(2) $K_X^3 = \sum_{\mathcal{B}} \frac{b^2}{r} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3$.

Proof. For a proof of (1), compute $p_3 - 5p_2$ using Proposition 1. Recall that $b \leq \frac{r}{2}$.

$$\begin{aligned} p_3 - 5p_2 &= 10\chi(\mathcal{O}_X) - 4l_1 + l_2 \\ &= 10\chi(\mathcal{O}_X) + \sum_{\mathcal{B}} \frac{2b(r-2b) - 4b(r-b)}{2r} \\ &= 10\chi(\mathcal{O}_X) - \sum_{\mathcal{B}} b. \end{aligned}$$

For a proof of (2), compute $3p_3 - 5p_2 = 5K_X^3 - 2l_1 + 3l_2$.

$$\begin{aligned} K_X^3 &= \frac{1}{5}(2l_1 - 3l_2) + \frac{1}{5}(3p_3 - 5p_2) \\ &= \sum_{\mathcal{B}} \frac{5b^2 - 2br}{5r} + \frac{1}{5}(3p_3 - 5p_2) \end{aligned}$$

$$= \sum_{\mathcal{B}} \frac{b^2}{r} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3,$$

since $\sum_{\mathcal{B}} b = 10\chi(\mathcal{O}_X) + 5p_2 - p_3$ by (1). $\square$

Lemma 3. *Let $\mathcal{B} = \{\frac{b}{r}\}$ be a basket of singularities of X. Then*

$$4\chi(\mathcal{O}_X) + (3p_2 - p_3) < \sum_{\mathcal{B}} \frac{b^2}{r} \leq 3 \sum_{\mathcal{B}} \frac{r^2-1}{r} - 68\chi(\mathcal{O}_X) + (3p_2 - p_3).$$

Proof. The left inequality is induced easily by (2) in Lemma 2 since $K_X^3 > 0$. To prove the right inequality, by the result of R. Barlow,

$$\rho^* K_X \cdot c_2(Y) = \sum_{\mathcal{B}} \frac{r^2-1}{r} - 24\chi(\mathcal{O}_X),$$

where $\rho : Y \to X$ is a resolution of singularities of X (see Reid [5]).

$$\begin{aligned}
\chi(\mathcal{O}_X) &= \frac{1}{24} \sum_{\mathcal{B}} \frac{r^2-1}{r} - \frac{1}{24} \rho^* K_X \cdot c_2(Y) \\
&\leq \frac{1}{24} \sum_{\mathcal{B}} \frac{r^2-1}{r} - \frac{1}{72} K_X^3 \\
&= \frac{1}{24} \sum_{\mathcal{B}} \frac{r^2-1}{r} - \frac{1}{72} \left(\sum_{\mathcal{B}} \frac{b^2}{r} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3 \right),
\end{aligned}$$

where the second inequality is Miyaoka-Yau inequality and the last equality is proved just above. Hence,

$$\sum_{\mathcal{B}} \frac{b^2}{r} \leq 3 \sum_{\mathcal{B}} \frac{r^2-1}{r} - 68\chi(\mathcal{O}_X) + (3p_2 - p_3).$$

$\square$

Even though the formula for p_n is known and the basket of singularities is given, it is complicate to express explicitly an equation form of p_n because the formula for p_n contains terms $\frac{\overline{ib}(r-\overline{ib})}{2r}$ for $i = 1, \ldots, i-1$ in $L(n)$.

More precisely, a term $\frac{\overline{ib}(r-\overline{ib})}{2r}$ varies: for $\frac{b}{r}$ in a basket of singularities,

$$\frac{\overline{ib}(r-\overline{ib})}{2r} = \begin{cases} \frac{ib(r-ib)}{2r} & \text{if } ib \leq r & \text{i.e., } 0 < \frac{b}{r} \leq \frac{1}{i} \\ \frac{(ib-r)(2r-ib)}{2r} & \text{if } r \leq ib \leq 2r & \text{i.e., } \frac{1}{i} \leq \frac{b}{r} \leq \frac{2}{i} \\ \frac{(ib-2r)(3r-ib)}{2r} & \text{if } 2r \leq ib \leq 3r & \text{i.e., } \frac{2}{i} \leq \frac{b}{r} \leq \frac{3}{i} \\ \vdots & \vdots & \vdots \end{cases}$$

Thus, to find an explicit expression for $L(n)$, we need to consider all the subintervals in $(0, \frac{1}{2}]$ determined by

$$ED_n \stackrel{\text{def}}{:=} \left\{ \frac{x}{i} \in (0, \frac{1}{2}] \,\middle|\, 2 \leq i \leq n-1,\ (x,i) = 1 \right\}.$$

Notice that the smallest point is $\frac{1}{n-1}$ and the largest point is $\frac{1}{2}$ in ED_n.

As an example, to express $L(7)$ explicitly, it is enough to consider all the subintervals of $(0, \frac{1}{2}]$ determined by $ED_7 = \left\{ \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2} \right\}$, i.e., $(0, \frac{1}{6}]$, $[\frac{1}{6}, \frac{1}{5}], \ldots, [\frac{2}{5}, \frac{1}{2}]$ because $L(7) = l_1 + l_2 + \cdots + l_6$.

Let's consider $ED_n = \{\frac{b_j}{r_j}\}$.

Now, l_i $(i \leq n-1)$ is expressed uniquely on each subinterval I determined by ED_n because $\overline{ib}$ is given as follows:

$$\exists!\ k \in \mathbf{Z} \text{ such that } \overline{ib} = ib - kr \ \text{ for all } \frac{b}{r} \in I.$$

Notice that the above constant k depends only on a given subinterval I and a multiple i, not on the points $\frac{b}{r}$ in a subinterval I. Thus, l_i over I is

$$l_i|_I = \sum \frac{\overline{ib}(r - \overline{ib})}{2r} = \sum \frac{(ib - kr)((k+1)r - ib)}{2r},$$

where the summation is over the points of the basket of singularities in I.

Let's consider a special linear combination $\sum_{j=1}^n c_j p_j$ of p_j $(c_j \in \mathbb{Z})$ which satisfies the following (1) and (2):

(1) Suppose that the terms $\chi(\mathcal{O}_X)$ and K_X^3 are eliminated in $\sum_{j=1}^n c_j p_j$.

Then in a linear combination $\sum_{j=1}^n c_j p_j$, there are terms only related to l_i, i.e., $\sum_{j=1}^n c_j p_j$ is given as follows: for some $q_i \in \mathbf{Z}$

$$\sum_{j=1}^{n} c_j p_j = \sum_{i=1}^{n-1} q_i l_i.$$

We can express equations forms of l_i over subintervals determined by ED_n.

(2) Suppose that $\sum_{j=1}^n c_j p_j$ is expressed explicitly over the subintervals of $(0, \frac{1}{2}]$ determined by ED_n as follows:

$$\sum_j c_j p_j = \sum_{i=1}^{n-1} q_i l_i = \begin{cases} \sum a_1 b & \text{on a subinterval } (0, \frac{b_1}{r_1}] \\ \sum a_2 b + d_2 r & \text{on a subinterval } [\frac{b_1}{r_1}, \frac{b_2}{r_2}] \\ \sum a_3 b + d_3 r & \text{on a subinterval } [\frac{b_2}{r_2}, \frac{b_3}{r_3}] \\ \quad\vdots & \quad\vdots \\ \sum a_m b + d_m r & \text{on a subinterval } [\frac{b_{m-1}}{r_{m-1}}, \frac{1}{2}], \end{cases}$$

where the summation is over points $\frac{b}{r}$ of a basket of singularities in each subinterval and a_j, d_j are integers. Suppose more that equations on a right side have same value at the boundary point $\frac{b_j}{r_j}$.

Notice that there is no term related to r on the subinterval $(0, \frac{b_1}{r_1}]$.

Definition 1. If a linear combination $\sum_{j=1}^n c_j p_j$ satisfies the above conditions (1) and (2), we say that a linear combination $\sum_{j=1}^n c_j p_j$ is a linearized equation form on $\{\frac{b_1}{r_1}, \ldots, \frac{b_{m-1}}{r_{m-1}}, \frac{1}{2}\}$ or linearized on $(0, \frac{1}{2}]$ for short.

