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JACOBI SPECTRAL GALERKIN METHODS FOR VOLTERRA INTEGRAL EQUATIONS WITH WEAKLY SINGULAR KERNEL

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Abstract. We propose and analyze spectral and pseudo-spectral Jacobi-Galerkin approaches for weakly singular Volterra integral equations (VIEs). We provide a rigorous error analysis for spectral and pseudo-spectral Jacobi-Galerkin methods, which show that the errors of the approximate solution decay exponentially in L^∞ norm and weighted L^2-norm. The numerical examples are given to illustrate the theoretical results.

1. Introduction

Volterra integral equations (VIEs) arise widely in mathematical models of certain biological and physical phenomena. Due to the wide application of these equations, they must be solved successfully with efficient numerical methods.

In this article, we are concerned with the numerical study of the following Volterra integral equations with weakly singular kernel:

$$y(t)=\int_0^t (t-\tau)^{-\gamma}K(t,\tau)y(\tau)d\tau+f(t),\quad 0<\gamma<1,\quad t\in[0,T], \tag{1}$$

where the source function f and the kernel function K are given and the function $y(t)$ is the unknown function. Here, f and K are assumed to be sufficiently smooth on their respective domains $[0,T]$ and $0\le\tau\le t\le T$.

The numerical treatment of the VIEs (1) is a challenge work, mainly due to the fact that the solutions of (1) usually have a weak singularity at $t=0$, even when the inhomogeneous term $f(t)$ is regular. Collocation spectral methods and the corresponding error analysis have been provided recently [14, 16] for (1) without the singular kernel (i.e., $\gamma=0$) in case of the underlying

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solutions are smooth. Chen and Tang [4, 17] developed a novel spectral Jacobi-collocation method to solve (1) and provided a rigorous error analysis which theoretically justifies the spectral rate of convergence, see also [19] for Jacobi spectral-collocation method for fractional integro-differential equations. In [20], the authors extended the Legendre-collocation methods to nonlinear Volterra integral equations. Recently, in [16], the authors provided a Legendre spectral Galerkin method for second-kind Volterra integral equations, [15, 18] provide general spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches for the second kind Volterra integro-differential equations. Inspired by those work, we extend spectral Galerkin approach to Volterra integral equations with weakly singular kernels and provide a rigorous convergence analysis for the spectral and pseudo-spectral Jacobi-Galerkin methods, which indicates that the proposed methods converges exponentially provided that the data in the given VIEs are smooth.

This paper is organized as follows. In Section 2, we demonstrate the implementation of the spectral and pseudo-spectral Galerkin approaches for Volterra integral equations with weakly singular kernels. Some lemmas useful for establishing the convergence result will be provided in Section 3. The convergence analysis for both spectral and pseudo-spectral Jacobi-Galerkin methods will be given in Section 4. Numerical results will be carried out in Section 5, which will be used to verify the theoretical result obtained in Section 4. Finally, in Section 6, we end with conclusion and future work.

2. Spectral and pseudo-spectral Galerkin methods

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t = \frac{1}{2}T(1+x), \quad x = \frac{2t}{T} - 1,$$

$$\tau = \frac{1}{2}T(1+s), \quad s = \frac{2\tau}{T} - 1,$$

and let

$$u(x) = y(\frac{1}{2}T(1+x)), \quad g(x) = f(\frac{1}{2}T(1+x)),$$

$$k(x,s) = \left(\frac{T}{2}\right)^{1-\gamma} K(\frac{1}{2}T(1+x)\frac{1}{2}T(1+s)).$$

Then we get the following one dimension VIEs with weakly singular kernel defined on $[-1,1]$

$$(2)\quad u(x) = \int_{-1}^{x} (x-s)^{-\gamma} k(x,s)u(s)ds + g(x), \quad 0 < \gamma < 1, \quad x \in \Lambda := [-1,1].$$

