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TWISTED TORUS KNOTS WITH GRAPH MANIFOLD DEHN SURGERIES

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Abstract. In this paper, we classify all twisted torus knots which are doubly middle Seifert-fibered. Also we show that all of these knots possibly except a few admit Dehn surgery producing a non-Seifert-fibered graph manifold which consists of two Seifert-fibered spaces over the disk with two exceptional fibers, glued together along their boundaries. This provides another infinite family of knots in S^3 admitting Dehn surgery yielding such manifolds as done in [5].

1. Introduction

Throughout this paper, we denote by $S(a_1, \ldots, a_n)$ the Seifert-fibered space over a surface S with n exceptional fibers of indexes $a_1, \ldots, a_n$.

A simple closed curve k in the boundary of a genus two handlebody H is said to be *Seifert* if $H[k]$, i.e., the 3-manifold obtained by adding a 2-handle to H along k, is a Seifert-fibered space and not a solid torus. It follows from Proposition 1.1 that there are only two possibilities of $H[k]$ for a Seifert curve k: either $D^2(a, b)$ or an orientable Seifert-fibered space over the Möbius band with at most one exceptional fiber.

Suppose K is a knot in S^3 which lies in a genus two Heegaard surface Σ of S^3 bounding handlebodies H and H'. The knot K in Σ is called *Seifert/Seifert* if it is Seifert with respect to both H and H'. Let γ be a surface slope with respect to the Heegaard surface Σ, i.e., the isotopy class of a component of $\partial N(K) \cap \Sigma$, where $N(K)$ is a tubular neighborhood of K in S^3. Lemma 2.1 in [3] implies that if K is Seifert/Seifert, then the γ-Dehn surgery $K(\gamma)$ is either $S^2(a, b, c, d), \mathbb{RP}^2(a, b, c)$, $K^2(a, b)$, or a graph manifold, where K^2 is a Klein bottle. However $K^2(a, b)$ can be ruled out for homological reasons.

In [3], Dean introduced twisted torus knots which lie in a genus two Heegaard surface standardly embedded in S^3. He also provided three criteria for twisted torus knots to be Seifert as a curve lying in the boundary of a genus two

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handlebody. The three criteria give rise to three types of Seifert curves called: hyper Seifert-fibered; middle Seifert-fibered; end Seifert-fibered. Since twisted torus knots lie in a genus two Heegaard surface bounding two handlebodies, one can consider several kinds of twisted torus knots which are Seifert/Seifert. For example, a twisted torus knot is called *hyper/middle* if it is hyper Seifert-fibered in one handlebody and middle Seifert-fibered in the other handlebody. If a twisted torus knot is middle Seifert-fibered in both of the handlebodies, then it is said to be a *middle/middle* twisted torus knot.

In [5] and [6], the author classified all hyper/hyper and all hyper/middle twisted torus knots respectively. Furthermore it is shown that hyper/hyper twisted torus knots admit a Dehn surgery producing $S^2(a, b, c, d)$, while hyper/middle twisted torus knots admit Dehn surgery producing a graph manifold which consists of two Seifert-fibered spaces over the disk with two exceptional fibers, glued together along their boundaries.

The goal of this paper is to find all middle/middle twisted torus knots and to show that all of these knots possibly except a few admit Dehn surgery yielding a non-Seifert-fibered graph manifold whose decomposing pieces are manifolds $D^2(a, b)$ and $D^2(c, d)$.

As will be explained in Section 2, twisted torus knots are parameterized by p, q, r, m, and n, and are denoted by $K(p, q, r, m, n)$, where p, q and m, n come from a (p, q)-torus knot and an (m, n)-torus knot respectively, and r means the number of parallel arcs of a (p, q)-torus knot with $0 < r \leq p + q$. If a twisted torus knot is hyper/middle, then $m > 1$ and $n = 1$. However if a twisted torus knot is middle/middle, then $m = n = 1$. In this sense, it is worthwhile to show that middle/middle twisted torus knots admit Dehn surgery producing graph manifolds, which provide another infinite family of knots in S^3 admitting Dehn surgery yielding such manifolds as done in [5].

We finish this section by giving the following proposition mentioned earlier.

Proposition 1.1. *Let H be a genus two handlebody and k a simple closed curve on ∂H. If $H[k]$ is a Seifert-fibered space, then $H[k]$ is a Seifert-fibered space over the disk with two exceptional fibers, or $H[k]$ is an orientable Seifert-fibered space over the Möbius band with at most one exceptional fiber.*

Proof. Let M be a 3-manifold obtained from $H[k]$ by attaching a 2-handle along a regular fiber of $H[k]$ and then filling in its boundary sphere with a 3-ball. Since H is a genus two handlebody, this construction of M implies that M admits a genus two Heegaard splitting. If M contains an essential sphere, then $M = M_1 \# M_2$ and by Haken's lemma [4], M_1 and M_2 are lens spaces.

Suppose for contradiction first that $H[k]$ is a Seifert-fibered space over the disk with three exceptional fibers. Figure 1 shows the base space disk with three cone points and two properly embedded arcs (dotted arcs) each of which with a boundary arc contains one cone point.

Each arc gives rise to an essential annulus in the Seifert-fibered space $H[k]$ whose boundary components are regular fibers. Then attaching a 2-handle

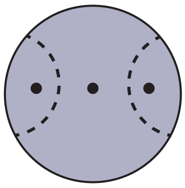

FIGURE 1. The base space disk with three cone points and two properly embedded arcs.

along a regular fiber makes these two essential annuli to be two essential spheres in M. This implies that M is the connected sum of three manifolds. This is a contradiction that M is the connected sum of two lens spaces. For the case that there are more than three exceptional fibers, the similar argument can apply.

Second, suppose $H[k]$ is an orientable Seifert-fibered space over the Möbius band with two exceptional fibers. Then as we did in the disk case, we consider two properly embedded arcs each of which with a boundary arc contains one cone point. These two arcs yield two essential annuli in $H[k]$ and attaching a 2-handle along a regular fiber yields two essential spheres in M. This implies that M is the connected sum of three manifolds, which is a contradiction. For the case that there are more than two exceptional fibers, the similar argument can apply.

If the base space of $H[k]$ is neither a disk nor the Möbius band, then we can consider two nonparallel nonseparating properly embedded arcs in the base space. These arcs yield two nonseparating annuli in $H[k]$ and attaching a 2-handle along a regular fiber yields two nonseparating spheres in M. These spheres induce $S^2 \times S^1$ summands in M. Therefore M is the connected sum of more than two manifolds, a contradiction. $\square$

2. Three types of a Seifert curve for twisted torus knots

The definition and properties of twisted torus knots were introduced by Dean in [3], and the brief explanation on how to construct twisted torus knots was given in [5]. Also the definitions, lemmas and propositions in [3] related to Seifert/Seifert twisted torus knots were given in [5]. However for the purpose of this paper that they will be applied significantly several times in the rest of the paper, we describe them here in the same manner as in [5] for the convenience of readers.

Let V_1 and V_2 be two standardly embedded disjoint unlinked solid tori in S^3. Let $T(p,q)$ be a (p,q)-torus knot which lies in the boundary of V_1. Let $rT(m,n)$ be the r parallel copies of $T(m,n)$ which lie in the boundary of V_2. Here we may assume that $0 < q < p$ and $m, n > 0$. Let D_1 be a disk in ∂V_1 so that $T(p,q)$ intersects D_1 in r disjoint parallel arcs, where $0 < r \leq p+q$, and D_2 a disk in ∂V_2 so that $rT(m,n)$ intersects D_2 in r disjoint parallel arcs, one for

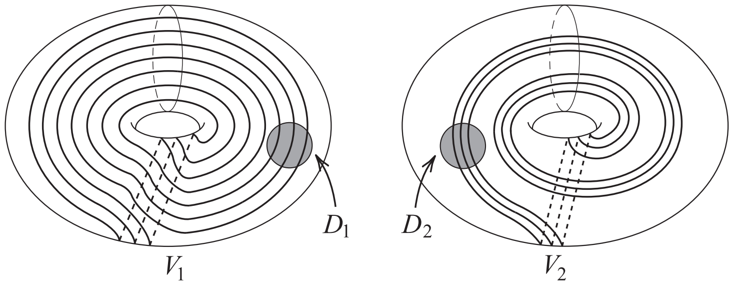

FIGURE 2. A $(7, 3)$-torus knot $T(7, 3)$ and 3 parallel copies $3T(2, 1)$ of a $(2, 1)$-torus knot.

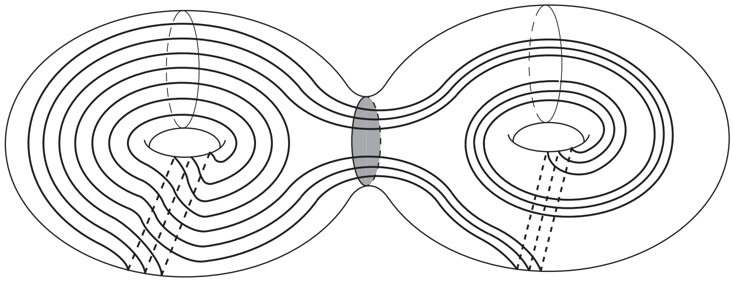

FIGURE 3. A twisted torus knot $K(7, 3, 3, 2, 1)$.

each component of $rT(m, n)$. Figure 2 illustrates a $(7, 3)$-torus knot $T(7, 3)$, 3 parallel copies $3T(2, 1)$ of a $(2, 1)$-torus knot, and disks D_1 and D_2. We excise the disks D_1 and D_2 from their respective tori and glue the punctured tori together along their boundaries so that the orientations of $T(p, q)$ and $rT(m, n)$ align correctly. The resulting one must yield a knot which lies in the boundary of a genus two handlebody H which is obtained from V_1 and V_2 by gluing the disks D_1 and D_2. This knot is called a twisted torus knot, which is denoted by $K(p, q, r, m, n)$. Figure 3 shows a twisted torus knot $K(7, 3, 3, 2, 1)$.

Let $H' = \overline{S^3 - H}$ and $\Sigma = \partial H = \partial H'$. Then $(H, H'; \Sigma)$ forms a genus two Heegaard splitting of S^3. In the rest of the paper we regard all twisted torus knots as lying in this genus two Heegaard surface Σ bounding the two handlebodies H and H' of S^3 as described above.

Proposition 2.1. *The surface slope γ of a twisted torus knot $K(p, q, r, m, n)$ with respect to the Heegaard surface Σ is $pq + r^2mn$.*

Proof. This is Proposition 3.1 in [3]. □

Let $G_{a,b} = \langle x, y \mid x^a y^b \rangle$ be a group presentation with two generators x, y and one relator $x^a y^b$. An element w in the free group $\langle x, y \rangle$ is said to be (a, b)

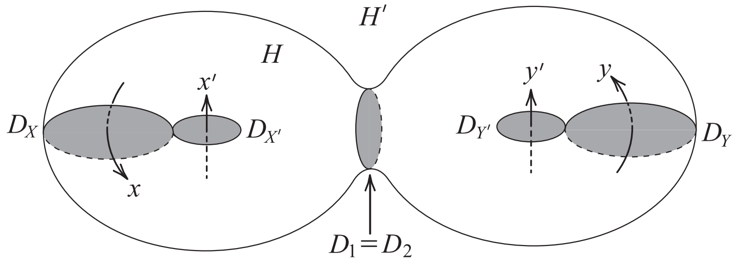

FIGURE 4. The generators of $\pi_1(H)$ and $\pi_1(H')$.

Seifert-fibered if $\langle\, x, y \mid w \,\rangle$ is isomorphic to $G_{a,b}$. The following lemma indicates that geometric and algebraic definitions of a Seifert curve are equivalent.

Lemma 2.2. *Let k be a simple closed curve in the boundary of a genus two handlebody H. k is a Seifert curve in H with $H[k] = D^2(a,b)$ if and only if k in $\pi_1(H)$ is (a,b) Seifert-fibered.*

Proof. This is Lemma 2.2 in [3]. □

Let $w_{p,q,r,m,n}$ and $w'_{p,q,r,m,n}$ be the conjugacy classes of a twisted torus knot $K(p,q,r,m,n)$ in $\pi_1(H) = \langle x, y\rangle$ and $\pi_1(H') = \langle x', y'\rangle$ respectively, where x and y are generators in H and x' and y' are generators in H', which are dual to the cutting disks as shown in Figure 4. Note that $w'_{p,q,r,m,n}$ is equal to $w_{q,p,r,n,m}$ with x replaced by x' and y replaced by y', and by the construction of a twisted torus knot, $w_{p,q,r,m,n}$ ($w'_{p,q,r,m,n}$, resp.) does not depend on the parameter n (m, resp.). Therefore we often omit n (m, resp.).