As an example, consider the following linear combination:

$$LE_{1,7}:\ 3p_2 + p_3 - p_4 - p_5 - p_6 + p_7 = 2l_1 - l_2 - 2l_3 - l_4 + l_6.$$

Compute each l_i on the subintervals determined by ED_7 and add up. Then,

$$3p_2 + p_3 - p_4 - p_5 - p_6 + p_7 = \begin{cases} \sum(-2b) & \text{on } (0, \frac{1}{6}] \\ \sum(-r+4b) & \text{on } [\frac{1}{6}, \frac{1}{4}] \\ 0 & \text{on } [\frac{1}{4}, \frac{1}{2}] \end{cases}$$

which the summation is over points $\frac{b}{r}$ of a basket of singularities in each subinterval. Thus we have a linearized equation form $LE_{1,7}$ on $(0, \frac{1}{2}]$. Notice that $LE_{1,7}$ has a non-positive value at every point in $(0, \frac{1}{2}]$.

Definition 2. For the sake of simplicity, let's denote 'n points of type $\frac{b}{r}$' by $n \times \frac{b}{r}$ or $n\frac{b}{r}$.

Define an operation $\uplus$ by

$$n_1 \frac{b_1}{r_1} \uplus n_2 \frac{b_2}{r_2} = \frac{n_1 b_1 + n_2 b_2}{n_1 r_1 + n_2 r_2}.$$

Since the next lemma is easily obtained from the construction of ED_n, we just state the lemma without a proof.

Lemma 4. *Let $ED_n = \{\, \frac{b_i}{r_i} \,\}$. Suppose that $\frac{b_{i-1}}{r_{i-1}} \le \frac{b}{r} \le \frac{b_i}{r_i}$ with $(r, b) = 1$.*

Then there are unique nonnegative integers m_{i-1}, m_i such that

$$\frac{b}{r} = m_{i-1} \frac{b_{i-1}}{r_{i-1}} \uplus m_i \frac{b_i}{r_i}.$$

Moreover, $m_{i-1} = -br_i + b_i r$ and $m_i = br_{i-1} - b_{i-1} r$.

Suppose that $\sum_{j=1}^{n} c_j p_j$ is linearized on $ED_n = \{\, \frac{b_i}{r_i} \,\}$.

Now, we are going to construct a new basket of singularities from the original basket $\mathcal{B} = \{\frac{b}{r}\}$ of singularities.

(i) For a point $\frac{b}{r}$ in $\mathcal{B}$ with $\frac{b_{i-1}}{r_{i-1}} \le \frac{b}{r} \le \frac{b_i}{r_i}$, there exist m_{i-1}, m_i by the above lemma. Then put these points $m_{i-1} \times \frac{b_{i-1}}{r_{i-1}}$ and $m_i \times \frac{b_i}{r_i}$ in the new basket. Simply we may think that a point $\frac{b}{r}$ in $\mathcal{B}$ is transformed into $\{m_{i-1} \times \frac{b_{i-1}}{r_{i-1}}, m_i \times \frac{b_i}{r_i}\}$.

(ii) For a point $\frac{b}{r}$ in $\mathcal{B}$ with $\frac{b}{r} \le \frac{1}{n-1}$, put points $b \times \frac{1}{n-1}$ in the new basket. Simply speaking, a point $\frac{b}{r}$ in $\mathcal{B}$ is transformed into $b \times \frac{1}{n-1}$.

Let's consider the case (i).

Since $\sum_j c_j p_j$ is linearized on ED_n, the equation form of $\sum_j c_j p_j$ on the subinterval $[\frac{b_{i-1}}{r_{i-1}}, \frac{b_i}{r_i}]$ is given as follows:

$$\sum a_i b + d_i r,$$

where the summation is over the points of $\mathcal{B}$ in $[\frac{b_{i-1}}{r_{i-1}}, \frac{b_i}{r_i}]$.

The contribution of $\frac{b}{r}$ to $\sum_j c_j p_j$ is equal to the sum of two contributions of $m_{i-1}\,\frac{b_{i-1}}{r_{i-1}}$ and $m_i\,\frac{b_i}{r_i}$ to $\sum_j c_j p_j$, i.e.,

$$\begin{aligned}\left(\sum_j c_j p_j\right)|_{\frac{b}{r}} &= a_i b + d_i r \\ &= a_i\left(m_{i-1}b_{i-1} + m_i b_i\right) \,+\, d_i\left(m_{i-1}r_{i-1} + m_i r_i\right) \\ &= m_{i-1}\left(a_i b_{i-1} + d_i r_{i-1}\right) \,+\, m_i\left(a_i b_i + d_i r_i\right) \\ &= m_{i-1}\left(\sum_j c_j p_j\right)|_{\frac{b_{i-1}}{r_{i-1}}} + m_i\left(\sum_j c_j p_j\right)|_{\frac{b_i}{r_i}}.\end{aligned}$$

In the case (ii), the both contributions of a point $\frac{b}{r}$ in $\mathcal{B}$ and $b \times \frac{1}{n-1}$ to $\sum_j c_j p_j$ are also same since

$$\left(\sum_j c_j p_j\right)|_{\frac{b}{r}} = a_1 b = b\left(\sum_j c_j p_j\right)|_{\frac{1}{n-1}}.$$

Therefore, to compute $\sum_{j=1}^{n} c_j p_j$ which is linearized on ED_n, it is not necessary to use the original basket $\mathcal{B}$ of singularities. Instead, it is enough to use a newly constructed basket from the original basket $\mathcal{B}$ which is described above.

Definition 3. Denote by $\mathcal{B}_n$ a basket which is newly constructed above from the original basket $\mathcal{B}$. Let's call $\mathcal{B}_n$ 'the n-th linearized basket' of $\mathcal{B}$ on ED_n.

In fact, $\mathcal{B}_n$ consists of points in ED_n.

From now on, as a notation we are going to use $\mathcal{B}$ for the original basket of singularities and $\mathcal{B}_n$ for the n-th linearized basket of $\mathcal{B}$ on ED_n.

The following lemma is useful to see the gap between the original basket $\mathcal{B}$ and the newly constructed basket $\mathcal{B}_n$.

Lemma 5. *Let $\mathcal{B}_n = \{\frac{b_i}{r_i}\}$ be the n-th linearized basket of $\mathcal{B} = \{\frac{b}{r}\}$ on ED_n. Then*

$$\sum_{\mathcal{B}} \frac{b^2}{r} \le \sum_{\mathcal{B}_n} \frac{b_i^2}{r_i}.$$

Proof. Let's consider the following two cases

$$(1)\ \frac{b}{r} \le \frac{1}{n-1}, \quad (2)\ \frac{b_{i-1}}{r_{i-1}} \le \frac{b}{r} \le \frac{b_i}{r_i}.$$

For the case (1), it is enough to show $\frac{b^2}{r} \le b\frac{1^2}{n-1}$ since a point $\frac{b}{r}$ in $\mathcal{B}$ is transformed into $b \times \frac{1}{n-1}$ by Lemma 4. Thus,

$$b\frac{1^2}{n-1} - \frac{b^2}{r} = b\left(\frac{1}{n-1} - \frac{b}{r}\right) \ge 0.$$

For the case (2), there are m_{i-1} and m_i such that $\frac{b}{r} = m_{i-1}\frac{b_{i-1}}{r_{i-1}} \uplus m_i\frac{b_i}{r_i}$, where $m_{i-1} = -br_i + b_i r$ and $m_i = br_{i-1} - b_{i-1}r$.

Thus, it is enough to check $\frac{b^2}{r} \le m_{i-1}\frac{b_{i-1}^2}{r_{i-1}} + m_i\frac{b_i^2}{r_i}$.

$$\begin{aligned}
& m_{i-1}\frac{b_{i-1}^2}{r_{i-1}} + m_i\frac{b_i^2}{r_i} - \frac{b^2}{r} \\
= & \frac{(-br_i + b_i r)b_{i-1}^2 r r_i + (br_{i-1} - b_{i-1}r)b_i^2 r_{i-1} r - r_{i-1}r_i b^2}{r_{i-1}r_i r} \\
= & \frac{-b_{i-1}b_i r^2 + (b_i r_{i-1} + b_{i-1}r_i)rb - r_{i-1}r_i b^2}{r_{i-1}r_i r} \\
= & -\frac{(b_{i-1}r - r_{i-1}b)(b_i r - r_i b)}{r_{i-1}r_i r} \\
= & -r\left(\frac{b_{i-1}}{r_{i-1}} - \frac{b}{r}\right)\left(\frac{b_i}{r_i} - \frac{b}{r}\right) \ge 0.
\end{aligned}$$

Recall that the construction of ED_n shows $b_i r_{i-1} - b_{i-1} r_i = 1$. □

Remark. By the construction of $\mathcal{B}_n$, $\sum_{\mathcal{B}} b = \sum_{\mathcal{B}_n} b_i$. By Lemma 2,

$$\chi(\mathcal{O}_X) = \sum_{\mathcal{B}} \frac{b}{10} + \frac{-5p_2 + p_3}{10} = \sum_{\mathcal{B}_n} \frac{b_i}{10} + \frac{-5p_2 + p_3}{10}.$$

Remark. One of main tools is using appropriate linearized equation forms $\sum_j c_j p_j$ for our situation. Most of them have non-positive values at every point $\frac{b}{r}$ in the interval $(0, \frac{1}{2}]$. We are going to denote by LE_i non-positive linearized equation forms, most of which will be shown up later.