We formulate the Jacobi-spectral Galerkin schemes and investigate the global convergence properties for the problem (2). For this purpose, we first define a linear integral operator $G : C(\Lambda) \to C(\Lambda)$ by

$$(G\phi)(x) := \int_{-1}^{x} (x-s)^{-\gamma} k(x,s)\phi(s)ds.$$

Then, the problem (2) reads: find $u = u(x)$ such that

$$u(x) = (Gu)(x) + g(x), \quad x \in \Lambda \tag{3}$$

and its weak form is to find $u \in L^2(\Lambda)$ such that

$$(u, v) = (Gu, v) + (g, v), \quad \forall v \in L^2(\Lambda), \tag{4}$$

where $(\cdot,\cdot)$ denotes the usual inner product in the L^2-space.

Let us demonstrate the numerical implementation of the spectral Jacobi-Galerkin approach first. Denote by $\mathbb{N}$ the set of all nonnegative integers. For any $N \in \mathbb{N}$, P_N denotes the set of all algebraic polynomials of degree at most N in Λ, $\phi_j(x)$ is the j-th Jacobi polynomial corresponding to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$. As a result,

$$P_N = \text{span}\{\phi_0(x), \phi_1(x), \ldots, \phi_N(x)\}.$$

Our spectral Jacobi-Galerkin approximation of (3) is now defined as: Find $u_N \in P_N$ such that

$$(u_N, v_N)_{\omega^{\alpha,\beta}} = (Gu_N, v_N)_{\omega^{\alpha,\beta}} + (g, v_N)_{\omega^{\alpha,\beta}}, \quad \forall v \in P_N, \tag{5}$$

where

$$(u, v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x)v(x)\omega^{\alpha,\beta}(x)dx$$

is the continuous inner product. Set $u_N(x) = \sum_{j=0}^{N} \xi_j \phi_j(x)$. Substituting it into (5) and taking $v_N = \phi_i(x)$, we obtain

$$\sum_{j=0}^{N} \xi_j (\phi_i, \phi_j)_{\omega^{\alpha,\beta}} = \sum_{j=0}^{N} \xi_j (\phi_i, G\phi_j)_{\omega^{\alpha,\beta}} + \sum_{j=0}^{N} (\phi_i, g)_{\omega^{\alpha,\beta}}, \tag{6}$$

which leads to an equation of the matrix form

$$A\xi = B\xi + C, \tag{7}$$

where

$$\xi = [\xi_0, \xi_1, \ldots, \xi_N]^T, \quad A_{i,j} = (\phi_i, \phi_j)_{\omega^{\alpha,\beta}},$$
$$B_{i,j} = (\phi_i, G\phi_j)_{\omega^{\alpha,\beta}}, \quad C_i = (\phi_i, g)_{\omega^{\alpha,\beta}}.$$

Next we propose the pseudo-spectral Jacobi-Galerkin method. Set

$$s(x,\theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}, \quad -1 \le \theta \le 1,$$

it is clear that

$$
\begin{aligned}
Gu(x) &= \int_{-1}^{x} (x-s)^{-\gamma} k(x,s) u(s) ds \\
&= \int_{-1}^{1} (1-\theta)^{-\gamma} \tilde{k}(x, s(x,\theta)) u(s(x,\theta)) d\theta
\end{aligned} \tag{8}
$$

with

$$\tilde{k}(x, s(x,\theta)) = \left(\frac{1+x}{2}\right)^{1-\gamma} k(x, s(x,\theta)).$$

Using $(N+1)$-point Gauss quadrature formula to approximate (8) yields

$$Gu(x) \approx G_N u(x) := \sum_{k=0}^{N} \tilde{k}(x, s(x,\theta_k)) u(s(x,\theta_k)) \omega_k^{-\gamma,0}, \tag{9}$$

where $\{\theta_k\}_{k=0}^N$ are the $(N+1)$-degree Jacobi-Gauss points corresponding to the weights $\{\omega_k^{-\gamma,0}\}_{k=0}^N$.