There are more properties in $w_{p,q,r,m,n}$. For g and h in a group G, we say g is *equivalent* to h, denoted by $g \equiv h$, if there is an automorphism of G carrying g to h.

Lemma 2.3. *The word $w_{p,q,r,m}$ has the following properties.*

(1) $w_{p,q,r,m} \equiv w_{p,q',r,m}$ *if* $q \equiv \pm q' \mod p$.
(2) $w_{p,q,r,m} \equiv w_{p,q,r',m}$ *if* $r \equiv \pm r' \mod p$.

Proof. This is Lemma 3.3 in [3]. □

For integers p and q, $\hat{q}^{-1}$ is defined to be the smallest positive integer congruent to $\pm q^{-1} \mod p$. For a real number x, $\tilde{x}$ denotes the least integer function. The following proposition gives three criteria to determine which $w_{p,q,r,m,n}$ are Seifert-fibered in $\pi_1(H)$. The three criteria were introduced and proved in [3].

Proposition 2.4. *Let $w = w_{p,q,r,m}$ be a conjugacy class in $\pi_1(H)$ of a twisted torus knot $K(p,q,r,m,n)$. Let q' be an integer such that $q \equiv \pm q' \mod p$ with $0 < q' < p/2$.*

(1) *If $m > 1$ and $r \equiv \pm 1$ or $\pm q' \mod p$, then w is (p,m) Seifert-fibered.*

(2) *If* $m = 1$ *and* $r \equiv \pm\beta q' \mod p$, *where* $1 < \beta \le p/q'$ *with* $p - \beta q' > 1$, *then* w *is* $(\beta, p - \beta q')$ *Seifert-fibered.*

(3) *If* $m = 1$ *and* $r \equiv \pm\bar{r} \mod p$, *where* $1 < \bar{r} \le \widetilde{p/\hat{q'}^{-1}}$ *with* $p - \bar{r}\hat{q'}^{-1} > 1$, *then* w *is* $(\bar{r}, p - \bar{r}\hat{q'}^{-1})$ *Seifert-fibered.*

Proof. The parts (1), (2), and (3) are Propositions 3.6, 3.8, and 3.10 respectively in [3]. □

The first type (1), the second type (2), and the third type (3) of Seifert-fibered $w_{p,q,r,m}$ (or $K(p,q,r,m,n)$) in Proposition 2.4 are called *hyper Seifert-fibered, middle Seifert-fibered*, and *end Seifert-fibered* in H respectively. Also it is conjectured in [3] that these three types describe all $w_{p,q,r,m}$ that are Seifert-fibered. With respect to the other handlebody H' we can apply Proposition 2.4 by switching p and q, and m and n to say that $w'_{p,q,r,n}$ ($w_{q,p,r,n}$ or $K(p,q,r,m,n)$) is *hyper Seifert-fibered, middle Seifert-fibered*, or *end Seifert-fibered* in H'.

In this paper, we consider a twisted torus knot $K(p,q,r,m,n)$ which is middle Seifert-fibered in both H and H'. This knot is called a *middle/middle* twisted torus knot. In Section 3 we will find all possible values of the parameters p,q,r,m, and n such that $K(p,q,r,m,n)$ is middle/middle. And then in Section 4 we will show that the γ-Dehn surgery $K(\gamma)$, where γ is a surface slope, is a non-Seifert-fibered graph manifold whose decomposing pieces are $D^2(a,b)$ and $D^2(c,d)$.

3. Finding the parameters p, q, r, m, and n

In this section we find all possible values of the parameters p,q,r,m, and n for which $K(p,q,r,m,n)$ is middle Seifert-fibered in both H and H'.

Theorem 3.1. *Let* K *be a twisted torus knot* $K(p,q,r,m,n)$ *lying in a genus two Heegaard splitting* $(H,H';\Sigma)$ *of* S^3 *with* $0 < q < p$, $\gcd(p,q) = 1$, *and* $0 < r \le p+q$. K *is a middle/middle twisted torus knot if and only if the parameters* p,q,r,m, *and* n *take one of the following values in Table* 1. *Table* 2 *describes* $H[K]$ *and* $H'[K]$ *explicitly.*

Proof. Since K is middle Seifert-fibered in both H and H', by Proposition 2.4 $m = n = 1$. If $q = 1$, then K is primitive in H' by Theorem 3.4 in [3], i.e., $H'[K]$ is a solid torus and thus K is not Seifert. If $r = 1$, then K becomes a torus knot. Therefore we may assume that $1 < q < p$ and $1 < r \le p+q$.

Let $\bar{p} = p - iq$ such that $0 < \bar{p} < q$. Note that since $\gcd(p,q) = 1$, $\gcd(q,\bar{p}) = 1$. Then we divide the argument into four cases:

1. $p > 2q$ and $q > 2\bar{p}$.
2. $p > 2q$ and $q < 2\bar{p}$.
3. $p < 2q$ and $q > 2\bar{p}$.
4. $p < 2q$ and $q < 2\bar{p}$.

TABLE 1. All possible values of parameters p, q, r, m, and n for which $K(p, q, r, m, n)$ is middle Seifert-fibered in both H and H'.

	(p, q, r, m, n)	satisfying
I	$(\beta q + \bar{p}, q, \beta q + 2\bar{p}, 1, 1)$	$1 < \bar{p} < q$, $\|q - 2\bar{p}\| > 1$, $\beta > 1$
II	$((\alpha+1)\bar{p} + \bar{q}, \alpha\bar{p} + \bar{q}, \beta\bar{p}, 1, 1)$	$0 \leq \bar{q} < \bar{p}$, $1 < \beta \leq \alpha$
III	$((\alpha+1)\bar{p} + \bar{q}, \alpha\bar{p} + \bar{q}, (\alpha - \beta)\bar{p} + \bar{q}, 1, 1)$	$0 < \bar{q} < \bar{p}$, $1 < \beta \leq \alpha$
IV	$((\alpha+1)\bar{p} + \bar{q}, \alpha\bar{p} + \bar{q}, (\alpha + \beta + 1)\bar{p} + \bar{q}, 1, 1)$	$0 \leq \bar{q} < \bar{p}$, $1 < \beta < \alpha$
V	$((\alpha+1)\bar{p} + \bar{q}, \alpha\bar{p} + \bar{q}, (2\alpha - \beta)\bar{p} + 2\bar{q}, 1, 1)$	$0 < \bar{q} < \bar{p}$, $1 < \beta < \alpha$
VI	$(2\bar{p} + \bar{q}, \bar{p} + \bar{q}, 2\bar{p}, 1, 1)$	$0 < \bar{q} < \bar{p}$

TABLE 2. $H[K]$ and $H'[K]$ when K is middle Seifert-fibered in both H and H'.

	$H[K]$	$H'[K]$
I	$D^2(\beta, \bar{p})$	$D^2(2, \|q - 2\bar{p}\|)$
II	$D^2(\beta, (\alpha - \beta + 1)\bar{p} + \bar{q})$	$D^2(\beta, (\alpha - \beta)\bar{p} + \bar{q})$
III	$D^2(\beta + 1, (\alpha - \beta)\bar{p} + \bar{q})$	$D^2(\beta, (\alpha - \beta)\bar{p} + \bar{q})$
IV	$D^2(\beta, (\alpha - \beta + 1)\bar{p} + \bar{q})$	$D^2(\beta + 1, (\alpha - \beta - 1)\bar{p} + \bar{q})$
V	$D^2(\beta + 2, (\alpha - \beta - 1)\bar{p} + \bar{q})$	$D^2(\beta, (\alpha - \beta)\bar{p} + \bar{q})$
VI	$D^2(2, \bar{q})$	$D^2(2, \bar{p} - \bar{q})$

Case 1: Suppose $p > 2q$ and $q > 2\bar{p}$.

On H, by Proposition 2.4 $w_{p,q,r,1}$ satisfies $r \equiv \pm\beta_1 q \bmod p$, where $1 < \beta_1 < p/q$, and $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1 q)$ Seifert-fibered provided that $\beta_1, p - \beta_1 q > 1$. Thus the possible values of r are

$$r = \beta_1 q,\ p - \beta_1 q \text{ or } 2p - \beta_1 q.$$

On H', by Lemma 2.3 $w_{q,p,r,1}$ is equivalent to $w_{q,\bar{p},r,1}$. Since $p \equiv \bar{p} \bmod q$ and $\bar{p} < q/2$, by Proposition 2.4 $w_{q,\bar{p},r,1}$ satisfies $r \equiv \pm\beta_2\bar{p} \bmod q$, where $1 < \beta_2 < q/\bar{p}$, and $w_{q,\bar{p},r,1}$ is $(\beta_2, q - \beta_2\bar{p})$ Seifert-fibered provided that $\beta_2, q - \beta_2\bar{p} > 1$. Thus the possible values of r are

$$r = kq \pm \beta_2\bar{p} \quad \text{for some nonnegative integer } k.$$

We need to find the parameters p, q, r by equating both values of r on H and H'. In other words, we need to consider the following three subcases:

(1) $r = \beta_1 q = kq \pm \beta_2\bar{p}$,

(2) $r = p - \beta_1 q = kq \pm \beta_2\bar{p}$,

(3) $r = 2p - \beta_1 q = kq \pm \beta_2\bar{p}$.

Subcase (1): $r = \beta_1 q = kq \pm \beta_2\bar{p}$. Replacing $\bar{p}$ by $p - iq$, we obtain that $\pm\beta_2 p = (\beta_1 - k \pm i\beta_2)q$. Since $\gcd(p, q) = 1$, q divides β_2, i.e., $q \leq \beta_2$. This is a contradiction to $q - \beta_2\bar{p} > 1$.

Subcase (2): $r = p - \beta_1 q = kq \pm \beta_2 \bar{p}$. Similarly, we get that $(-1 \pm \beta_2)p = (-\beta_1 - k \pm i\beta_2)q$ and then $q \leq \beta_2 \pm 1$, which is a contradiction to $q - \beta_2\bar{p} > 1$.

Subcase (3): $r = 2p - \beta_1 q = kq \pm \beta_2 \bar{p}$. First, we assume that $2p - \beta_1 q = kq - \beta_2 \bar{p}$. Then by replacing $\bar{p}$ by $p - iq$, we obtain

$$(2 + \beta_2)p = (\beta_1 + k + i\beta_2)q.$$

From the conditions that $\gcd(p, q) = 1$ and $q - \beta_2\bar{p} > 1$, $q = \beta_2 + 2$ and $\bar{p} = 1$. Since $p = iq + \bar{p}$ and $r = 2p - \beta_1 q$, $p = iq + 1$ and $r = (2i - \beta_1)q + 2$. The inequality $1 < r \leq p+q$ becomes $1 < (2i-\beta_1)q+2 \leq (i+1)q+1$, which implies that $i < \beta_1 + 1$. On the other hand, $1 < p - \beta_1 q$ becomes $1 < (i - \beta_1)q + 1$, which implies that $\beta_1 < i$, a contradiction.

Second, we assume that $2p - \beta_1 q = kq + \beta_2 \bar{p}$. Then

$$(2 - \beta_2)p = (\beta_1 + k - i\beta_2)q.$$

If $\beta_2 \neq 2$, then $q < \beta_2$, a contradiction. Therefore $\beta_2 = 2$. Also $k = 2i - \beta_1$ and $r = kq + 2\bar{p}$. Using the equations $p = iq + \bar{p}$ and $r = 2p - \beta_1 q$, and the inequalities $1 < r \leq p+q$ and $1 < p-\beta_1 q$, we obtain that $1 < (i-\beta_1)q+\bar{p} \leq q$. Thus $i = \beta_1$ and $k = 2i-\beta_1 = \beta_1$. Putting these values of i and k in p and r, we obtain that $p = \beta_1 q+\bar{p}, r = \beta_1 q+2\bar{p}$. This belongs to the solution (I) in Table 1 with $\beta = \beta_1$. For $H[K]$ and $H'[K]$, since $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1 q)$ Seifert-fibered and $w_{q,p,r,1}$ is $(\beta_2, q-\beta_2\bar{p})$ Seifert-fibered, by Lemma 2.2 $H[K] = D^2(\beta, \bar{p})$ and $H'[K] = D^2(2, q - 2\bar{p})$ as desired in Table 2.

Case 2: Suppose $p > 2q$ and $q < 2\bar{p}$.