In proving the results, there are some parts which are very difficult to do without using mathematical software or computer programming, such as finding linearized equation forms, computing explicit expressions, checking non-positiveness of LE_i on subintervals, and solving a system of linear equations. These not only can be done easily by computer software, but also require huge space to write in details. Thus, we are not going to present them here. We will explain the method through the example $LE_{1,7}$.

Proposition 2. *For $n = 2m + 1$ $(m \ge 3)$, consider the following non-positive linearized equation form on $(0, \frac{1}{2}]$:*

$$LE_{1,n}: \ 2p_2 + p_{m-1} + p_m - p_{m+1} - p_{m+2} - p_{n-1} + p_n.$$

If $p_{m-1} \ge k$ for a positive integer k, then $p_{n-1} \ge k$. In particular, when $n = 7$, $p_2 \ge 1$ implies $p_6 \ge 2$.

Proof. To get a contradiction, suppose that $p_{n-1} < k$.

If $p_{m+1} > 0$, then $p_{n-1} \geq k$ since $p_{m-1} \geq k \geq 1$. Thus, $p_{m+1} = 0$. Also, $p_n \geq p_{m+2}$ since $p_{m-1} \geq k$. Rearrange terms in $LE_{1,n}$ as follows:

$$LE_{1,n}: \; 2p_2 + (p_{m-1} - p_{n-1}) + p_m + (p_n - p_{m+2}).$$

Each term is non-negative. Thus, each term must be zero since $LE_{1,n}$ is non-positive on $(0, \frac{1}{2}]$. It means $p_{n-1} = p_{m-1} \geq k$. It is a contradiction.

When $n = 7$, $LE_{1,7} = (3p_2 - p_6 - p_4) + p_3 + (p_7 - p_5)$. If $p_6 = 1$, then $p_4 = 1$. It implies that $LE_{1,7}$ is positive on $(0, \frac{1}{2}]$. Thus, $p_6 \geq 2$. □

Proposition 3. *If $p_3 \geq 1$, then there exists $n \in \{4, 5, 6\}$ such that $p_n \geq 2$ except the following cases.*

case	p_2	p_3	p_4	p_5	p_6	p_7
(1)	0	1	0	0	1	0
(2)	0	1	0	1	1	0
(3)	0	1	0	1	1	1
(4)	0	1	1	0	1	1
(5)	0	1	1	1	1	1
(6)	0	1	1	1	1	2

Proof. Our claim holds true since $p_6 \geq 2$ if $p_3 \geq 2$. We also know $p_6 \geq 2$ if $p_2 \geq 1$. Thus, it is enough to consider the case that $p_2 = 0$ and $p_3 = 1$.

To find all the possible exceptional cases, suppose that $p_i \leq 1\,(i = 4, 5, 6)$.

Since $p_3 = 1$, $p_6 = 1$ clearly. Since p_i $(i = 2, \ldots, 6)$ is given, p_7 should be determined to keep $LE_{1,7} \leq 0$. Hence we have 8 possible exceptional cases, i.e., the above 6 cases plus the following two more cases:

(7)	0	1	1	0	1	0
(8)	0	1	1	1	1	0

But cases (7) and (8) can't happen since $p_3 = p_4 = 1$ imply $p_7 \geq 1$. □

There is an easy way to find a linearized form $\sum_j^n c_j p_j$, which is replacing K_X^3 and $\chi(\mathcal{O}_X)$ in p_n $(n \geq 4)$ by terms given in Lemma 2.

As an example, consider $p_4 = 7K_X^3 - 7\chi(\mathcal{O}_X) + L(4)$.

After replacing K_X^3 and $\chi(\mathcal{O}_X)$ by terms given in Lemma 2 and simplify. Then we have the following:

$$p_4 = -\frac{7}{2}p_2 + \frac{7}{2}p_3 + \begin{cases} \sum -\frac{1}{2}b & \text{on } (0, \frac{1}{3}] \\ \sum \frac{5}{2}b - r & \text{on } [\frac{1}{3}, \frac{1}{2}], \end{cases}$$

where each summation is over points of $\mathcal{B}$ in each subinterval. Then, we obtain a linearized form $p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2$ on $ED_4 = \{\frac{1}{3}, \frac{1}{2}\}$.

Now, construct $\mathcal{B}_4$ on ED_4 from the original basket $\mathcal{B}$, i.e.,

$$\mathcal{B}_4 = \{t(1) \times \frac{1}{3},\, t(2) \times \frac{1}{2}\},$$

where $t(i)$ is the number of each point.

As explained just above Definition 3, $p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2$ can be computed using $\mathcal{B}_4$ instead of the original basket $\mathcal{B}$ since it is linearized on $(0, \frac{1}{2}]$. In fact, since the values of $p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2$ are $-\frac{1}{2}$ and $\frac{1}{2}$ at points $\frac{1}{3}$ and $\frac{1}{2}$ respectively,

$$p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2 = -\frac{1}{2}t(1) + \frac{1}{2}t(2).$$

Construct $\mathcal{B}_7$ on $ED_7 = \{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}\}$ from the original basket $\mathcal{B}$.

$$\mathcal{B}_7 = \left\{ n(1) \times \frac{1}{6},\, n(2) \times \frac{1}{5},\, n(3) \times \frac{1}{4},\, n(4) \times \frac{1}{3},\, n(5) \times \frac{2}{5},\, n(6) \times \frac{1}{2} \right\},$$

where $n(i)$ is the number of each point in $\mathcal{B}_7$.

For $n\,(n = 4, \dots, 7)$, replace K_X^3 and $\chi(\mathcal{O}_X)$ in p_n by terms in Lemma 2 and apply the above processes to p_n. Then, we can get 4 linearized equation forms on ED_7. All these 4 linearized equation forms can be computed using $\mathcal{B}_7$ instead of $\mathcal{B}$. As an example, this time

$$p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2 = -\frac{1}{2}n(1) - \frac{1}{2}n(2) - \frac{1}{2}n(3) - \frac{1}{2}n(4) + 0\,n(5) + \frac{1}{2}n(6)$$

since an equation of $p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2$ on each subinterval is given above.

Find all the linearized equation forms obtained for $n = 4, \dots, 7$ using $\mathcal{B}_7$. Then we obtain a system of linear equations of $n(i)$:

$$\begin{pmatrix} p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2 \\ p_5 - \frac{81}{10}p_3 + \frac{21}{2}p_2 \\ p_6 - \frac{77}{5}p_3 + 22p_2 \\ p_7 - 26p_3 + 39p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{19}{10} & -\frac{19}{10} & -\frac{19}{10} & -\frac{9}{10} & \frac{1}{5} & \frac{11}{10} \\ -\frac{23}{5} & -\frac{23}{5} & -\frac{18}{5} & -\frac{8}{5} & -\frac{1}{5} & \frac{12}{5} \\ -9 & -8 & -6 & -3 & 0 & 4 \end{pmatrix} \begin{pmatrix} n(1) \\ n(2) \\ n(3) \\ n(4) \\ n(5) \\ n(6) \end{pmatrix}.$$

Solve the above equation and the solutions are:

$$\begin{aligned} n(1) &= 2n(6) - 3n(4) - 9p_2 + 14p_3 - 10p_4 + 2p_5 + 2p_6 - p_7, \\ n(2) &= -4n(6) + 6n(4) + 15p_2 - 29p_3 + 21p_4 - 3p_5 - 3p_6 + p_7, \\ n(3) &= 3n(6) - 4n(4) - 13p_2 + 22p_3 - 13p_4 + p_5 + p_6, \\ n(5) &= 4n(6) - 5n(4) - 14p_2 + 26p_3 - 19p_4 + 5p_5. \end{aligned}$$

Theorem 1. *There are inequalities between K_X^3, $\chi(\mathcal{O}_X)$ and p_n.*

(1) $K_X^3 \le \frac{1}{6}\chi(\mathcal{O}_X) - \frac{1}{3}p_2 + \frac{1}{6}p_4$.
(2) $K_X^3 \le \frac{1}{12}\chi(\mathcal{O}_X) - \frac{1}{12}p_2 - \frac{1}{12}p_3 + \frac{1}{12}p_5$.
(3) $K_X^3 \le \frac{1}{20}\chi(\mathcal{O}_X) - \frac{1}{20}p_2 - \frac{1}{20}p_4 + \frac{1}{20}p_6$.