On the other hand, instead of the continuous inner product, the discrete inner product will be implemented in (5) and (6), i.e.,

$$(u,v)_{\omega^{\alpha,\beta}} \approx (u,v)_{\omega^{\alpha,\beta},N} = \sum_{k=0}^{N} u(x_k) v(x_k) \omega_k^{\alpha,\beta}(x_k). \tag{10}$$

As a result,

$$(u,v)_{\omega^{\alpha,\beta}} = (u,v)_{\omega^{\alpha,\beta},N}, \quad \text{if } u, v \in P_{2N}.$$

Substitute (9) and (10) into (5), the pseudo-spectral Jacobi-Galerkin method is to find

$$\bar{u}_N(x) = \sum_{j=0}^{N} \bar{\xi}_j \phi_j(x)$$

such that

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta},N} = (G_N \bar{u}_N, v_N)_{\omega^{\alpha,\beta},N} + (g, v_N)_{\omega^{\alpha,\beta},N}, \quad \forall v_N \in P_N, \tag{11}$$

$\{\bar{\xi}_j\}_{j=0}^N$ are determined by

$$\sum_{j=0}^{N} \bar{\xi}_j (\phi_i, \phi_j)_{\omega^{\alpha,\beta},N} = \sum_{j=0}^{N} \bar{\xi}_j (\phi_i, G_N \phi_j)_{\omega^{\alpha,\beta},N} + (\phi_i, g)_{\omega^{\alpha,\beta},N}, \tag{12}$$

the matrix form

$$\bar{A}\bar{\xi} = \bar{B}\bar{\xi} + \bar{C}, \tag{13}$$

where

$$
\begin{aligned}
&\bar{\xi} = [\bar{\xi}_0, \bar{\xi}_1, \ldots, \bar{\xi}_N]^T, \quad \bar{A}_{i,j} = (\phi_i, \phi_j)_{\omega^{\alpha,\beta},N}, \\
&\bar{B}_{i,j} = (\phi_i, G\phi_j)_{\omega^{\alpha,\beta},N}, \quad \bar{C}_i = (\phi_i, g)_{\omega^{\alpha,\beta},N}.
\end{aligned}
$$

3. Some useful lemmas

In this section, we will provide some elementary lemmas, which are important for the derivation of the main results in the subsequent section.

First we define the Jacobi orthogonal projection operator $\Pi_N : L^2_\omega \to P_N$ which satisfies

$$(\Pi_N u, v_N)_{\omega^{\alpha,\beta}} = (u, v_N)_{\omega^{\alpha,\beta}}, \quad \forall u \in L^2_{\omega^{\alpha,\beta}}, v_N \in P_N, \tag{14}$$

$$L^2_{\omega^{\alpha,\beta}}(\Lambda) = \{u : u \text{ is measurable and } \|u\|_{\omega^{\alpha,\beta}} < \infty\},$$

$$\|u\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} u^2(x)\omega^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}}.$$

Further, define

$$H^m_{\omega^{\alpha,\beta}}(\Lambda) = \{u : D^k u \in L^2_{\omega^{\alpha,\beta}}(\Lambda), 0 \le k \le m\},$$

equipped with the norm

$$\|u\|_{H^m_{\omega^{\alpha,\beta}}} = \left(\sum_{k=0}^{m} \left\|\frac{d^k u}{dx^k}\right\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

Lemma 3.1 (see [15]). *Suppose that $u \in H^m_{\omega^{\alpha,\beta}}(\Lambda)$ and $m \ge 1$. Then*

$$\|u - \Pi_N u\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}, \tag{15}$$

$$\|u - \Pi_N u\|_\infty \le CN^{\frac{3}{4}-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}, \tag{16}$$

where $|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(\Lambda)}$ denotes the seminorm defined by

$$|u|_{H^{m;N}_{\omega^{\alpha,\beta}}} = \left(\sum_{k=\min(m,N+1)}^{m} \left\|\frac{d^k u}{dx^k}\right\|^2_{\omega^{\alpha,\beta}}\right)^{1/2}.$$