As in Case 1, on H $w_{p,q,r,1}$ satisfies $r \equiv \pm\beta_1 q \bmod p$, where $1 < \beta_1 < p/q$ and $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1 q)$ Seifert-fibered with $\beta_1, p - \beta_1 q > 1$. Thus the possible values of r are

$$r = \beta_1 q,\ p - \beta_1 q \text{ or } 2p - \beta_1 q.$$

However, we have different situation on H'. Since $q < 2\bar{p}$, we let $\bar{\bar{p}} = q-\bar{p}$ so that we can apply Proposition 2.4. Since $\bar{\bar{p}} < q/2$ and also $p \equiv -\bar{\bar{p}} \bmod q$, $w_{q,p,r,1}$ is equivalent to $w_{q,\bar{\bar{p}},r,1}$ which satisfies $r \equiv \pm\beta_2\bar{\bar{p}} \bmod q$, where $1 < \beta_2 < q/\bar{\bar{p}}$ and $w_{q,\bar{\bar{p}},r,1}$ is $(\beta_2, q - \beta_2\bar{\bar{p}})$ Seifert-fibered with $\beta_2, q - \beta_2\bar{\bar{p}} > 1$. Thus the possible values of r are

$$r = kq \pm \beta_2\bar{\bar{p}} \text{ for some nonnegative integer } k.$$

In order to find values of $p, q,$ and r, we consider the following three subcases:

$$r = \beta_1 q = kq \pm \beta_2\bar{\bar{p}},\ r = p - \beta_1 q = kq \pm \beta_2\bar{\bar{p}},\ r = 2p - \beta_1 q = kq \pm \beta_2\bar{\bar{p}}.$$

By applying the similar argument as in Case 1, it is easy to see that the first two subcases induce a contradiction to $q - \beta_2\bar{\bar{p}} > 1$.

For the third, we first assume that $r = 2p - \beta_1 q = kq - \beta_2\bar{\bar{p}}$. By replacing $\bar{\bar{p}}$ by $q - \bar{p}$ $(= (i + 1)q - p)$, we obtain the equation

$$(2 - \beta_2)p = (k - (i + 1)\beta_2 + \beta_1)q.$$

If $\beta_2 \neq 2$, then $q < \beta_2$, a contradiction. Therefore $\beta_2 = 2$ and thus $k = 2(i+1) - \beta_1$. Using the equations $p = iq + \bar{p}$ and $r = 2p - \beta_1 q$, and the inequalities $1 < r \leq p+q$ and $p - \beta_1 q > 1$, we obtain that $1 < (i-\beta_1)q + \bar{p} \leq q$. Thus $i = \beta_1$ and $k = \beta_1 + 2$. In conclusion, $p = \beta_1 q + \bar{p}$ and $r = \beta_1 q + 2\bar{p}$. This belongs to the solution (I) with $\beta = \beta_1$. For $H[K]$ and $H'[K]$, since $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1 q)$ Seifert-fibered and $w_{q,p,r,1}$ is $(\beta_2, q - \beta_2\bar{\bar{p}})$ Seifert-fibered, by Lemma 2.2 $H[K] = D^2(\beta, \bar{p})$ and $H'[K] = D^2(2, 2\bar{p} - q)$ as desired in Table 2.

Second, we assume that $r = 2p - \beta_1 q = kq + \beta_2\bar{\bar{p}}$. By replacing $\bar{\bar{p}}$ by $q - \bar{p}$ $(= (i+1)q - p)$, we obtain the equation

$$(2 + \beta_2)p = (k + (i+1)\beta_2 + \beta_1)q.$$

From the conditions that $\gcd(p, q) = 1$ and $q - \beta_2\bar{\bar{p}} > 1$, it follows that $q = \beta_2 + 2$, $\bar{\bar{p}} = 1$ and $p = k + (i+1)\beta_2 + \beta_1$. As in the case of $r = 2p - \beta_1 q = kq - \beta_2\bar{\bar{p}}$ above, the inequality $1 < (i - \beta_1)q + \bar{p} \leq q$ holds and then $i = \beta_1$. Since $q = \beta_2 + 2$ and $p = (i+1)q - 1 = k + (i+1)\beta_2 + \beta_1$, $k = \beta_1 + 1$. In conclusion, $p = \beta_1 q + \bar{p}, r = \beta_1 q + 2\bar{p}$, which belongs to the solution (I).

Case 3: Suppose $p < 2q$ and $q > 2\bar{p}$.

Since $p < 2q$, $i = 1$ and $\bar{p} = p - q$ in the equation $p = iq + \bar{p}$. Also $q \equiv -\bar{p}$ mod p and $\bar{p} < p/2$. On H $w_{p,q,r,1} \equiv w_{p,\bar{p},r,1}$, which satisfies $r \equiv \pm\beta_1\bar{p}$ mod p because of Proposition 2.4, where $1 < \beta_1 < p/\bar{p}$, and $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1\bar{p})$ Seifert-fibered with $\beta_1, p - \beta_1\bar{p} > 1$. Thus the possible values of r are

$$r = \beta_1\bar{p},\ p - \beta_1\bar{p},\ p + \beta_1\bar{p},\ \text{or } 2p - \beta_1\bar{p}.$$

On H' $w_{q,p,r,1} \equiv w_{q,\bar{p},r,1}$, which satisfies $r \equiv \pm\beta_2\bar{p}$ mod q, where $1 < \beta_2 < q/\bar{p}$, and $w_{q,\bar{p},r,1}$ is $(\beta_2, q - \beta_2\bar{p})$ Seifert-fibered with $\beta_2, q - \beta_2\bar{p} > 1$. Thus the possible values of r are

$$r = kq \pm \beta_2\bar{p} \ \text{ for some nonnegative integer } k.$$

We consider the following four subcases:

(1) $r = \beta_1\bar{p} = kq \pm \beta_2\bar{p}$,
(2) $r = p - \beta_1\bar{p} = kq \pm \beta_2\bar{p}$,
(3) $r = p + \beta_1\bar{p} = kq \pm \beta_2\bar{p}$,
(4) $r = 2p - \beta_1 = kq \pm \beta_2\bar{p}$.

Subcase (1): $r = \beta_1\bar{p} = kq \pm \beta_2\bar{p}$. First, we assume that $r = \beta_1\bar{p} = kq + \beta_2\bar{p}$. Then $(\beta_1 - \beta_2)\bar{p} = kq$. If $\beta_1 = \beta_2$, then $k = 0$, and thus $p = q + \bar{p}, r = \beta_1\bar{p} = \beta_2\bar{p}$. This gives the solution (II) in Table 1 by letting $q = \alpha\bar{p} + \bar{q}$ and $\beta = \beta_2$, where $0 \leq \bar{q} < \bar{p}$. Here $\alpha \geq \beta$ because $q - \beta_2\bar{p} > 1$. Suppose $\beta_1 \neq \beta_2$. Since $\gcd(\bar{p}, q) = 1$, from the equation $(\beta_1 - \beta_2)\bar{p} = kq$ we see that $\beta_1 - \beta_2 = lq$ and $k = l\bar{p}$ for some positive integer l. However the inequalities $p - \beta_1\bar{p} > 1$ and $p < 2q$ together with $\beta_1 = lq + \beta_2$ force $l = \bar{p} = 1$, which implies that $k = l\bar{p} = 1$, $\beta_1 = q + \beta_2$, and $p = q + \bar{p} = q + 1$. Then $\beta_1 \geq p$, which is a contradiction to $p - \beta_1\bar{p} > 1$.

Second, we assume that $r = \beta_1\bar{p} = kq - \beta_2\bar{p}$. Then $(\beta_1 + \beta_2)\bar{p} = kq$, which implies that $\beta_1 + \beta_2 = lq$ and $k = l\bar{p}$ for some positive integer l. The inequalities $p - \beta_1\bar{p} > 1$ and $q - \beta_2\bar{p} > 1$ induce $p + q > (\beta_1 + \beta_2)\bar{p} + 2$. Combining with $\beta_1 + \beta_2 = lq$ and $p < 2q$, we see that $k = l\bar{p}$ is either 1 or 2 and thus l is either 1 or 2. If $l = 2$, then $\beta_1 + \beta_2 = 2q$ and $\bar{p} = 1$, which lead to $\beta_1 = q + q - \beta_2$ and $p = q + 1$. Then $p < \beta_1$ because $q - \beta_2\bar{p} > 1$, a contradiction. Therefore $l = 1$ and then $\beta_1 + \beta_2 = q$ and $k = \bar{p}$ $(= 1$ or $2)$. If $k = \bar{p} = 2$, then we have the inequality $p + q > (\beta_1 + \beta_2)\bar{p} + 2 = 2q + 2$. However $p + q = 2q + \bar{p} = 2q + 2$, a contradiction. Therefore $k = \bar{p} = 1$. This yields the solution $p = \beta_1 + \beta_2 + 1, q = \beta_1 + \beta_2$, and $r = \beta_1$. This belongs to the solution (II) by letting $\bar{p} = 1$, $\bar{q} = 0$, $\beta = \beta_1$, and $\alpha = \beta_1 + \beta_2$ in Table 1.

Subcase (2): $r = p - \beta_1\bar{p} = kq \pm \beta_2\bar{p}$. First, assume that $r = p - \beta_1\bar{p} = kq + \beta_2\bar{p}$. By replacing p by $q + \bar{p}$, we obtain the equation

$$(1 - \beta_1 - \beta_2)\bar{p} = (k - 1)q.$$

Since $1 - \beta_1 - \beta_2 < 0$ and $\gcd(q, \bar{p}) = 1$, $k = 0$ and $\bar{p} = 1$. Therefore $q = \beta_1 + \beta_2 - 1$, $p = \beta_1 + \beta_2$, and $r = \beta_2$. This belongs to the solution (II) by letting $\bar{p} = 1$, $\bar{q} = 0$, $\beta = \beta_2$, and $\alpha = \beta_1 + \beta_2 - 1$ in Table 1.

Second, we assume that $r = p - \beta_1\bar{p} = kq - \beta_2\bar{p}$. Then we have the equation

$$(1 - \beta_1 + \beta_2)\bar{p} = (k - 1)q.$$

If $k > 1$, then $1 - \beta_1 + \beta_2 = lq$ for some positive integer l, which is a contradiction to $q > \beta_2$. Note that since $r > 0$ and $r = kq - \beta_2\bar{p}$, $k > 0$. Therefore $k = 1$ and $\beta_2 = \beta_1 - 1$. This gives rise to $p = q + \bar{p}$ and $r = q - \beta_2\bar{p}$, which induces the solution (III) in Table 1 by letting $q = \alpha\bar{p} + \bar{q}$ and $\beta = \beta_2$, where $0 \leq \bar{q} < \bar{p}$. However if $\bar{q} = 0$, then $\bar{p} = 1$, which belongs to the solution (II). Thus we may assume that $0 < \bar{q} < \bar{p}$ in the solution (III).

Subcase (3): $r = p + \beta_1\bar{p} = kq \pm \beta_2\bar{p}$. Assume that $r = p + \beta_1\bar{p} = kq + \beta_2\bar{p}$. Then

$$(1 + \beta_1 - \beta_2)\bar{p} = (k - 1)q.$$

Also since $0 < r \leq p + q$, $k = 0, 1$, or 2. If $k = 0$, then $\bar{p} = 1$ and $\beta_2 - \beta_1 - 1 = q$, which is a contradiction to $q > \beta_2$. If $k = 2$, then $\bar{p} = 1$ and $1 + \beta_1 - \beta_2 = q$, i.e., $\beta_1 = q + \beta_2 - 1$, which is a contradiction to $p = q + 1 > \beta_1$. Therefore k must be 1 and thus $\beta_2 = \beta_1 + 1$. We have the solution that $p = q + \bar{p}$ and $r = p + \beta_1\bar{p}$. This gives rise to the solution (IV) in Table 1 by letting $q = \alpha\bar{p} + \bar{q}$ and $\beta = \beta_1$, where $0 \leq \bar{q} < \bar{p}$, and $\alpha > \beta$ because $q - \beta_2\bar{p} > 1$.

Now we assume that $r = p + \beta_1\bar{p} = kq - \beta_2\bar{p}$. Then

$$(1 + \beta_1 + \beta_2)\bar{p} = (k - 1)q,$$

which implies $k > 1$. Furthermore, since $0 < r \leq p + q$, $k = 2$ or 3. If $k = 2$, then $\bar{p} = 1$ and $q = 1 + \beta_1 + \beta_2$. Also $p = 2 + \beta_1 + \beta_2$ and $r = p + \beta_1$. This solution belongs to the solution (IV) in Table 1 by letting $\bar{p} = 1$, $\bar{q} = 0$, $\beta = \beta_1$, and $\alpha = \beta_1 + \beta_2 + 1$. Suppose $k = 3$. Then from the equation $(1 + \beta_1 + \beta_2)\bar{p} = (k - 1)q$, $\bar{p}$ is either 1 or 2. If $\bar{p} = 1$, Then $1 + \beta_1 + \beta_2 = 2q$ and $p = q + 1$. This is a contradiction to the inequalities $p - \beta_1\bar{p} > 1$ and $q - \beta_2\bar{p} > 1$, which implies that $p + q > (\beta_1 + \beta_2)\bar{p} + 2$. If $\bar{p} = 2$, then $q = 1 + \beta_1 + \beta_2$ and thus $p = q + 2$. Putting $q = 1 + \beta_1 + \beta_2$ and $p = q + 2 = 3 + \beta_1 + \beta_2$ in the inequalities $p - \beta_1\bar{p} > 1$ and $q - \beta_2\bar{p} > 1$, we see that $\beta_2 < \beta_1 < \beta_2 + 2$, i.e., $\beta_1 = \beta_2 + 1$. This implies that $q = 2\beta_2 + 2$ and $p = 2\beta_2 + 4$, both of which are even, a contradiction to $\gcd(p, q) = 1$.