(4) $K_X^3 \leq -\frac{1}{30}\chi(\mathcal{O}_X) - \frac{1}{12}p_2 + \frac{1}{30}p_3 - \frac{1}{20}p_4 + \frac{1}{60}p_5 - \frac{1}{60}p_6 + \frac{1}{30}p_7 + \frac{1}{60}n(6)$, *where* $n(6)$ *is the number of the point* $\frac{1}{2}$ *in* $\mathcal{B}_7$ *as explained above.*

Proof. Let $\mathcal{B} = \{\frac{b}{r}\}$ be the original basket of singularities. Let $\mathcal{B}_n = \{\frac{b_i}{r_i}\}$ be the n-th linearized basket of singularities.

We are going to prove the case (4) first. The proofs for the other cases are almost same.

For a proof of (4), we are going to use $\mathcal{B}_7$ and $n(i)$ obtained just above. By the remark below Lemma 5,

$$\begin{aligned}
\chi(\mathcal{O}_X) &= \sum_{\mathcal{B}} \frac{b}{10} + \frac{-5p_2 + p_3}{10} \\
&= \sum_{\mathcal{B}_7} \frac{b_i}{10} + \frac{-5p_2 + p_3}{10} \\
&= \frac{1}{10}(n(1) + n(2) + n(3) + n(4) + 2n(5) + n(6)) + \frac{-5p_2 + p_3}{10} \\
&= n(6) - n(4) - 4p_2 + 6p_3 - 4p_4 + p_5.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
n(4) =& -\chi(\mathcal{O}_X) + n(6) - 4p_2 + 6p_3 - 4p_4 + p_5, \\
\sum_{\mathcal{B}_7} \frac{b_i^2}{r_i} =& \; n(1)\frac{1}{6} + n(2)\frac{1}{5} + n(3)\frac{1}{4} + n(4)\frac{1}{3} + n(5)\frac{2^2}{5} + n(6)\frac{1}{2} \\
=& \; \frac{1}{30}p_7 - \frac{1}{60}p_6 + \frac{239}{60}p_5 - \frac{191}{12}p_4 + \frac{137}{6}p_3 - \frac{259}{20}p_2 \\
& - \frac{119}{30}n(4) + \frac{239}{60}n(6).
\end{aligned}$$

Since $n(4) = -\chi(\mathcal{O}_X) + n(6) - 4p_2 + 6p_3 - 4p_4 + p_5$,

$$\sum_{\mathcal{B}_7} \frac{b_i^2}{r_i} = \frac{1}{30}p_7 - \frac{1}{60}p_6 + \frac{1}{60}p_5 - \frac{1}{20}p_4 - \frac{29}{30}p_3 + \frac{35}{12}p_2 + \frac{1}{60}n(6) + \frac{119}{30}\chi(\mathcal{O}_X).$$

Recall $\sum_{\mathcal{B}} \frac{b^2}{r} \leq \sum_{\mathcal{B}_7} \frac{b_i^2}{r_i}$ in Lemma 5. By Lemma 2,

$$\begin{aligned}
K_X^3 &= \sum_{\mathcal{B}} \frac{b^2}{r} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3 \\
&\leq \sum_{\mathcal{B}_7} \frac{b_i^2}{r_i} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3 \\
&\leq -\frac{1}{30}\chi(\mathcal{O}_X) - \frac{1}{12}p_2 + \frac{1}{30}p_3 - \frac{1}{20}p_4 + \frac{1}{60}p_5 - \frac{1}{60}p_6 + \frac{1}{30}p_7 + \frac{1}{60}n(6).
\end{aligned}$$

For (1), consider $p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2$ on ED_4. It can be computed as follows:

$$p_4 - \frac{7}{2}p_3 + \frac{7}{2}p_2 = -t(1)\frac{1}{2} + t(2)\frac{1}{2},$$

where $t(i)$ is the number of each point in $\mathcal{B}_4$. (See below Proposition 3.) From this, we have $t(2) = 2p_4 - 7p_3 + 7p_2 + t(1)$.

$$\begin{aligned}\chi(\mathcal{O}_X) &= \sum_{\mathcal{B}_4} \frac{b_i}{10} + \frac{-5p_2 + p_3}{10} \\ &= t(1)\frac{1}{10} + t(2)\frac{1}{10} + \frac{-5p_2 + p_3}{10} \\ &= \frac{t(1)}{5} + \frac{p_4 - 3p_3 + p_2}{5}.\end{aligned}$$

Thus, we have

$$t(1) = 5\chi(\mathcal{O}_X) - p_2 + 3p_3 - p_4.$$

Since $\sum_{\mathcal{B}_4} \frac{b_i^2}{r_i} = t(1)\frac{1}{3} + t(2)\frac{1}{2}$,

$$K_X^3 \leq \sum_{\mathcal{B}_4} \frac{b_i^2}{r_i} - 4\chi(\mathcal{O}_X) - 3p_2 + p_3 = \frac{1}{6}\chi(\mathcal{O}_X) - \frac{1}{3}p_2 + \frac{1}{6}p_4.$$

For (2) and (3), apply the same processes to p_n for $n\,(n = 4,\,5)$ on ED_5 and $n\,(4 \leq n \leq 6)$ on ED_6 respectively. Remaining steps are same. ☐

Now, we are going to investigate 6 exceptional cases in Proposition 3.

Lemma 6. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 0,\ p_5 = 0,\ p_6 = 1,\ p_7 = 0$, *i.e., the case* (1) *in Proposition 3. Then*

$$1 \leq p_9 \leq p_8 \leq p_{11}, \quad p_{12} \geq 2 \textit{ and } p_n \geq 2 \textit{ for } n \geq 14.$$

Proof. Consider the non-positive linearized equation form $LE_{1,9}$ on $(0, \frac{1}{2}]$:

$$LE_{1,9}:\ 2p_2 + p_3 + p_4 - p_5 - p_6 - p_8 + p_9 = -p_8 + p_9,$$

since $p_n\,(n = 2, \ldots, 7)$ are given. Since $LE_{1,9} \leq 0$ on $(0, \frac{1}{2}]$ and $p_3 = 1$,

$$1 \leq p_9 \leq p_8 \leq p_{11}.$$

To show $p_{12} \geq 2$, consider the linearized equation form:

$$LE_2:\ 3p_2 + 3p_3 + p_4 - p_5 - p_6 - p_7 - p_{12} + p_{13},$$

which is non-positive on $(0, \frac{1}{2}]$. By the conditions on $p_n\,(n = 2, \ldots, 7)$,

$$LE_2 : 3p_2 + 3p_3 + p_4 - p_5 - p_6 - p_7 - p_{12} + p_{13} = 2 - p_{12} + p_{13}.$$

Since $LE_2 \leq 0$ on $(0, \frac{1}{2}]$ and $p_{13} \geq 0$,

$$p_{12} \geq 2.$$

To show $p_n \geq 2$ for $n \geq 14$, we consider two cases $p_8 \geq 2$ and $p_8 = 1$.

Suppose $p_8 \geq 2$.

Clearly $p_{14} \geq 2$ and $p_{16} \geq 2$. $p_{12} \geq 2$ implies $p_{15} \geq 2$. Thus, since $p_3 = 1$,

$$p_n \geq 2 \quad \text{for } n \geq 14.$$

Now, suppose $p_8 = 1$.