Lemma 3.2 (see [6]). *Suppose that $u \in L^2_{\omega^{\alpha,\beta}}(\Lambda)$. Then*

$$\|\Pi_N u\|_{\omega^{\alpha,\beta}} \le C\|u\|_{\omega^{\alpha,\beta}},$$

$$\|\Pi_N u\|_\infty \le C\|u\|_\infty.$$

Lemma 3.3 (see [2]). *Assume that an $(N+1)$-point Gauss quadrature formula relative to the Jacobi weight is used to integrate the product $u\varphi$, where $u \in H^m(\Lambda)$ with I for some $m \ge 1$ and $\varphi \in \mathcal{P}_N$. Then there exists a constant C independent of N such that*

$$\left|\int_{-1}^{1} u(x)\varphi(x)\omega^{\alpha,\beta}(x)dx - (u,\varphi)_{\omega^{\alpha,\beta},N}\right| \le CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}}\|\varphi\|_{\omega^{\alpha,\beta}}. \tag{17}$$

Lemma 3.4 (see [2])**.** *Assume that $u \in H^m_{\omega^{\alpha,\beta}}(\Lambda)$, $m \geq 1$ $I_N^{\alpha,\beta}u$ denotes the interpolation operator of u based on $(N+1)$-degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$ with $-1 < \alpha, \beta < 1$. Then*

$$\|u - I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leq CN^{-m}|u|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \tag{18a}$$

$$\|u - I_N^{\alpha,\beta}u\|_{\infty} \leq \begin{cases} CN^{\frac{1}{2}-m}\log N|u|_{H^{m,N}_{\omega^c}}, & -1 \leq \alpha, \beta < -\frac{1}{2}, \\ CN^{\nu+1-m}|u|_{H^{m,N}_{\omega^c}}, & \text{otherwise}, \ \nu = \max(\alpha, \beta), \end{cases} \tag{18b}$$

where $\omega^c = \omega^{-\frac{1}{2},-\frac{1}{2}}$ denotes the Chebyshev weight function.

Lemma 3.5 (see [9])**.** *For every bounded function u, there exists a constant C, independent of u such that*

$$\|I_N^{\alpha,\beta}u(x_j)\|_{\omega^{\alpha,\beta}} \leq C\|u\|_{\infty},$$

where $I_N^{\alpha,\beta}u(x) = \sum_{j=0}^N u(x_j)F_j(x)$ is the Lagrange interpolation basis function associated with $(N+1)$-degree Jabobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$.

Lemma 3.6 (see [9])**.** *Assume that $\{F_j(x)\}_{j=0}^N$ are the N-th degree Lagrange basis polynomials associated with the Gauss points of the Jacobi polynomials. Then*

$$\begin{aligned} \|I_N^{\alpha,\beta}\|_{\infty} &\leq \max_{x\in[-1,1]} \sum_{j=0}^N |\ F_j(x)\ | \\ &= \begin{cases} \mathcal{O}(\log N), & -1 < \alpha, \beta \leq -\frac{1}{2}, \\ \mathcal{O}(N^{\nu+\frac{1}{2}}), & \nu = \max(\alpha, \beta), \ \text{otherwise}. \end{cases} \end{aligned} \tag{19}$$

Lemma 3.7 (see [11, 12])**.** *For a nonnegative integer r and $\kappa \in (0, 1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1, 1])$, there exists a polynomial function $\mathcal{T}_N v \in \mathcal{P}_N$ such that*

$$\|v - \mathcal{T}_N v\|_{\infty} \leq C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa}, \tag{20}$$

where $\|\cdot\|_{r,\kappa}$ is the standard norm in $C^{r,\kappa}([-1, 1])$, T_N is a linear operator from $C^{r,\kappa}([-1, 1])$ into $\mathcal{P}_N$, as stated in [11, 12].