Subcase (4): $r = 2p - \beta_1\bar{p} = kq \pm \beta_2\bar{p}$. Assume that $r = 2p - \beta_1\bar{p} = kq + \beta_2\bar{p}$. Then

$$(2 - \beta_1 - \beta_2)\bar{p} = (k - 2)q.$$

Since $2 - \beta_1 - \beta_2 < 0$, $k = 0$ or 1. If $k = 1$, then $\bar{p} = 1$ and $q = \beta_1 + \beta_2 - 2$. Also $p = \beta_1 + \beta_2 - 1$ and $r = q + \beta_2$, which belongs to the solution (IV) in Table 1 by letting $\bar{p} = 1$, $\bar{q} = 0$, $\beta = \beta_2 - 1$, and $\alpha = \beta_1 + \beta_2 - 2$. Suppose $k = 0$. Then $\bar{p} = 1$ or 2. If $\bar{p} = 1$, then $\beta_1 + \beta_2 = 2q + 2 > q + 1 + q = p + q$, a contradiction. If $\bar{p} = 2$, then $\beta_1 + \beta_2 = q + 2$ and $p = q + 2$. This is a contradiction to $p + q > (\beta_1 + \beta_2)\bar{p} + 2$.

Second, we assume that $r = 2p - \beta_1\bar{p} = kq - \beta_2\bar{p}$. Then

$$(2 - \beta_1 + \beta_2)\bar{p} = (k - 2)q.$$

Since $0 < r \le p + q$ and $r = kq - \beta_2\bar{p}$, $k = 1, 2$ or 3. If $k = 1$, then $\bar{p} = 1$ and $2 - \beta_1 + \beta_2 = -q$. This is a contradiction since $p = q + 1$ and $p > \beta_1$. If $k = 3$, then similarly $\beta_2 = q + \beta_1 - 2$, a contradiction to $q > \beta_2$. Thus $k = 2$ and then $\beta_1 = \beta_2 + 2$. This yields that $r = 2p - (\beta_2 + 2)\bar{p} = 2q - \beta_2\bar{p}$, which leads to the solution (V) in Table 1 by letting $q = \alpha\bar{p} + \bar{q}$ and $\beta = \beta_2$, where $0 \le \bar{q} < \bar{p}$, and $\alpha > \beta$ because $p - \beta_1\bar{p} > 1$. Furthermore, if $\bar{q} = 0$, then $\bar{p} = 1$, which belongs to the solution (IV). Thus we may assume that $0 < \bar{q} < \bar{p}$ in the solution (V).

Case 4: Suppose $p < 2q$ and $q < 2\bar{p}$.

Since $p < 2q$, $i = 1$ and $\bar{p} = p - q$ in the equation $p = iq + \bar{p}$. Also since $q < 2\bar{p}$, we let $\bar{\bar{p}} = q - \bar{p}$. Then $2\bar{\bar{p}} < q$ and $p \equiv -\bar{\bar{p}} \bmod q$. On H, $w_{p,q,r,1} \equiv w_{p,\bar{p},r,1}$ which satisfies $r \equiv \pm\beta_1\bar{p} \bmod p$, where $1 < \beta_1 < p/\bar{p}$ and $w_{p,q,r,1}$ is $(\beta_1, p - \beta_1\bar{p})$ Seifert-fibered with $\beta_1, p - \beta_1\bar{p} > 1$. The inequality $p - \beta_1\bar{p} > 1$ forces β_1 to be 2 as follows.

Claim. $\beta_1 = 2$.

Proof. The inequalities $p - \beta_1\bar{p} > 1$, i.e., $p > \beta_1\bar{p}$ and $q < 2\bar{p}$ imply that $\bar{p} = p - q > \beta_1\bar{p} - 2\bar{p} = (\beta_1 - 2)\bar{p}$. Therefore $\beta_1 = 2$. □

So $w_{p,q,r,1}$ is $(2, p - 2\bar{p})$ Seifert-fibered with $p - 2\bar{p} > 1$ and the possible values of r are

$$r = 2\bar{p},\ p - 2\bar{p},\ p + 2\bar{p}, \quad \text{or} \quad 2p - 2\bar{p}.$$

On H', $w_{q,p,r,1} \equiv w_{q,\bar{\bar{p}},r,1}$ which satisfies $r \equiv \pm\beta_2\bar{\bar{p}} \bmod q$, where $1 < \beta_2 < q/\bar{\bar{p}}$ and $w_{q,\bar{\bar{p}},r,1}$ is $(\beta_2, q - \beta_2\bar{\bar{p}})$ Seifert-fibered with $\beta_2, q - \beta_2\bar{\bar{p}} > 1$. Thus the possible values of r are

$$r = kq \pm \beta_2\bar{\bar{p}} \ \text{ for some nonnegative integer } k.$$

We consider the following four subcases:

(1) $r = 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$,

(2) $r = p - 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$,
(3) $r = p + 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$,
(4) $r = 2p - 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$.

Subcase (1): $r = 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$. First, we assume that $r = 2\bar{p} = kq + \beta_2\bar{\bar{p}}$. Then $(2+\beta_2)\bar{p} = (k+\beta_2)q$ and also $k = 0, 1$, or 2 because $0 < r \leq p+q$. Thus $2 + \beta_2 = lq$ and $k + \beta_2 = l\bar{p}$ for some positive integer l. Since $\beta_2 = lq - 2 = (l-1)q + q - 2$ and $\beta_2 < q$, l must be 1. Thus $q = 2 + \beta_2$ and $\bar{p} = \beta_2 + k$. The inequality $p - 2\bar{p} > 1$ implies that

$$p - 2\bar{p} = q - \bar{p} = 2 + \beta_2 - (\beta_2 + k) = 2 - k > 1 \Leftrightarrow k < 1.$$

Therefore $k = 0$ and thus $\bar{p} = \beta_2$ and $q = \beta_2 + 2$. And also $\bar{\bar{p}} = q - \bar{p} = 2$, which implies $q - \beta_2\bar{\bar{p}} = 2 - \beta_2 \leq 0$, a contradiction to $q - \beta_2\bar{\bar{p}} > 1$.

Second, we assume that $r = 2\bar{p} = kq - \beta_2\bar{\bar{p}}$. Then $(2-\beta_2)\bar{p} = (k-\beta_2)q$. Suppose $\beta_2 \neq 2$. Then $\beta_2 - 2 = lq$ for some positive integer l and thus $\beta_2 > q$, a contradiction. Therefore $\beta_2 = 2$ and $k = \beta_2 = 2$, and thus $r = 2\bar{p}$. Also $p = q + \bar{p}$ and $q = \bar{p} + \bar{\bar{p}}$. This gives the solution (VI) in Table 1 by letting $\bar{q} = \bar{\bar{p}}$.

Subcase (2): $r = p - 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$. Then $(-1 \pm \beta_2)\bar{p} = (k \pm \beta_2 - 1)q$. Thus $\beta_2 = lq \pm 1$ for some positive integer l. This is a contradiction to $q - \beta_2\bar{\bar{p}} > 1$.

Subcase (3): $r = p + 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$. Assume that $r = p + 2\bar{p} = kq + \beta_2\bar{\bar{p}}$. Then $(3+\beta_2)\bar{p} = (k+\beta_2-1)q$ and $k = 0, 1$, or 2. This implies that $3 + \beta_2 = lq$ and $k + \beta_2 - 1 = l\bar{p}$ for some positive integer l. Since $q - \beta_2\bar{\bar{p}} > 1$, $l = 1$. Thus $q = \beta_2 + 3$ and $\bar{p} = k + \beta_2 - 1 = q + k - 4$, which implies that $\bar{\bar{p}} = q - \bar{p} = 4 - k \geq 2$ because $k \leq 2$. Then $q - \beta_2\bar{\bar{p}} \leq q - 2\beta_2 \leq 1$, a contradiction to $q - \beta_2\bar{\bar{p}} > 1$.

We now assume that $r = p + 2\bar{p} = kq - \beta_2\bar{\bar{p}}$. Then $(3-\beta_2)\bar{p} = (k - \beta_2 - 1)q$ and $k = 1, 2$, or 3. If $\beta_2 = 2$, then $\bar{p} = (k-3)q$, a contradiction. If $\beta_2 = 3$, then $k = \beta_2 + 1 = 4$, a contradiction. If $\beta_2 > 3$, then $\beta_2 - 3 = lq$ for some positive integer l. Then $\beta_2 = lq + 3 > q$, a contradiction.

Subcase (4): $r = 2p - 2\bar{p} = kq \pm \beta_2\bar{\bar{p}}$. Then $\pm\beta_2\bar{p} = (k \pm \beta_2 - 2)q$. This implies $\beta_2 \geq q$, a contradiction. □

4. Twisted torus knots admitting graph manifold Dehn surgeries

Section 3 shows that there are six types of middle/middle twisted torus knots. By Lemma 2.1 in [3] it follows that for a twisted torus knot, the γ-Dehn surgery $K(\gamma)$ with γ a surface slope is homeomorphic to $H[K] \cup_\partial H'[K]$. Therefore middle/middle twisted torus knots admit either a Dehn surgery producing either $S^2(a, b, c, d)$ or a graph manifold whose decomposing pieces consist of two Seifert-fibered manifold over the disk with two exceptional fibers. In this section, we show that all of the six types admit the latter. The main tool to verify this is to use R-R diagrams and the key idea is to show using R-R diagrams that two regular fibers of $H[K]$ and $H'[K]$ intersect transversally in at least one point.

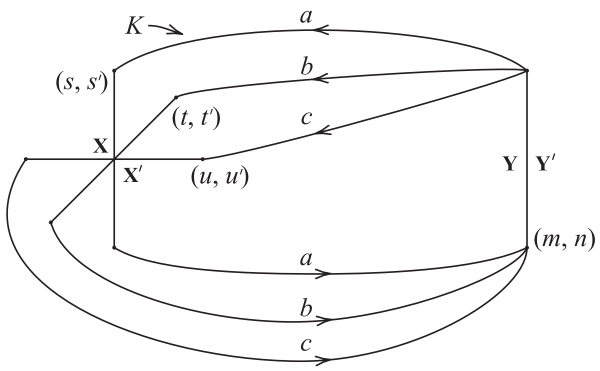

FIGURE 5. R-R diagram of $K(p, q, r, m, n)$ with respect to H and H', where $as + bt + cu = p, as' + bt' + cu' = q$, and $a + b + c = r$.

R-R diagrams were originally introduced by Osborne and Stevens in [7], and developed by Berge. R-R diagrams are a type of planar diagram related to Heegaard diagrams of simple closed curves in the boundary of a genus two handlebody and in particular useful for describing embeddings of simple closed curves in the boundary of a handlebody so that the embedded curves represent certain conjugacy classes in the fundamental group of the handlebody. For the definition and properties of R-R diagrams, see [1] or [5]. The R-R diagram of a twisted torus knot $K(p, q, r, m, n)$ is shown in Figure 5. The details of how to make the R-R diagram of a twisted torus knot $K(p, q, r, m, n)$ are given in [5].

The following lemma and the remarks were given in [5]. However since they are essential in this paper, we describe here as well.

Lemma 4.1. *If k is a nonseparating simple closed curve on the boundary of a genus two handlebody H such that $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers, then k has an R-R diagram with the form of Figure* 6a*, with $n, s > 1$, or* 6b *with $n > 0$, $s > 1$, $a, b > 0$, and* $\gcd(a, b) = 1$.

Conversely, if k has an R-R diagram with the form of Figure 6a*, with $n, s > 1$, or Figure* 6b *with $n > 0, s > 1$, $a, b > 0$, and* $\gcd(a, b) = 1$*, then $H[k]$ is Seifert-fibered over D^2 with two exceptional fibers of indexes n and s, or indexes $n(a + b) + b$ and s respectively.*

In addition, the curves τ_1 and τ_2 in Figure 6a *and the curve τ in Figure* 6b *are regular fibers of $H[k]$.*

Proof. This is Theorem 3.2 in [2]. □

Remarks. (1) Algebraically in $\pi_1(H) = \langle x, y \rangle$ k in Figure 6a represents $x^n y^s$, while k in Figure 6b is the product of $x^n y^s$ and $x^{n+1} y^s$ with $|x^n y^s| = a$ and $|x^{n+1} y^s| = b$. Here $|x^n y^s|$ denotes the total number of appearances of $x^n y^s$ in the word of k in $\pi_1(H)$, etc.