Consider another linearized equation form:

$$LE_3 : \ 7p_2 + 4p_3 + 2p_4 - p_5 - 2p_6 - 2p_8 - p_{10} + p_{13},$$

which is non-positive on $(0, \frac{1}{2}]$ and is zero only at $\frac{b}{r} = \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}$.

$$LE_3 : 7p_2 + 4p_3 + 2p_4 - p_5 - 2p_6 - 2p_8 - p_{10} + p_{13} = -p_{10} + p_{13} \leq 0,$$

since p_n are known for $n = 2, \ldots, 8$. Thus, $p_{13} \leq p_{10}$. Since $p_3 = 1$, $p_{10} \leq p_{13}$ clearly. Hence $p_{10} = p_{13}$.

It means that the equation form LE_3 is identically zero on $(0, \frac{1}{2}]$.

Since $LE_3|_{\frac{b}{r}}$ is zero only at $\frac{b}{r} = \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}$, the original basket $\mathcal{B}$ of singularities must consist of points $\frac{1}{4}$, $\frac{2}{7}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{1}{2}$ only, i.e.,

$$\mathcal{B} = \{t(1) \times \frac{1}{4},\, t(2) \times \frac{2}{7},\, t(3) \times \frac{1}{3},\, t(4) \times \frac{2}{5},\, t(5) \times \frac{3}{7},\, t(6) \times \frac{1}{2}\},$$

where $t(i)$ is the number of each point.

Now, p_n is known for $n = 2, \ldots, 9$. Recall that $1 \leq p_9 \leq p_8$.

Applying the following steps (1)$\sim$(4) to p_n for $n = 4, \ldots, 9$ with the basket $\mathcal{B}$, we obtain the system of linear equations of $t(i)$:

(step 1) replace $\chi(\mathcal{O}_X)$ and K_X^3 in p_n by terms in Lemma 2.

(step 2) rearrange the terms and obtain a linearized form for each n.

(step 3) construct a system of linear equations of $t(i)$ using the step 2.

(step 4) solve the system of linear equations of $t(i)$ in step 3.

In fact, the system of linear equations of $t(i)$ is given as follows:

$$\begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{19}{10} & -\frac{14}{5} & -\frac{9}{10} & \frac{1}{5} & \frac{13}{10} & \frac{11}{10} \\ -\frac{18}{5} & -\frac{26}{5} & -\frac{8}{5} & -\frac{1}{5} & \frac{11}{5} & \frac{12}{5} \\ -6 & -9 & -3 & 0 & 4 & 4 \\ -\frac{19}{2} & -15 & -\frac{9}{2} & 0 & \frac{11}{2} & \frac{13}{2} \\ -\frac{29}{2} & -22 & -\frac{13}{2} & 0 & \frac{17}{2} & \frac{19}{2} \end{pmatrix} \begin{pmatrix} t(1) \\ t(2) \\ t(3) \\ t(4) \\ t(5) \\ t(6) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{81}{10} \\ -\frac{72}{5} \\ -26 \\ -\frac{79}{2} \\ -\frac{117}{2} \end{pmatrix}.$$

Then we obtain solutions $t(i)$ as follows:

$$\begin{cases} t(1) = 2t(6) - 3t(4), & t(2) = 1 - t(6) + 2t(4), \\ t(3) = 3 + 3t(6) - 4t(4), & t(5) = -2 + 2t(6) - 3t(4). \end{cases}$$

Using $t(i)$, compute $\chi(\mathcal{O}_X)$ and K_X^3 by Lemma 2. Then

$$\chi(\mathcal{O}_X) = -t(4) + t(6),\ K_X^3 = \frac{t(4)}{420}.$$

Thus, $t(4) \geq 1$ and $t(6) \geq 2$ since $\chi(\mathcal{O}_X)$ and K_X^3 are positive.

Now, we are ready to compute p_n for $n = 14$, 15, 16 since all the information, i.e., $\chi(\mathcal{O}_X)$, K_X^3 and the basket of singularities are known.

$$p_{14} = 1 + t(4),\ p_{15} = 2 + t(4),\ \text{and } p_{16} = 1 + t(4).$$

Since $t(4) \geq 1$, $p_n \geq 2$ for $n = 14, 15, 16$. Therefore, since $p_3 = 1$,

$$p_n \geq 2 \text{ for } n \geq 14. \qquad \square$$

The proofs for the remaining cases are similar to Lemma 6. Almost the same processes are going to be applied.

Lemma 7. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 0,\ p_5 = 1,\ p_6 = 1,\ p_7 = 0$, *i.e., the case* (2) *in Proposition 3. Then*

$$p_n \geq 1 \textit{ for } n \geq 8, \quad p_9 \geq 2,\ p_{12} \geq 2 \textit{ and } p_n \geq 2 \textit{ for } n \geq 14.$$

Proof. Clearly, $p_n \geq 1$ for $n = 8, 9, 10$ by the assumption. Since $p_3 = 1$,

$$p_n \geq 1 \text{ for } n \geq 8.$$

Consider the following linearized equation form:

$$LE_4 :\ 5p_2 + 3p_3 + p_4 + p_5 - p_6 - p_7 - 2p_8 - 2p_9 + 2p_{11},$$

which is non-positive on $(0, \frac{1}{2}]$. By the given conditions for $p_n\,(n = 2, \ldots, 7)$,

$$LE_4 : 5p_2 + 3p_3 + p_4 + p_5 - p_6 - p_7 - 2p_8 - 2p_9 + 2p_{11} = 3 - 2p_8 - 2p_9 + 2p_{11} \leq 0.$$

Since $p_3 = 1$, we have $-2p_8 + 2p_{11} \geq 0$. Thus, $3 - 2p_9 \leq 0$. Hence

$$p_9 \geq 2 \text{ and } p_{12} \geq 2.$$

Next, we are going to show $p_n \geq 2$ for $n \geq 14$.

Since $p_5 = p_6 = 1$ and $p_9 \geq 2$, we have $p_{14} \geq 2$ and $p_{15} \geq 2$. To show $p_n \geq 2$ for $n \geq 14$, it's enough to show $p_{16} \geq 2$.

If $p_8 \geq 2$, then $p_{16} \geq 2$. Hence it is enough to consider the case $p_8 = 1$.

Suppose $p_8 = 1$.

Consider the following non-positive linearized equation form on $(0, \frac{1}{2}]$:

$$LE_5 : 15p_2 + 10p_3 + 10p_4 - 2p_5 - 5p_6 - 3p_7 - 8p_8 + 2p_9 - 2p_{10} + p_{11} + 2p_{13},$$

which is zero only at $\frac{b}{r} = \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}$ in the interval $(0, \frac{1}{2}]$.

$$LE_5 = -5 + 2p_9 + p_{11} + (-2p_{10} + 2p_{13}) \leq 0$$

by the conditions on $p_n\,(n = 2, \ldots, 8)$. Since $p_3 = 1$, $-2p_{10} + 2p_{13} \geq 0$. Thus, $p_9 = 2$, $p_{11} = 1$ and $p_{10} = p_{13}$ since that $p_9 \geq 2$ and $p_{11} \geq 1$. Then, LE_5 is identically zero on $(0, \frac{1}{2}]$.

It means that the original basket $\mathcal{B}$ is given as follows:

$$\mathcal{B} = \{t(1) \times \frac{1}{5}, t(2) \times \frac{1}{4}, t(3) \times \frac{2}{7}, t(4) \times \frac{1}{3}, t(5) \times \frac{2}{5}, t(6) \times \frac{3}{7}, t(7) \times \frac{1}{2}\},$$

where $t(i)$ is the number of each point.

Now apply the step (1)$\sim$(4) to $p_n\,(n = 2, \ldots, 9)$ and obtain the system of linear equations of $t(i)$. Then

$$t(1) = 1, \quad t(2) = t(6), \quad t(3) = -6 + t(4) - 2t(6),$$
$$t(5) = -11 + 2t(4) - 3t(6), \quad t(7) = -18 + 3t(4) - 4t(6).$$

Using $t(i)$, get $\chi(\mathcal{O}_X)$ and K_X^3 by Lemma 2. Then

$$\chi(\mathcal{O}_X) = -5 + t(4) - t(6),\ K_X^3 = -\frac{1}{35} - \frac{1}{140}t(6) + \frac{1}{210}t(4).$$

Compute p_{16} using all the information $\chi(\mathcal{O}_X)$, K_X^3 and $\mathcal{B}$.

$$p_{16} = -11 + 2t(4) - 3t(6) = 2t(3) + 1 + t(6).$$

If $t(3) = t(6) = 0$, then $t(4) = 6$ since $t(3) = -6 + t(4) - 2t(6)$. It means $K_X^3 = 0$. It contradicts since K_X^3 is positive. Thus,

$$p_{16} \geq 2.$$

□

Lemma 8. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 0,\ p_5 = 1,\ p_6 = 1,\ p_7 = 1$, *i.e., the case* (3) *in Proposition 3. Then*

$$p_n \geq 1 \quad \textit{for} \quad n \geq 5, \qquad p_{12} \geq 2 \quad \textit{and} \quad p_n \geq 2 \quad \textit{for} \quad n \geq 14.$$

Proof. Clearly, $p_n \geq 1$ for $n \geq 5$ since $p_3 = 1$ and $p_5 = p_6 = p_7 = 1$.