Lemma 3.8 (see [5])**.** *Let $\kappa \in (0, 1)$ and let G be defined by*

$$(Gv)(x) = \int_{-1}^x (x - \tau)^{-\mu}K(x, \tau)v(\tau)d\tau.$$

Then, for any function $v \in C([-1, 1])$, there exists a positive constant C such that

$$\frac{|Gv(x') - Gv(x'')|}{|x' - x''|} \leq C \max_{x\in[-1,1]} |v(x)|,$$

under the assumption that $0 < \kappa < 1 - \mu$*, for any* $x', x'' \in [-1, 1]$ *and* $x' \neq x''$*. This implies that*

$$\| Gv \|_{0,\kappa} \leq C \max_{x \in [-1,1]} |v(x)|, \quad 0 < \kappa < 1 - \mu.$$

Lemma 3.9 ([7], Gronwall inequality)**.** *Suppose* $L \geq 0$ *and* $G(x)$ *is a non-negative, locally integrable function defined on* $[-1, 1]$ *satisfying*

$$E(x) \leq G(x) + L \int_{-1}^{x} E(\tau) d\tau.$$

Then there exists a constant C *such that*

$$\|E\|_{L^p(I)} \leq L \|G\|_{L^p(I)}, \quad p \geq 1.$$

Here and below, C denotes a positive constant which is independent of N, and whose particular meaning will become clear by the context in which it arises.

4. Convergence analysis for spectral and pseudo-spectral Jacobi-Galerkin method

According to (5) and the definition of the projection operator Π_N, the spectral Jacobi-Galerkin solution u_N satisfies

$$u_N = \Pi_N G u_N + \Pi_N g. \tag{21}$$

Theorem 4.1. *Suppose that* u_N *is the spectral Jacobi-Galerkin solution determined by* (5)*, if the solution* u *of* (2) *satisfies* $u \in H^{m,N}_{\omega^{\alpha,\beta}}(\Lambda)$*, then we have the following error estimate*

$$\begin{aligned} &\|u - u_N\|_\infty \leq C N^{\frac{3}{4} - m} |u|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \\ &\|u - u_N\|_{\omega^{\alpha,\beta}} \leq C N^{-m} \left(C N^{\frac{3}{4} - \kappa} + 1 \right) |u|_{H^{m,N}_{\omega^{\alpha,\beta}}}, \quad \kappa \in (0, \gamma). \end{aligned} \tag{22}$$

Proof. Subtracting (21) from (3), yields

$$u - u_N = Gu - \Pi_N G u_N + g - \Pi_N g. \tag{23}$$

Set $e = u - u_N$, direct computation shows that

$$\begin{aligned} &Gu - \Pi_N G u_N \\ &= Gu - \Pi_N G u + \Pi_N G(u - u_N) \\ &= Gu - \Pi_N G u + G(u - u_N) - [G(u - u_N) - \Pi_N G(u - u_N)] \\ &= (u - g) - \Pi_N (u - g) + G(u - u_N) - [G(u - u_N) - \Pi_N G(u - u_N)] \\ &= (u - g) - \Pi_N (u - g) + Ge - [Ge - \Pi_N Ge]. \end{aligned} \tag{24}$$

The insertion of (24) into (23) yields

$$
\begin{aligned}
e(x) &= \int_{-1}^{x} (x-s)^{-\gamma} k(x,s) e(s) ds + u - \Pi_N u + [\Pi_N Ge - Ge] \\
&= \int_{-1}^{x} (x-s)^{-\gamma} k(x,s) e(s) ds + I_1 + I_2,
\end{aligned} \tag{25}
$$

where

$$
I_1 = u - \Pi_N u, \quad I_2 = \Pi_N Ge - Ge.
$$

It follows from the Gronwall inequality that

$$
\| e(x) \|_\infty \leq C \sum_{i=1}^{2} \| I_i \|_\infty . \tag{26}
$$

By Lemma 3.1,

$$
\|I_1\|_\infty \leq C N^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}}, \tag{27}
$$