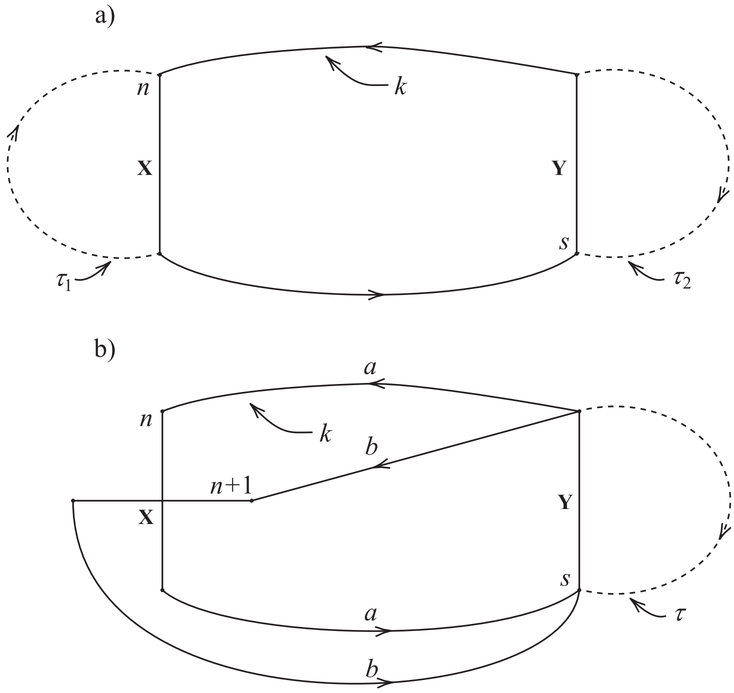

FIGURE 6. Two types of R-R diagrams of a Seifert curve k with $n, s > 1$ in Figure 6a, and $n > 0, s > 1$, $a, b > 1$, and $\gcd(a, b) = 1$ in Figure 6b, and regular fibers τ, τ_1, and τ_2 of $H[k]$.

(2) Algebraically the regular fibers τ_1 and τ_2 in Figure 6a of $H[k]$ represent x^n and y^s respectively, while the regular fiber τ in Figure 6b represents y^s in $\pi_1(H)$ with $n, s > 1$. In other words, the regular fibers correspond to the generator in k which has only one exponent.

(3) If a curve disjoint from k in Figure 6a represents x^n (y^s, resp.), then this curve is isotopic to the curve τ_1 (τ_2, resp.) and thus can be a regular fiber of $H[k]$. Similarly if a curve disjoint from k in Figure 6b represents y^s, then this curve is isotopic to the curve τ_2 and thus can be a regular fiber of $H[k]$.

Making use of R-R diagrams of a twisted torus knot $K = K(p, q, r, m, n)$ which allows one to compute K in $\pi_1(H)$ and $\pi_1(H')$, and applying Lemma 4.1 and the remarks above, we are able to find regular fibers of $H[K]$ and $H'[K]$ for all the types of middle/middle twisted torus knots in Table 1.

Lemma 4.2. *Let $K = K(p, q, r, m, n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* I *in Table* 1*, i.e., $(p, q, r, m, n) = (\beta q + \bar{p}, q, \beta q + 2\bar{p}, 1, 1)$ with $1 < \bar{p} < q$, $|q - 2\bar{p}| > 1$,*

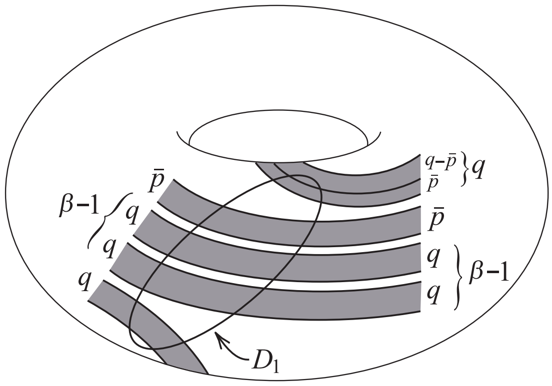

FIGURE 7. A torus knot $T(p, q)$ and the disk D_1 containing r parallel arcs of $T(p, q)$, where $p = \beta q + \bar{p}$ and $r = \beta q + 2\bar{p}$.

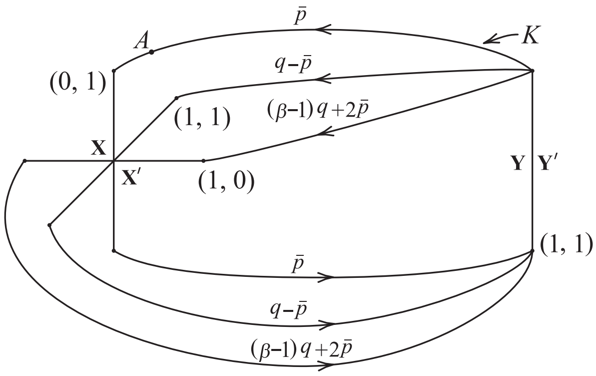

FIGURE 8. R-R diagram of K.

and $\beta > 1$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, \bar{p})$ and $D^2(2, |q - 2\bar{p}|)$.

Proof. Theorem 3.1 says that for the type I, $H[K] = D^2(\beta, \bar{p})$ and $H'[K] = D^2(2, |q - 2\bar{p}|)$. By Lemma 2.1 in [3] $K(\gamma) \cong H[K] \cup_\partial H'[K]$. Therefore in order to prove this lemma, it suffices to figure out how regular fibers of $H[K]$ and $H'[K]$ lie in Σ.

Recall the solid torus V_1 and the disk D_1 to construct a twisted torus knot in Section 2. Figure 7 shows a torus knot $T(p, q)$ and the disk D_1 containing r parallel arcs of $T(p, q)$ in V_1, where $p = \beta q + \bar{p}$ and $r = \beta q + 2\bar{p}$. It follows by figuring out nonparallel bands of connections in the once-punctured torus $\overline{\partial V_1 - D_1}$ that K has an R-R diagram of the form shown in Figure 8. As a special case, Figure 9 illustrates the R-R diagram of K when $\beta = 3$, which will

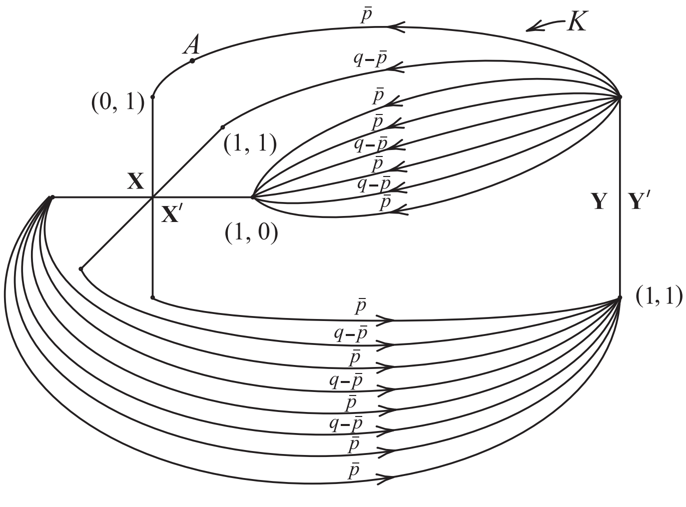

FIGURE 9. R-R diagram of K when $\beta = 3$.

be used later to find algebraic expressions of K in $\pi_1(H)$ and $\pi_1(H')$ as well as to depict regular fibers of $H[K]$ and $H'[K]$.

We record the curve K algebraically by starting the $\bar{p}$ parallel arcs entering into the $(0,1)$-connection in the (X, X')-handle. In other words, in Figure 8, we read off a word of K from the point A lying on $\bar{p}$ parallel edges entering into the $(0,1)$-connection in the (X, X')-handle. Let $q = a\bar{p} + b$, where $a > 0, 0 < b < \bar{p}$.

First we record K with respect to H, where $\pi_1(H) = \langle x, y \rangle$. The $\bar{p}$ parallel edges trace out

$$x^0 y(xy)^\beta (xy(xy)^{\beta-1})^{a-1} \cdots$$

and then they split into two subsets of parallel edges, one of which has $\bar{p} - b$ parallel edges and the other has b parallel edges. The $\bar{p} - b$ parallel edges trace out xy while the b parallel edges trace out $xy(xy)^{\beta-1}xy$ before they come back to the starting point A. To make sure, one can use Figure 9 when $\beta = 3$. Therefore it follows that K is the product of two subwords

$$x^0 y(xy)^\beta (xy(xy)^{\beta-1})^{a-1} xy = xy^2(xy)^{\beta a} \text{ and}$$
$$x^0 y(xy)^\beta (xy(xy)^{\beta-1})^{a-1} xy(xy)^{\beta-1}xy = xy^2(xy)^{\beta(a+1)}$$

with $|xy^2(xy)^{\beta a}| = \bar{p} - b$ and $|xy^2(xy)^{\beta(a+1)}| = b$.

We perform a change of cutting disks of the handlebody H underlying the R-R diagram, which induces an automorphism of $\pi_1(H)$ that takes $x \mapsto xy^{-1}$ and leaves y fixed. Then by this change of cutting disks, $xy^2(xy)^{\beta a}$ and $xy^2(xy)^{\beta(a+1)}$ are sent to $yx^{\beta a+1}$ and $yx^{\beta(a+1)+1}$ respectively. We perform another change of cutting disks of H inducing an automorphism $y \mapsto yx^{-\beta a-1}$ to send $yx^{\beta a+1}$ and $yx^{\beta(a+1)+1}$ to y and yx^β respectively. Then only β appears

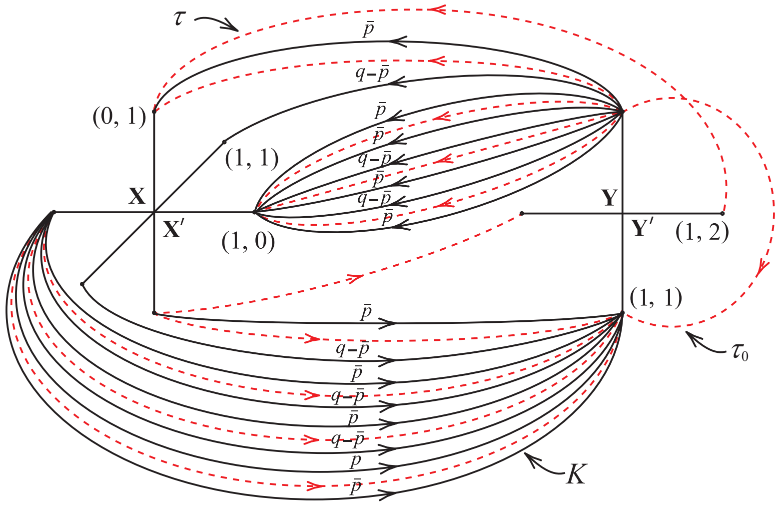

FIGURE 10. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 3$, where the curve τ' is obtained from τ by twisting $\beta + 3$ times about τ_0.

in the exponent of x. From Lemma 4.1 and the remarks below Lemma 4.1, it follows that a curve representing x^β is a regular fiber of $H[K]$. This can also be guaranteed from Theorem 3.1 and Table 2 showing that $H[K] = D^2(\beta, \bar{p})$ so that β is the index of one of the two exceptional fibers.

Consider the curve τ, which is the dotted line, in the original R-R diagram of K shown in Figure 10 when $\beta = 3$. τ is disjoint from K and represents

$$x^0 y(xy)^{\beta-1} xyx^0 y^{-1} = (xy)^\beta$$

in $\pi_1(H)$, which is sent to x^β after performing the two automorphisms $x \mapsto xy^{-1}$ and $y \mapsto yx^{-\beta a-1}$ consecutively as performed to K. Therefore by the remark (3) below Lemma 4.1, τ is a regular fiber of $H[K]$.