Consider the following two non-positive linearized forms on $(0, \frac{1}{2}]$:

$$LE_2 : 3p_2 + 3p_3 + p_4 - p_5 - p_6 - p_7 - p_{12} + p_{13}$$
$$LE_6 : 4p_2 + 2p_3 - p_6 - p_8 - p_9 + p_{11}.$$

By the conditions on p_n for n $(4 \leq n \leq 7)$,

$$p_{13} \leq p_{12} \text{ from } LE_2$$
$$1 + p_{11} \leq p_8 + p_9 \text{ from } LE_6.$$

If $p_8 \geq 2$, then $p_{13} \geq 2$, $p_{14} \geq 2$ and $p_{12} \geq 2$ since $p_5 = p_6 = 1$ and $p_{13} \leq p_{12}$. Thus, if $p_8 \geq 2$, then we have

$$p_n \geq 2 \quad \text{for } n \geq 12.$$

If $p_9 \geq 2$, then $p_{12} \geq 2$ since $p_3 = 1$. $p_{14} \geq 2$, $p_{15} \geq 2$ and $p_{16} \geq 2$ since $p_5 = p_6 = p_7 = 1$. Thus, if $p_9 \geq 2$, then we have

$$p_{12} \geq 2 \quad \text{and} \quad p_n \geq 2 \quad \text{for } n \geq 14.$$

To complete the proof, it is enough to consider the case $p_8 = p_9 = 1$.

Then $p_{11} = 1$ since $1 + p_{11} \leq p_8 + p_9 = 2$ from LE_6.

First, let's prove $p_{12} \geq 2$. To get a contradiction, assume $p_{12} = 1$.

Then $p_{10} = p_{13} = 1$ since $p_3 = 1$ and $p_{10} \leq p_{13} \leq p_{12} = 1$. Thus,

$$p_2 = 0,\ p_3 = 1,\ p_4 = 0, \ \text{ and } p_n = 1 \text{ for } n = 5, \dots, 13.$$

Consider the non-positive linearized equation form $LE_{1,7}$ on $(0, \frac{1}{2}]$:

$$LE_{1,7}:\ 3p_2 + p_3 - p_4 - p_5 - p_6 + p_7,$$

which is zero only at points in $[\frac{1}{4}, \frac{1}{2}]$. By the condition on p_n for $n = 2, \dots, 7$, the linearized form $LE_{1,7}$ is identically zero on $(0, \frac{1}{2}]$. Thus, the original basket $\mathcal{B}$ consists of the points in $[\frac{1}{4}, \frac{1}{2}]$.

Consider the non-positive linearized equation form LE_7 on $(0, \frac{1}{2}]$:

$$LE_7: \ 7p_2 + 5p_3 + 3p_4 + p_5 - 3p_7 - 2p_8 - p_9 - p_{10} - 2p_{12} + 3p_{13},$$

which is zero only at points in $\{\frac{1}{5}, \frac{2}{7}, \frac{1}{3}\} \cup [\frac{3}{8}, \frac{2}{5}] \cup \{\frac{1}{2}\}$. By the condition on p_n for $n = 2, \ldots, 13$, the linearized form LE_7 is identically zero on $(0, \frac{1}{2}]$. Thus, the original basket $\mathcal{B}$ consists of the points in $\{\frac{1}{5}, \frac{2}{7}, \frac{1}{3}\} \cup [\frac{3}{8}, \frac{2}{5}] \cup \{\frac{1}{2}\}$.

Hence, the original basket $\mathcal{B}$ of singularities must consist of the points in

$$\{\frac{2}{7}, \frac{1}{3}\} \cup [\frac{3}{8}, \frac{2}{5}] \cup \{\frac{1}{2}\}.$$

It means that the 9-th linearized basket $\mathcal{B}_9$ from $\mathcal{B}$ must be given as follows:

$$\mathcal{B}_9 = \{t(1) \times \frac{2}{7}, t(2) \times \frac{1}{3}, t(3) \times \frac{3}{8}, t(4) \times \frac{2}{5}, t(5) \times \frac{1}{2}\},$$

where $t(i)$ is the number of each point in $\mathcal{B}_9$.

To obtain more information about $\mathcal{B}$, consider another non-positive linearized equation form LE_8 on $(0, \frac{1}{2}]$:

$$LE_8: \ 6p_2 + p_4 + 3p_5 + p_6 - 3p_7 - 2p_8 - p_{10} + p_{11} - 2p_{12} + 2p_{13}.$$

The value of LE_8 at a point $\frac{b}{r}$ in $\mathcal{B}$ is given as follows:

$$LE_8|_{\frac{b}{r}} = \begin{cases} -1 & \text{at } \frac{b}{r} = \frac{2}{7} \\ 0 & \text{at } \frac{b}{r} = \frac{1}{3} \\ 0 & \text{at } \frac{b}{r} \in [\frac{3}{8}, \frac{2}{5}] \\ 0 & \text{at } \frac{b}{r} = \frac{1}{2} \end{cases}$$

LE_8 are identically -1 on $(0, \frac{1}{2}]$ by the conditions on p_n for $n = 2, \ldots, 13$. Thus, the original basket $\mathcal{B}$ must contain only one point of $\frac{2}{7}$, since $\frac{2}{7}$ is the only point which gives -1 to LE_8. It means that $t(1) = 1$ in $\mathcal{B}_9$.

Now, we are going to apply the step (1)$\sim$(4) in Lemma 6 to $p_4, \ldots, p_9$ using $\mathcal{B}_9$. Then, we have the system of linear equations of $t(i)$:

$$\begin{pmatrix} -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{14}{5} & -\frac{9}{10} & -\frac{7}{10} & \frac{1}{5} & \frac{11}{10} \\ -\frac{26}{5} & -\frac{8}{5} & -\frac{9}{5} & -\frac{1}{5} & \frac{12}{5} \\ -9 & -3 & -3 & 0 & 4 \\ -15 & -\frac{9}{2} & -\frac{9}{2} & 0 & \frac{13}{2} \\ -22 & -\frac{13}{2} & -\frac{15}{2} & 0 & \frac{19}{2} \end{pmatrix} \begin{pmatrix} t(1) \\ t(2) \\ t(3) \\ t(4) \\ t(5) \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{71}{10} \\ -\frac{72}{5} \\ -25 \\ -\frac{79}{2} \\ -\frac{117}{2} \end{pmatrix}.$$

The solutions $t(i)$ are given as follows:

$$t(2) = t(1) + 2, \quad t(3) = 1, \quad t(4) = 2t(1) - 1, \quad t(5) = 3t(1) - 4.$$

Since $t(1) = 1$, $t(5) = -1$. It contradicts since $t(i)$ is nonnegative. Therefore, if $p_8 = p_9 = 1$, then we have

$$p_{12} \geq 2.$$

Next, we are going to prove $p_n \geq 2$ for $n \geq 14$ in the case $p_8 = p_9 = 1$.

The fact $p_{12} \geq 2$ implies that $p_{15} \geq 2$ and $p_n \geq 2$ for $n \geq 17$ since $p_3 = 1$ and $p_n \geq 1$ for $n \geq 5$.

Consider the following two non-positive linearized forms on $(0, \frac{1}{2}]$:

$$LE_{1,15} : 2p_2 + p_6 + p_7 - p_8 - p_9 - p_{14} + p_{15}$$
$$LE_{1,17} : 2p_2 + p_7 + p_8 - p_9 - p_{10} - p_{16} + p_{17}.$$

By the conditions on p_n for n $(2 \leq n \leq 9)$,

$$p_{15} \leq p_{14} \text{ from } LE_{1,15}$$
$$1 + p_{17} \leq p_{10} + p_{16} \text{ from } LE_{1,17}.$$

Since $p_{15} \geq 2$, $p_{14} \geq 2$. Since $p_{17} \geq 2$ and $p_{16} \geq p_{10}$, $p_{16} \geq 2$. Therefore,

$$p_n \geq 2 \quad \text{for } n \geq 14. \qquad \square$$

Lemma 9. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 1,\ p_5 = 0,\ p_6 = 1,\ p_7 = 1$, *i.e., the case* (4) *in Proposition 3. Then*

$$p_n \geq 1 \textit{ for } n \geq 6, \quad p_8 \geq 2, p_{11} \geq 2, p_{12} \geq 2 \quad \textit{and } p_n \geq 2 \textit{ for } n \geq 14.$$

Proof. Since $p_4 = 1$, we have $p_8 \geq 1$. Thus, $p_n \geq 1$ for $n \geq 6$ clearly.

Consider the non-positive linearized form $LE_{1,9}$ on $(0, \frac{1}{2}]$:

$$LE_{1,9} :\ 2p_2 + p_3 + p_4 - p_5 - p_6 - p_8 + p_9 = 1 - p_8 + p_9 \leq 0$$

by the conditions on p_n for $n = 4, \dots, 7$. Since $p_9 \geq 1$, $p_8 \geq 2$. Thus, since $p_3 = 1$, $p_4 = 1$ and $p_n \geq 1$ for $n \geq 6$,

$$p_{11} \geq 2, p_{12} \geq 2 \quad \text{and} \quad p_n \geq 2 \quad \text{for } n \geq 14. \qquad \square$$

Lemma 10. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 1,\ p_5 = 1,\ p_6 = 1,\ p_7 = 1$, *i.e., the case* (5) *in Proposition 3. Then*

$$p_n \geq 1 \quad \textit{for } n \geq 3, \quad p_n \geq 2 \quad \textit{for } n \geq 8.$$

Proof. It is clear that $p_n \geq 1$ for $n \geq 3$.

Consider the following three non-positive $LE_{1,9}$, LE_4 and $LE_{1,11}$ on $(0, \frac{1}{2}]$:

$$LE_{1,9} : 2p_2 + p_3 + p_4 - p_5 - p_6 - p_8 + p_9 = -p_8 + p_9$$
$$LE_{1,11} : 2p_2 + p_4 + p_5 - p_6 - p_7 - p_{10} + p_{11} = -p_{10} + p_{11}$$
$$LE_4 : 5p_2 + 3p_3 + p_4 + p_5 - p_6 - p_7 - 2p_8 - 2p_9 + 2p_{11} = 3 + 2(p_{11} - p_8) - 2p_9$$

by the assumptions on p_n. Since $p_3 = 1$, $p_{11} - p_8 \geq 0$ clearly. Then we have

$$p_9 \geq 2$$

from LE_4. The non-positiveness of each LE_i implies

$$2 \leq p_9 \leq p_8 \leq p_{11} \leq p_{10}.$$

Therefore, since $p_3 = 1$,

$$p_n \geq 2 \quad \text{for } n \geq 8. \qquad \square$$

Lemma 11. *Assume that* $p_2 = 0,\ p_3 = 1,\ p_4 = 1,\ p_5 = 1,\ p_6 = 1,\ p_7 = 2$, *i.e., the case* (6) *in Proposition 3. Then*

$$p_n \geq 1 \quad \textit{for } n \geq 3 \quad \textit{and} \quad p_n \geq 2 \quad \textit{for } n \geq 10.$$

Proof. Clearly, $p_n \geq 1$ for $n \geq 3$ and also $p_n \geq 2$ for $n \geq 10$ since $p_7 = 2$. □

Theorem 2. $p_n \geq 1$ *for at least one* n *in* $\{6, 8, 10\}$.

Proof. To derive a contradiction, assume $p_6 = p_8 = p_{10} = 0$. Then,

$$p_2 = p_3 = p_4 = p_5 = 0.$$

From non-positive linearized equation forms $LE_{1,7}$ and $LE_{1,9}$ on $(0, \frac{1}{2}]$, we obtain $p_7 = 0$ and $p_9 = 0$ respectively.

Consider the following non-positive LE_9 on $(0, \frac{1}{2}]$:

$$LE_9 : \ 9p_2 + 4p_3 + 3p_4 - 3p_6 - p_7 - 2p_8 - 2p_{10} + p_{11} + p_{13},$$

which is zero only at $\frac{b}{r} = \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}$. By conditions on p_n for $n = 2, \ldots, 10$, $LE_9 = p_{11} + p_{13}$, which implies $p_{11} = p_{13} = 0$.

It means that LE_9 is identically zero on $(0, \frac{1}{2}]$. Thus, the original basket $\mathcal{B}$ of singularities must be given as follows:

$$\mathcal{B} = \{t(1) \times \frac{1}{4},\ t(2) \times \frac{1}{3},\ t(3) \times \frac{2}{5},\ t(4) \times \frac{3}{7},\ t(5) \times \frac{1}{2}\},$$

where $t(i)$ means the number of each point. Apply the step (1)∼(4) in the proof of Lemma 6 to p_n for $n = 4, \ldots, 9$ with the basket $\mathcal{B}$ of singularities. Then we have a system of linear equations of $t(i)$:

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{19}{10} & -\frac{9}{10} & \frac{1}{5} & \frac{13}{10} & \frac{11}{10} \\ -\frac{18}{5} & -\frac{8}{5} & -\frac{1}{5} & \frac{11}{5} & \frac{12}{5} \\ -6 & -3 & 0 & 4 & 4 \\ -\frac{19}{2} & -\frac{9}{2} & 0 & \frac{11}{2} & \frac{13}{2} \\ -\frac{29}{2} & -\frac{13}{2} & 0 & \frac{17}{2} & \frac{19}{2} \end{pmatrix} \begin{pmatrix} t(1) \\ t(2) \\ t(3) \\ t(4) \\ t(5) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The solutions $t(i)$ and $\chi(\mathcal{O}_X)$ are given as follows:

$$t(2) = 2t(1),\ t(3) = t(1),\ t(4) = t(1),\ t(5) = 2t(1),\ \chi(\mathcal{O}_X) = t(1).$$

From Lemma 3, we have the following:

$$\sum_{\frac{b}{r} \in \mathcal{B}} \frac{b^2}{r} = \frac{1681}{420} t(1) \quad \text{and} \ 3 \sum_{\frac{b}{r} \in \mathcal{B}} \frac{r^2 - 1}{r} - 68\chi(\mathcal{O}_X) + 3p_2 - p_3 = \frac{1353}{420} t(1).$$

Thus,

$$\sum_{\frac{b}{r} \in \mathcal{B}} \frac{b^2}{r} > 3 \sum_{\frac{b}{r} \in \mathcal{B}} \frac{r^2 - 1}{r} - 68\chi(\mathcal{O}_X) + 3p_2 - p_3,$$

which is a contradiction by Lemma 3.

Therefore, there is at least one n in $\{6, 8, 10\}$ such that $p_n \geq 1$. □

Theorem 3. *Suppose that* $p_2 \geq 1$ *or* $p_3 \geq 1$. *Then*

(1) $p_{12} \geq 2$.

(2) $p_n \geq 2$ *for* $n \geq 14$ *with a possible exceptional case which must satisfy*:

i) $p_2 \geq 1$, $p_3 = p_5 = p_7 = p_9 = 0$ *and* $p_{15} \leq 1$.

ii) $p_n \geq 2$ *for an even integer* $n\,(n \geq 6)$.

iii) $K_X^3 \leq \frac{1}{12}\chi(\mathcal{O}_X) - \frac{1}{12}p_2$.

Proof. To prove the theorem, let's consider the following two cases:

$$\text{Case (1): } p_2 \geq 1, \quad \text{Case (2): } p_2 = 0 \text{ and } p_3 \geq 1.$$

Case (1) $p_2 \geq 1$.

By Proposition 2, $p_6 \geq 2$. Since $p_2 \geq 1$, $p_n \geq 2$ for an even integer $n \geq 6$ clearly. In particular, $p_{12} \geq 2$, $p_{14} \geq 2$ and $p_{16} \geq 2$.

If $p_n \geq 1$ for at least one $n \in \{3, 5, 7, 9\}$, then we have $p_{15} \geq 2$ since $p_n \geq 2$ for an even integer $n \geq 6$. Thus, $p_n \geq 2$ for $n \geq 14$.