Similarly, if we record K with respect to H', where $\pi_1(H') = \langle x', y' \rangle$, then K is the product of two subwords

$$x'y'(x'^0 y')^\beta (x'y'(x'^0 y')^{\beta-1})^{a-1} x'^0 y' = x'y'^{\beta+1}(x'y'^\beta)^{a-1} y' \text{ and}$$
$$x'y'(x'^0 y')^\beta (x'y'(x'^0 y')^{\beta-1})^{a-1} x'y'(x'^0 y')^{\beta-1} x'^0 y' = x'y'^{\beta+1}(x'y'^\beta)^a y'$$

with $|x'y'^{\beta+1}(x'y'^\beta)^{a-1}y'| = \bar{p} - b$ and $|x'y'^{\beta+1}(x'y'^\beta)^a y'| = b$. After performing two automorphisms $x' \mapsto x'y'^{-\beta}$ and $y' \mapsto x'^{-1}y'$ consecutively, then the two subwords $x'y'^{\beta+1}(x'y'^\beta)^{a-1}y'$ and $x'y'^{\beta+1}(x'y'^\beta)^a y'$ are sent to $y'^2 x'^{a-2}$ and $y'^2 x'^{a-1}$. Therefore a curve representing y'^2 is a regular fiber of $H'[K]$. Consider the curve τ' which is a simple closed curve on the surface Σ obtained from the curve τ by twisting $\beta + 3$ times about the oriented curve τ_0 in the

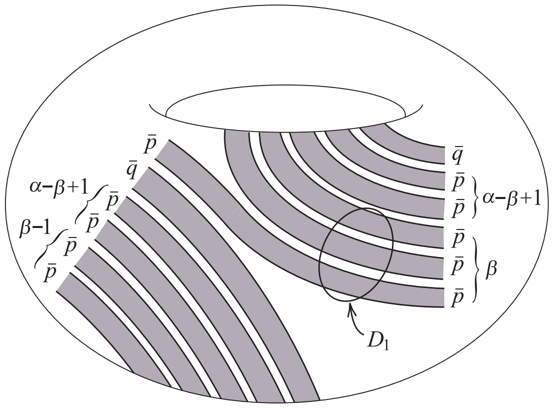

FIGURE 11. A torus knot $T(p,q)$ and the disk D_1 containing r parallel arcs of $T(p,q)$, where $p=(\alpha+1)\bar{p}+\bar{q}, q=\alpha\bar{p}+\bar{q}$, and $r=\beta\bar{p}$.

original R-R diagram of K. Figure 10 illustrates τ' when $\beta=3$. Note τ' is disjoint from K. Also τ' represents $x'y'^{\beta+1}x'y'^{\beta+1}$ in $\pi_1(H')$, which is sent to y'^2 after performing $x'\mapsto x'y'^{-\beta}$ and $y'\mapsto x'^{-1}y'$ consecutively as performed to K. Therefore by the remark (3) below Lemma 4.1, τ' is a regular fiber of $H'[K]$.

We conclude from Figure 10 that the two regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively intersect transversely $\beta+3$ times, which implies that $K(\gamma)\cong H[K]\cup_\partial H'[K]$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta,\bar{p})$ and $D^2(2,|q-2\bar{p}|)$, where γ is a surface slope. $K(\gamma)$ may be a Seifert-fibered space over the projective plane. However, this case happens only when either $D^2(\beta,\bar{p})$ or $D^2(2,|q-2\bar{p}|)$ is $D^2(2,2)$ which admits another Seifert fibering. $\square$

Lemma 4.3. *Let $K=K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* II *in Table* 1*, i.e.,* $(p,q,r,m,n)=((\alpha+1)\bar{p}+\bar{q},\alpha\bar{p}+\bar{q},\beta\bar{p},1,1)$ *with* $0\le\bar{q}<\bar{p}$ *and* $1<\beta\le\alpha$. *Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta,(\alpha-\beta+1)\bar{p}+\bar{q})$ and $D^2(\beta,(\alpha-\beta)\bar{p}+\bar{q})$.*

Proof. Figure 11 shows a torus knot $T(p,q)$ and the disk D_1 containing $r=\beta\bar{p}$ parallel arcs of $T(p,q)$ in V_1. Figure 12 shows an R-R diagram of K, where $s=\alpha-\beta+1, t=\alpha-\beta+2, u=\alpha-\beta+3$.

We first handle with respect to H. We assume that $\bar{q}=0$. Since $\gcd(\bar{p},\bar{q})=1$, $\bar{p}=1$. Then it is easy to see from the R-R diagram of K that K represents algebraically

$$x^{\alpha-\beta+2}y(xy)^{\beta-1},$$

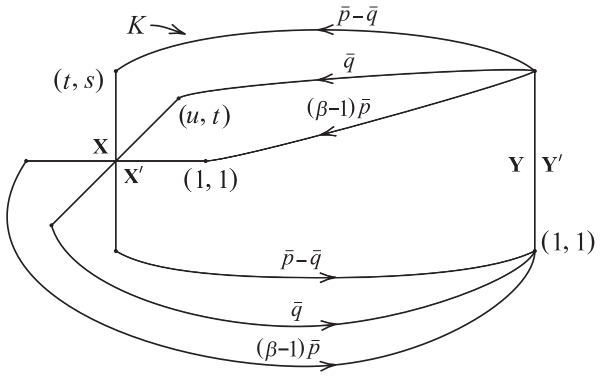

FIGURE 12. R-R diagram of K. Here $s = \alpha - \beta + 1, t = \alpha - \beta + 2, u = \alpha - \beta + 3$.

which can be sent to $x^{\alpha-\beta+1}y^{\beta}$ under an automorphism $y \mapsto x^{-1}y$. Therefore a curve representing $x^{\alpha-\beta+1}$ or y^{β} is a regular fiber of $H[K]$. We take y^{β} as a regular fiber of $H[K]$.

We now assume that $\bar{q} > 0$. Let $\bar{p} = a\bar{q} + b$, where $a > 0, 0 \leq b < a$. We record the curve K algebraically by starting the $\bar{q}$ parallel arcs entering into the (u, t)-connection in the (X, X')-handle. The $\bar{q}$ parallel edges trace out

$$x^{\alpha-\beta+3}(yx)^{\beta-1}(yx^{\alpha-\beta+2}(yx)^{\beta-1})^{a-1}\cdots$$

and then they split into two subsets of parallel edges, one of which has $\bar{q} - b$ parallel edges and the other has b parallel edges. The $\bar{q} - b$ parallel edges trace out y while the b parallel edges trace out $yx^{\alpha-\beta+2}(yx)^{\beta-1}y$ before they come back to the starting point. This implies that K is the product of two subwords

$$x^{\alpha-\beta+3}(yx)^{\beta-1}(yx^{\alpha-\beta+2}(yx)^{\beta-1})^{a-1}y \text{ and}$$
$$x^{\alpha-\beta+3}(yx)^{\beta-1}(yx^{\alpha-\beta+2}(yx)^{\beta-1})^{a}y$$

with

$$|x^{\alpha-\beta+3}(yx)^{\beta-1}(yx^{\alpha-\beta+2}(yx)^{\beta-1})^{a-1}y| = \bar{q} - b \text{ and}$$
$$|x^{\alpha-\beta+3}(yx)^{\beta-1}(yx^{\alpha-\beta+2}(yx)^{\beta-1})^{a}y| = b.$$

By performing an automorphism of $\pi_1(H)$ that takes $y \mapsto yx^{-1}$, they are carried into $x^{\alpha-\beta+2}y^{\beta}(x^{\alpha-\beta+1}y^{\beta})^{a-1}$ and $x^{\alpha-\beta+2}y^{\beta}(x^{\alpha-\beta+1}y^{\beta})^{a}$ respectively. Thus only β appears in the exponent of y and a curve representing y^{β} is a regular fiber of $H[K]$.

We have shown that a curve representing y^{β} can be taken as a regular fiber of $H[K]$ in both cases that $\bar{q} = 0$ and $\bar{q} > 0$. Consider the curve τ in the original R-R diagram of K shown in Figure 13, where $\beta = 4$. τ is disjoint

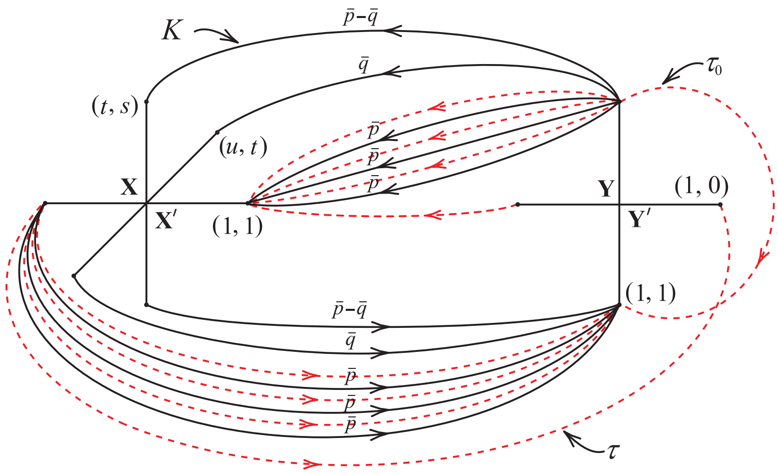

FIGURE 13. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 4$, where the curve τ' is obtained from τ by twisting once about τ_0.

from K and represents $(xy)^\beta$ in $\pi_1(H)$, which is sent to y^β after performing the automorphism $y \mapsto yx^{-1}$. Therefore τ is a regular fiber of $H[K]$.

For a regular fiber of $H'[K]$, if we record K with respect to H', only difference from H happens in the exponents of x and x'. In other words, we can get K with respect to H' by replacing $\alpha-\beta+3, \alpha-\beta+2$ by $\alpha-\beta+2, \alpha-\beta+1$ respectively in K with respect to H. Therefore applying the same argument as in H we see that a curve representing $(x'y')^\beta$ in the original R-R diagram of K is a regular fiber of $H'[K]$. Consider the curve τ' which is a simple closed curve on the surface Σ obtained from the curve τ by twisting once about the oriented curve τ_0 in the original R-R diagram of K. Figure 13 illustrates τ and τ_0 when $\beta = 4$. By Figure 13, τ' represents $(x'y')^\beta$, whence τ' is a regular fiber of $H'[K]$.

Since the two regular fibers τ and τ' intersect transversely once, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, (\alpha-\beta+1)\bar{p}+\bar{q})$ and $D^2(\beta, (\alpha-\beta)\bar{p}+\bar{q})$, where γ is a surface slope. As in Lemma 4.2 $K(\gamma)$ may be a Seifert-fibered space over the projective plane. $\square$

Lemma 4.4. *Let $K = K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* III *in Table* 1*, i.e.,* $(p,q,r,m,n) = ((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (\alpha-\beta)\bar{p}+\bar{q}, 1, 1)$ *with* $0 < \bar{q} < \bar{p}$ *and* $1 < \beta \leq \alpha$. *Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta+1, (\alpha-\beta)\bar{p}+\bar{q})$ and $D^2(\beta, (\alpha-\beta)\bar{p}+\bar{q})$.*

Proof. Figure 14 shows a torus knot $T(p,q)$ and the disk D_1 containing $r = (\alpha-\beta)\bar{p}+\bar{q}$ parallel arcs of $T(p,q)$ in V_1. Figure 15 shows R-R diagrams of K

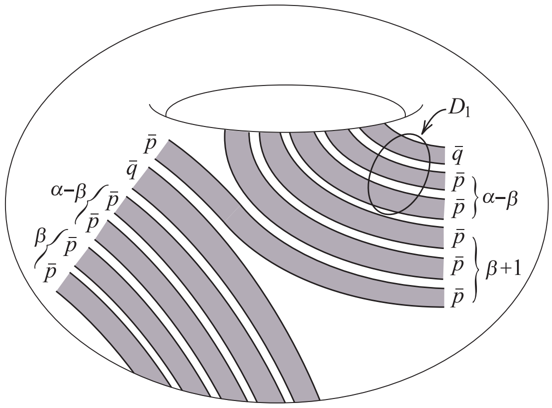

FIGURE 14. A torus knot $T(p,q)$ and the disk D_1 containing r parallel arcs of $T(p,q)$, where $p = (\alpha+1)\bar{p}+\bar{q}, q = \alpha\bar{p}+\bar{q}$, and $r = (\alpha-\beta)\bar{p}+\bar{q}$.

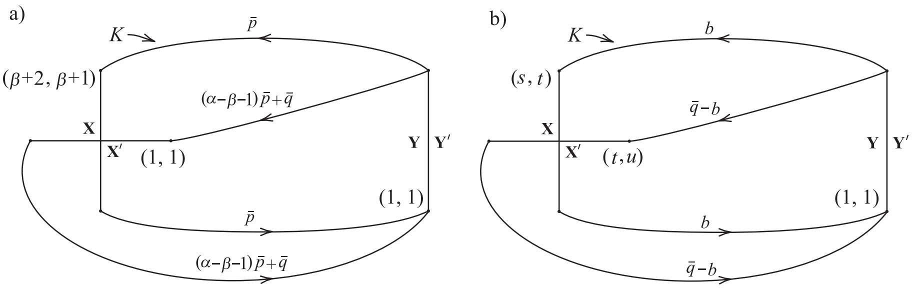

FIGURE 15. R-R diagrams of K when a) $\alpha > \beta$ and b) $\alpha = \beta$, where $(s,t) = ((a+1)(\beta+1)+1, (a+1)\beta+1)$ and $(t,u) = (a(\beta+1)+1, a\beta+1)$.

when $\alpha > \beta$ and $\alpha = \beta$. In Figure 15b, $(s,t) = ((a+1)(\beta+1)+1, (a+1)\beta+1)$ and $(t,u) = (a(\beta+1)+1, a\beta+1)$, where $\bar{p} = a\bar{q}+b$ with $a > 0$, $0 \leq b < \bar{q}$. Note that the two R-R diagrams have the same form. Therefore finding regular fibers when $\alpha = \beta$ can be achieved in the same manner as when $\alpha > \beta$.

Assume that $\alpha > \beta$. By starting the $\bar{p}$ parallel arcs entering into the $(\beta+2, \beta+1)$-connection in the (X, X')-handle, we observe that with respect to H, K is the product of two subwords

$$x^{\beta+2}(yx)^{\alpha-\beta-1}y \text{ and } x^{\beta+2}(yx)^{\alpha-\beta-1}yxy$$

with $|x^{\beta+2}(yx)^{\alpha-\beta-1}y| = \bar{p}-\bar{q}$ and $|x^{\beta+2}(yx)^{\alpha-\beta-1}yxy| = \bar{q}$.

By performing an automorphism $y \mapsto yx^{-1}$, they are carried into $x^{\beta+1}y^{\alpha-\beta}$ and $x^{\beta+1}y^{\alpha-\beta+1}$ respectively. Thus only $\beta+1$ appears in the exponent of x

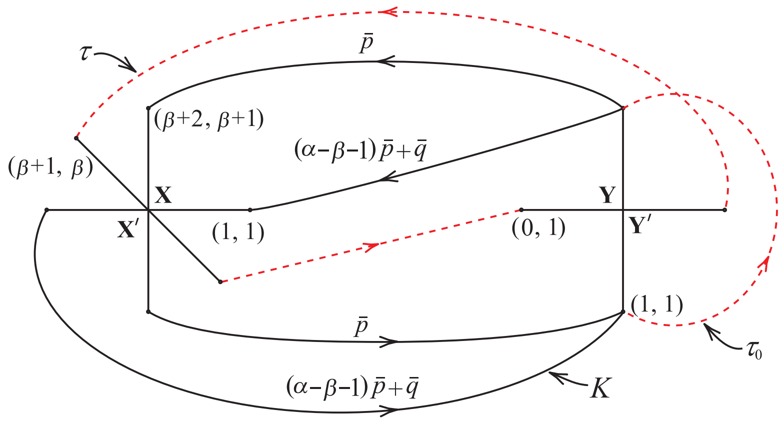

FIGURE 16. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively, where the curve τ' is obtained from τ by twisting once about τ_0.

and a curve representing $x^{\beta+1}$ is a regular fiber of $H[K]$. The curve τ in the original R-R diagram of K shown in Figure 16 represents $x^{\beta+1}$ in $\pi_1(H)$, which is sent to $x^{\beta+1}$ after performing the automorphism $y \mapsto yx^{-1}$. Therefore τ is a regular fiber of $H[K]$.

For a regular fiber of $H'[K]$, we apply the similar argument to see that a curve representing x'^{β} is a regular fiber of $H'[K]$ and a simple closed curve τ' represents x'^{β}, where τ' is a simple closed curve obtained from the curve τ by twisting once about τ_0 in Figure 16.

Since the two regular fibers τ and τ' intersect transversely once, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta+1, (\alpha-\beta)\bar{p}+\bar{q})$ and $D^2(\beta, (\alpha-\beta)\bar{p}+\bar{q})$, where γ is a surface slope. As in Lemma 4.2 $K(\gamma)$ may be a Seifert-fibered space over the projective plane. □

Lemma 4.5. *Let $K = K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* IV *in Table* 1*, i.e., $(p,q,r,m,n) = ((\alpha+1)\bar{p}+\bar{q}, \alpha\bar{p}+\bar{q}, (\alpha+\beta+1)\bar{p}+\bar{q}, 1, 1)$ with $0 \le \bar{q} < \bar{p}$ and $1 < \beta < \alpha$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, (\alpha-\beta+1)\bar{p}+\bar{q})$ and $D^2(\beta+1, (\alpha-\beta-1)\bar{p}+\bar{q})$.*

Proof. A torus knot $T(p,q)$ and the disk D_1 containing $r = (\alpha+\beta+1)\bar{p}+\bar{q}$ parallel arcs of $T(p,q)$ in V_1 are illustrated in Figure 17. The corresponding R-R diagram of K is depicted in Figure 18.

We start the $\bar{p}$ parallel arcs which are innermost in the $\beta\bar{p}$ parallel arcs entering into the $(0,1)$-connection in the (X, X')-handle. Then it follows that for H, K is the product of two subwords

$$x^0yxy(xy)^{\alpha-\beta}xy(x^0yxy)^{\beta-1} = (xy)^{\alpha-\beta+1}(xy^2)^{\beta} \text{ and}$$

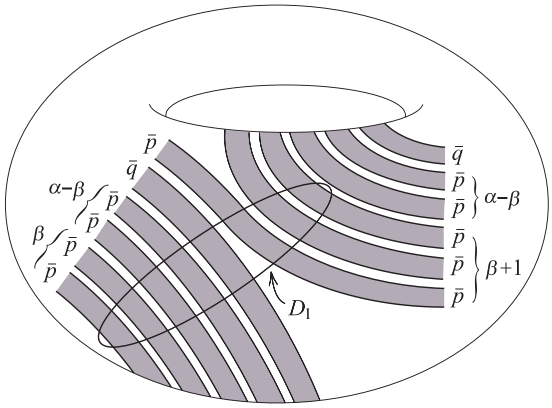

FIGURE 17. A torus knot $T(p,q)$ and the disk D_1 containing r parallel arcs of $T(p,q)$, where $p=(\alpha+1)\bar{p}+\bar{q}, q=\alpha\bar{p}+\bar{q}$, and $r=(\alpha+\beta+1)\bar{p}+\bar{q}$.

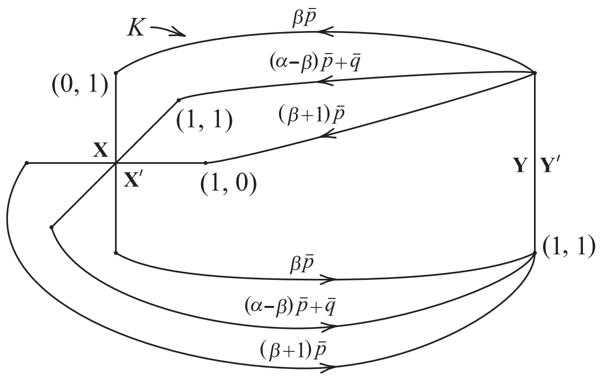

FIGURE 18. R-R diagram of K.

$$x^0yxy(xy)^{\alpha-\beta}xyxy(x^0yxy)^{\beta-1}=(xy)^{\alpha-\beta+2}(xy^2)^{\beta}$$

with $|(xy)^{\alpha-\beta+1}(xy^2)^{\beta}|=\bar{p}-\bar{q}$ and $|(xy)^{\alpha-\beta+2}(xy^2)^{\beta}|=\bar{q}$. This can be confirmed from Figure 19, which shows the R-R diagram of K when $\beta=2$.

After applying two automorphisms $y\mapsto x^{-1}y$ and $x^{-1}\mapsto x^{-1}y^{-2}$ consecutively, they are sent to $y^{\alpha-\beta+1}x^{-\beta}$ and $y^{\alpha-\beta+2}x^{-\beta}$ respectively. Therefore a curve representing $x^{-\beta}$ is a regular fiber of $H[K]$. The curve τ in the original R-R diagram of K shown in Figure 20, where $\beta=2$, represents $(x^0yxy)^{\beta}$ in $\pi_1(H)$, which is sent to $x^{-\beta}$ after performing $y\mapsto x^{-1}y$ and $x^{-1}\mapsto x^{-1}y^{-2}$ consecutively as performed to K. Thus τ is a regular fiber of $H[K]$.

Similarly, for H' K is the product of two subwords

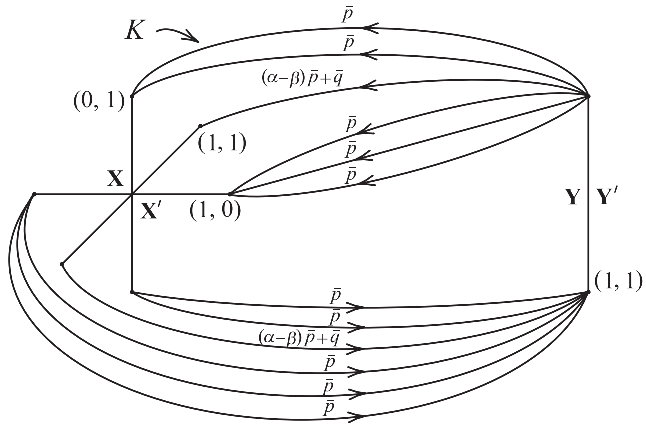

FIGURE 19. R-R diagram of K when $\beta = 2$.

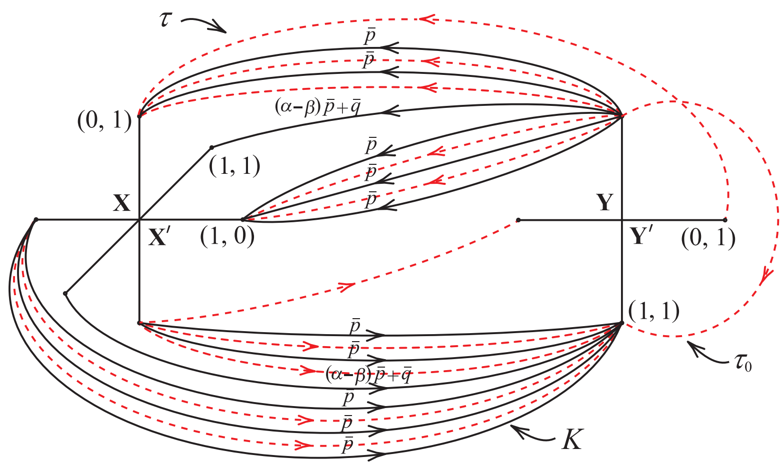

FIGURE 20. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 2$, where the curve τ' is obtained from τ by twisting three times about τ_0.

$$x'y'x'^0y'(x'y')^{\alpha-\beta}x'^0y'(x'y'x'^0y')^{\beta-1} = (x'y')^{\alpha-\beta-1}(x'y'^2)^{\beta+1} \text{ and}$$
$$x'y'x'^0y'(x'y')^{\alpha-\beta}x'y'x'^0y'(x'y'x'^0y')^{\beta-1} = (x'y')^{\alpha-\beta}(x'y'^2)^{\beta+1}$$

with $|(x'y')^{\alpha-\beta-1}(x'y'^2)^{\beta+1}| = \bar{p} - \bar{q}$ and $|(x'y')^{\alpha-\beta}(x'y'^2)^{\beta+1}| = \bar{q}$.

By performing automorphisms $y' \mapsto x'^{-1}y'$ and $x'^{-1} \mapsto x'^{-1}y'^{-2}$, it follows that a curve representing $x'^{-\beta-1}$ is a regular fiber of $H'[K]$. Let τ' be a simple closed curve obtained from the curve τ by twisting three times about τ_0 in

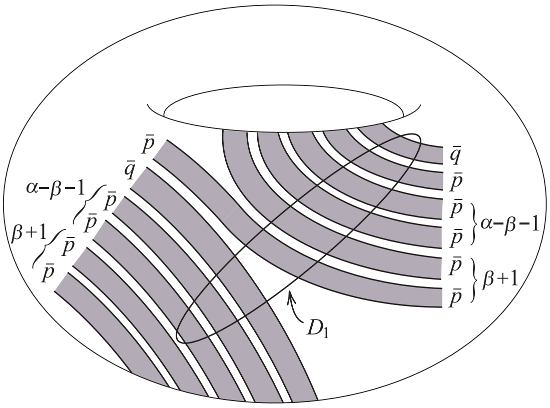

FIGURE 21. A torus knot $T(p,q)$ and the disk D_1 containing r parallel arcs of $T(p,q)$, where $p = (\alpha+1)\bar p + \bar q, q = \alpha\bar p + \bar q$, and $r = (2\alpha - \beta)\bar p + 2\bar q$.