Therefore, we can conclude that

$$p_{12} \geq 2 \text{ and } p_n \geq 2 \text{ for } n \geq 14$$

with a possible exceptional case which is described in the theorem. The inequality $K_X^3 \leq \frac{1}{12}\chi(\mathcal{O}_X) - \frac{1}{12}p_2$ comes from (2) of Theorem 1.

Case (2) $p_2 = 0$ and $p_3 \geq 1$.

By Proposition 2, $p_8 \geq k$ if $p_3 \geq k$.

Now, let's divide this case into the following three subcases:

[(2-1) case]: $p_2 = 0$ and $p_3 \geq 2$

[(2-2) case]: $p_2 = 0$ and $p_3 = 1$ and $\exists\, n$ in $\{4, 5, 6\}$ such that $p_n \geq 2$

[(2-3) case]: $p_2 = 0$ and $p_3 = 1$ and $p_n \leq 1$ for all $n = 4, 5, 6$

Subcase (2-1) $p_2 = 0$ and $p_3 \geq 2$.

Since $p_3 \geq 2$, $p_8 \geq 2$. Then $p_n \geq 2$ for $n = 6, 8, 9, 11, 12$, and $n \geq 14$.

Subcase (2-2) $p_2 = 0$, $p_3 = 1$ and $\exists\, n$ in $\{4, 5, 6\}$ such that $p_n \geq 2$

If $p_4 \geq 2$, then $p_n \geq 2$ for $n = 4, 7, 8$ and $n \geq 10$ since $p_3 = 1$.

If $p_5 \geq 2$, then $p_n \geq 2$ for $n = 5, 8, 10, 11$ since $p_3 = 1$. By Proposition 2, $p_{12} \geq 2$. Thus, $p_n \geq 2$ for $n = 5, 8$ and $n \geq 10$.

To complete the subcase (2-2), suppose $p_6 \geq 2$.

We obtain $p_n \geq 2$ for $n = 6, 9, 12, 14, 15$ since $p_3 = 1$ and $p_8 \geq 1$.

If $p_{16} \geq 2$, then we have $p_n \geq 2$ for $n = 6, 9, 12$ and $n \geq 14$ since $p_3 = 1$.

To derive a contradiction, assume $p_{16} = 1$.

Clearly $p_8 = 1$. Moreover, $p_4 = p_5 = p_7 = p_{10} = 0$. If not, we have $p_{16} \geq 2$ since $p_n \geq 2$ for $n = 6, 9, 12$.

Since $p_3 = 1$ and $p_6 \geq 2$, we have $p_6 < p_{12} \leq p_{15}$ easily.

Recall the equation expression of $LE_{1,7}$, which is strictly negative on $(0, \frac{1}{4})$ and is identically zero on $[\frac{1}{4}, \frac{1}{2}]$. By the conditions on p_n for $n = 2, \ldots, 7$,

$$LE_{1,7} = 1 - p_6 < 0.$$

It means that the original basket $\mathcal{B}$ must contain at least one point in $(0, \frac{1}{4})$.

Consider the non-positive linearized form $LE_{1,9}$ on $(0, \frac{1}{2}]$:

$$LE_{1,9}: \ 2p_2 + p_3 + p_4 - p_5 - p_6 - p_8 + p_9,$$

which is zero only at points in $[\frac{1}{5}, \frac{1}{3}] \cup [\frac{2}{5}, \frac{1}{2}]$. By the conditions on p_n for $n = 2$, 3, 4, 5, 7 and 8, we obtain

$$LE_{1,9}: \ 2p_2 + p_3 + p_4 - p_5 - p_6 - p_8 + p_9 = -p_6 + p_9 \leq 0.$$

Since $p_6 \leq p_9$, $p_6 = p_9$. It means that $LE_{1,9}$ is identically zero on $(0, \frac{1}{2}]$. Therefore, the original basket $\mathcal{B}$ must consist of points in $[\frac{1}{5}, \frac{1}{3}] \cup [\frac{2}{5}, \frac{1}{2}]$ only.

Consider the non-positive linearized form $LE_{1,17}$ on $(0, \frac{1}{2}]$:

By the conditions on p_n for $n = 2$, 7, 8, 10 and 16,

$$LE_{1,17}: \ 2p_2 + p_7 + p_8 - p_9 - p_{10} - p_{16} + p_{17} = -p_9 + p_{17} \leq 0.$$

Since $p_9 \geq 2$ and $p_8 = 1$, we have $p_9 \leq p_{17}$. Thus, $p_9 = p_{17}$.

Since $p_3 = 1$ and $p_8 = 1$, $p_6 \leq p_{14} \leq p_{17}$ clearly. Since $p_6 = p_9 = p_{17}$,

$$p_6 = p_9 = p_{14} = p_{17}.$$

Consider the non-positive linearized form $LE_{1,15}$ on $(0, \frac{1}{2}]$:

$$LE_{1,15}: \ 2p_2 + p_6 + p_7 - p_8 - p_9 - p_{14} + p_{15},$$

which is zero only at points in $[\frac{1}{8}, \frac{1}{6}] \cup [\frac{1}{4}, \frac{1}{3}] \cup [\frac{3}{8}, \frac{1}{2}]$. By the conditions on p_n for $n = 2, \ldots, 9$,

$$LE_{1,15}: \ 2p_2 + p_6 + p_7 - p_8 - p_9 - p_{14} + p_{15} = -1 - p_{14} + p_{15} \leq 0.$$

Thus, $p_{15} \leq 1 + p_{14}$. Since $p_6 = p_{14}$ and $p_6 < p_{15}$, we obtain

$$p_{15} = p_{14} + 1.$$

Then $LE_{1,15}$ is identically zero on $(0, \frac{1}{2}]$. It means that the original basket $\mathcal{B}$ of singularities must consist of points in $[\frac{1}{8}, \frac{1}{6}] \cup [\frac{1}{4}, \frac{1}{3}] \cup [\frac{3}{8}, \frac{1}{2}]$ only.

We already showed that the original basket $\mathcal{B}$ of singularities must consist of points in $[\frac{1}{5}, \frac{1}{3}] \cup [\frac{2}{5}, \frac{1}{2}]$. Thus, we conclude that the basket $\mathcal{B}$ must consist of points in $[\frac{1}{4}, \frac{1}{3}] \cup [\frac{2}{5}, \frac{1}{2}]$ only. But, it contradicts since the original basket $\mathcal{B}$ must contain a point in $(0, \frac{1}{4})$ by $LE_{1,7}$. Thus, if $p_6 \geq 2$, then we have

$$p_{16} \geq 2.$$

In conclusion, $p_{12} \geq 2$ and $p_n \geq 2$ for $n \geq 14$ in the subcase (2-2).

Subcase (2-3) $p_2 = 0, \ p_3 = 1$ and $p_n \leq 1$ for all $n = 4$, 5, 6

The subcase (2-3) is already described in Proposition 3 and investigated through Lemma 6 to Lemma 11. We can conclude that

$$p_{12} \geq 2 \ \text{ and } \ p_n \geq 2 \ \text{ for } n \geq 14. \qquad \square$$

Table for LE_i used in the proofs

$LE_{1,n}$	$2p_2 + p_{m-1} + p_m - p_{m+1} - p_{m+2} - p_{n-1} + p_n \quad (n = 2m+1)$
LE_2	$3p_2 + 3p_3 + p_4 - p_5 - p_6 - p_7 - p_{12} + p_{13}$
LE_3	$7p_2 + 4p_3 + 2p_4 - p_5 - 2p_6 - 2p_8 - p_{10} + p_{13}$
LE_4	$5p_2 + 3p_3 + p_4 + p_5 - p_6 - p_7 - 2p_8 - 2p_9 + 2p_{11}$
LE_5	$15p_2 + 10p_3 + 10p_4 - 2p_5 - 5p_6 - 3p_7 - 8p_8 + 2p_9 - 2p_{10} + p_{11} + 2p_{13}$
LE_6	$4p_2 + 2p_3 - p_6 - p_8 - p_9 + p_{11}$
LE_7	$7p_2 + 5p_3 + 3p_4 + p_5 - 3p_7 - 2p_8 - p_9 - p_{10} - 2p_{12} + 3p_{13}$
LE_8	$6p_2 + p_4 + 3p_5 + p_6 - 3p_7 - 2p_8 - p_{10} + p_{11} - 2p_{12} + 2p_{13}$
LE_9	$9p_2 + 4p_3 + 3p_4 - 3p_6 - p_7 - 2p_8 - 2p_{10} + p_{11} + p_{13}$

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