Figure 20, where $\beta = 2$. Then τ' represents $x'^{-\beta-1}$, whence it is a regular fiber of $H'[K]$.

Since the two regular fibers τ and τ' intersect transversely three times, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta, (\alpha-\beta+1)\bar p + \bar q)$ and $D^2(\beta+1, (\alpha-\beta-1)\bar p + \bar q)$, where γ is a surface slope. As in Lemma 4.2 $K(\gamma)$ may be a Seifert-fibered space over the projective plane. $\square$

Lemma 4.6. *Let $K = K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* V *in Table* 1*, i.e., $(p,q,r,m,n) = ((\alpha+1)\bar p + \bar q, \alpha\bar p + \bar q, (2\alpha-\beta)\bar p + 2\bar q, 1, 1)$ with $0 < \bar q < \bar p$ and $1 < \beta < \alpha$. Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(\beta+2, (\alpha-\beta-1)\bar p+\bar q)$ and $D^2(\beta, (\alpha-\beta)\bar p+\bar q)$.*

Proof. Figures 21 and 22 show a torus knot $T(p,q)$ and the disk D_1 containing $r = (2\alpha-\beta)\bar p + 2\bar q$ parallel arcs of $T(p,q)$ in V_1 and its corresponding R-R diagram of K respectively.

As did in Type IV, we start the $\bar p$ parallel arcs which are innermost in the $(\beta+1)\bar p$ parallel arcs entering into the $(1,1)$-connection in the (X, X')-handle. Then for H, K is the product of two subwords

$$xyxy(x^0yxy)^{\alpha-\beta-1}(xy)^{\beta} = (xy^2)^{\alpha-\beta-1}(xy)^{\beta+2} \text{ and}$$
$$xyxy(x^0yxy)^{\alpha-\beta-1}x^0yxy(xy)^{\beta} = (xy^2)^{\alpha-\beta}(xy)^{\beta+2}$$

with $|(xy^2)^{\alpha-\beta-1}(xy)^{\beta+2}| = \bar p - \bar q$ and $|(xy^2)^{\alpha-\beta}(xy)^{\beta+2}| = \bar q$.

For H', K is the product of two subwords

$$x'y'x'^0y'(x'y'x'^0y')^{\alpha-\beta-1}(x'y')^{\beta} = (x'y'^2)^{\alpha-\beta}(x'y')^{\beta} \text{ and}$$

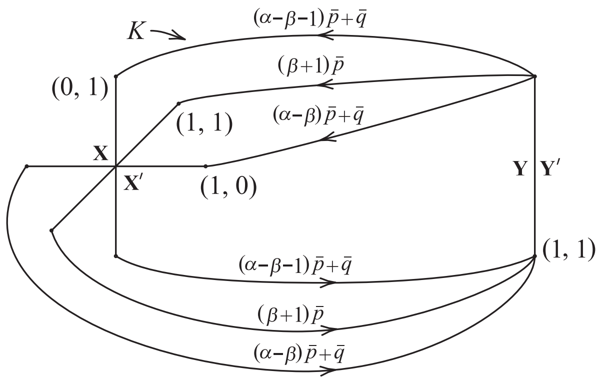

FIGURE 22. R-R diagram of K.

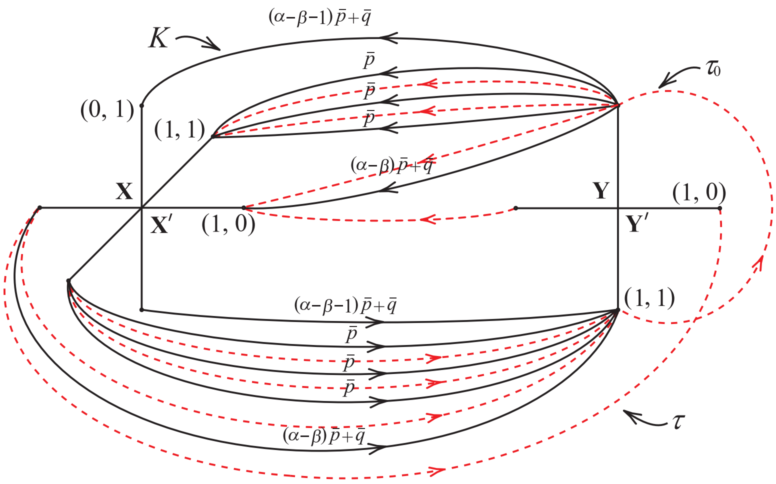

FIGURE 23. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively when $\beta = 2$, where the curve τ' is obtained from τ by twisting three times about τ_0.

$$x'y'x'^0y'(x'y'x'^0y')^{\alpha-\beta-1}x'y'x'^0y'(x'y')^{\beta} = (x'y'^2)^{\alpha-\beta+1}(x'y')^{\beta}$$

with $|(x'y'^2)^{\alpha-\beta}(x'y')^{\beta}| = \bar{p} - \bar{q}$ and $|(x'y'^2)^{\alpha-\beta+1}(x'y')^{\beta}| = \bar{q}$.

Observe that K in this type has the same algebraic expression with respect to H and H' as in the type IV. Applying the same argument, we see that the regular fibers of $H[K]$ and $H'[K]$ can be depicted as τ and τ' respectively in Figure 23 which intersect three times, where τ' is a simple closed curve obtained from the curve τ by twisting three times about τ_0. This implies that the Dehn surgery at a surface slope is a non-Seifert-fibered graph manifold consisting of

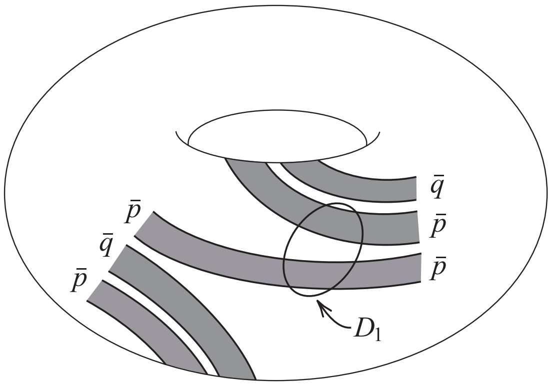

FIGURE 24. A torus knot $T(p,q)$ and the disk D_1 containing r parallel arcs of $T(p,q)$, where $p = 2\bar{p} + \bar{q}, q = \bar{p} + \bar{q}$, and $r = 2\bar{p}$.

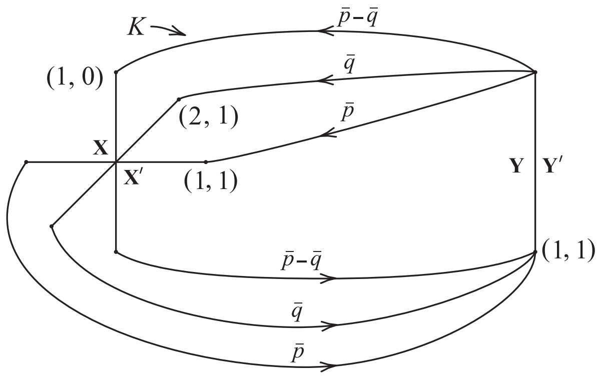

FIGURE 25. R-R diagram of K.

$D^2(\beta+2, (\alpha-\beta-1)\bar{p}+\bar{q})$ and $D^2(\beta, (\alpha-\beta)\bar{p}+\bar{q})$. As in Lemma 4.2 $K(\gamma)$ may be a Seifert-fibered space over the projective plane. □

Lemma 4.7. *Let $K = K(p,q,r,m,n)$ be a twisted torus knot lying in a genus two Heegaard surface Σ of S^3 such that K is of type* VI *in Table* 1, *i.e.,* $(p,q,r,m,n) = (2\bar{p}+\bar{q}, \bar{p}+\bar{q}, 2\bar{p}, 1, 1)$ *with* $0 < \bar{q} < \bar{p}$. *Then for the surface slope γ, $K(\gamma)$ is a non-Seifert-fibered graph manifold consisting of $D^2(2,\bar{q})$ and $D^2(2,\bar{p}-\bar{q})$.*

Proof. Figures 24 and 25 show a torus knot $T(p,q)$ and the disk D_1 containing $r = 2\bar{p}$ parallel arcs of $T(p,q)$ in V_1 and its corresponding R-R diagram of K respectively.

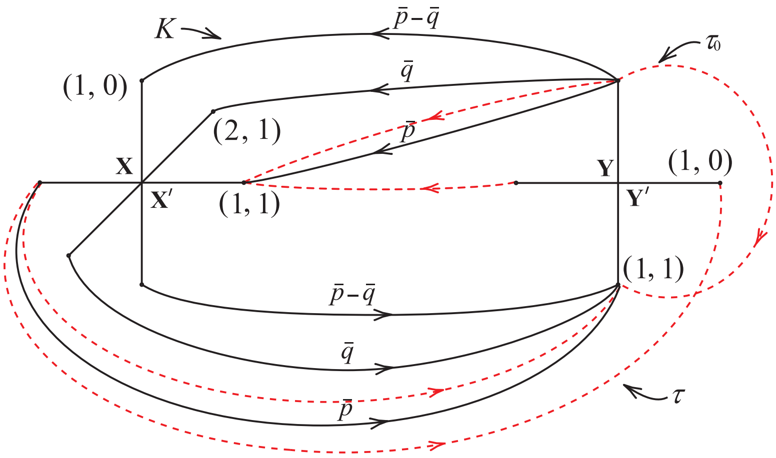

FIGURE 26. The regular fibers τ and τ' of $H[K]$ and $H'[K]$ respectively, where the curve τ' is obtained from τ by twisting once about τ_0.

Let $\bar{p} = a\bar{q} + b$, where $a > 0$, $0 < b < \bar{q}$. By starting the $\bar{q}$ parallel arcs entering into the $(2, 1)$-connection in the (X, X')-handle, we see that for H, K is the product of two subwords

$$x^2y(xyxy)^{a-1}xy = x^2y(xy)^{2a-1} \text{ and } x^2y(xyxy)^a xy = x^2y(xy)^{2a+1}$$

with $|x^2y(xy)^{2a-1}| = \bar{q} - b$ and $|x^2y(xy)^{2a+1}| = b$.

After performing two automorphisms $y \mapsto x^{-1}y$ and $x \mapsto xy^{-2a}$ consecutively, they are sent to x and xy^2 respectively. Therefore a curve representing y^2 is a regular fiber of $H[K]$. The curve τ in the original R-R diagram of K as shown in Figure 26 represents $(xyxy)^2$ in $\pi_1(H)$, which is sent to y^2 after performing $y \mapsto x^{-1}y$ and $x \mapsto xy^{-2a}$ consecutively as performed to K. Thus τ is a regular fiber of $H[K]$.

For H', K is the product of two subwords

$$x'y'(x'y'x'^0y')^{a-1}x'y' = (x'y')^2(x'y'^2)^{a-1} \text{ and}$$

$$x'y'(x'y'x'^0y')^a x'y' = (x'y')^2(x'y'^2)^a$$

with $|(x'y')^2(x'y'^2)^{a-1}| = \bar{q} - b$ and $|(x'y')^2(x'y'^2)^a| = b$.

After performing two automorphisms $y' \mapsto x'^{-1}y'$ and $x'^{-1} \mapsto x'^{-1}y'^{-2}$ consecutively, they are sent to $x'^{-a+1}y'^2$ and $x'^{-a}y'^2$ respectively. Therefore a curve representing y'^2 is a regular fiber of $H'[K]$. Consider the curve τ' which is a simple closed curve obtained from the curve τ by twisting once about τ_0 in Figure 26. The curve τ' represents $(x'y'x'y')^2$ in $\pi_1(H')$, which is sent to y'^2 after $y' \mapsto x'^{-1}y'$ and $x'^{-1} \mapsto x'^{-1}y'^{-2}$ consecutively as performed to K. Thus τ' is a regular fiber of $H'[K]$.

From Figure 26 it follows that the two regular fibers τ and τ' intersect once, whence the Dehn surgery at a surface slope is a non-Seifert-fibered graph manifold consisting of $D^2(2, \bar{q})$ and $D^2(2, \bar{p} - \bar{q})$. As in Lemma 4.2 $K(\gamma)$ may be a Seifert-fibered space over the projective plane. □

By Lemmas 4.2 $\sim$ 4.7 we obtain the main theorem of this paper combining with Theorem 3.1 as follows, which provides another infinite family of knots in S^3 admitting Dehn surgery yielding graph manifolds as done in [5].

Theorem 4.8. *All the middle/middle twisted torus knots possibly except a few in Theorem* 3.1 *admit Dehn surgeries producing non-Seifert-fibered graph manifolds.*

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