In the virtue of Lemma 3.2, Lemma 3.7 and Lemma 3.8,

$$
\begin{aligned}
\| I_2 \|_\infty &= \| (\Pi_N - I) Ge \|_\infty \\
&= \| (\Pi_N - I)(Ge - \mathcal{T}_N Ge) \|_\infty \\
&\leq \| \Pi_N (Ge - \mathcal{T}_N Ge) \|_\infty + \| Ge - \mathcal{T}_N Ge \|_\infty \\
&\leq C \| Ge - \mathcal{T}_N Ge \|_\infty \\
&\leq C N^{-\kappa} \| Ge \|_{0,\kappa}, \quad \kappa \in (0, 1-\mu) = (0, \gamma) \\
&\leq C N^{-\kappa} \| e \|_\infty .
\end{aligned} \tag{28}
$$

Combing (26), (27) and (28), we obtain, when N is large enough,

$$
\|u - u_N\|_\infty \leq C N^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}} .
$$

Now we investigate the $\| \cdot \|_{\omega^{\alpha,\beta}}$-error estimate. It follows from (25) and Gronwall inequality Lemma 3.9 that

$$
\| e(x) \|_{\omega^{\alpha,\beta}} \leq C \sum_{i=1}^{2} \| I_i \|_{\omega^{\alpha,\beta}} . \tag{29}
$$

Due to Lemma 3.1,

$$
\|I_1\|_{\omega^{\alpha,\beta}} \leq C N^{-m} |u|_{H^{m,N}_{\omega^{\alpha,\beta}}} . \tag{30}
$$

It follows from Lemma 3.2, Lemma 3.7 and Lemma 3.8 that

$$
\begin{aligned}
\| I_2 \|_{\omega^{\alpha,\beta}} &= \| (\Pi_N - I) Ge \|_{\omega^{\alpha,\beta}} \\
&= \| (\Pi_N - I)(Ge - \mathcal{T}_N Ge) \|_{\omega^{\alpha,\beta}} \\
&\leq \| \Pi_N (Ge - \mathcal{T}_N Ge) \|_{\omega^{\alpha,\beta}} + \| Ge - \mathcal{T}_N Ge \|_{\omega^{\alpha,\beta}} \\
&\leq C \| Ge - \mathcal{T}_N Ge \|_\infty \\
&\leq C N^{-\kappa} \| Ge \|_{0,\kappa}
\end{aligned} \tag{31}
$$

$$\leq CN^{-\kappa} \parallel e \parallel_{\infty}, \quad \kappa \in (0, 1-\mu) = (0, \gamma).$$

The combination of (29), (30) and (31) yields,

$$\|u - u_N\|_{\omega^{\alpha,\beta}} \leq CN^{-m}\left(1 + N^{\frac{3}{4}-\kappa}\right)|u|_{H^{m,N}_{\omega^{\alpha,\beta}}},$$

provided N is large enough. Hence, the theorem is proved. □

As $I_N^{\alpha,\beta}$ is the interpolation operator which is based on the $(N+1)$-degree Jacobi-Guass points, in terms of (11), the pseudo-spectral Galerkin solution $\bar{u}_N$ satisfies

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta}} = (I_N^{\alpha,\beta} G_N \bar{u}_N, v_N)_{\omega^{\alpha,\beta}} + (I_N^{\alpha,\beta} g, v_N)_{\omega^{\alpha,\beta}}. \tag{32}$$

Let

$$\begin{aligned} I(x) &= G\bar{u}_N - G_N\bar{u}_N \\ &= \int_{-1}^{1} (1-\theta)^{-\gamma} \tilde{k}(x, s(x,\theta)) u_N(s(x,\theta)) d\theta \\ &\quad - \sum_{k=0}^{N} \tilde{k}(x, s(x,\theta_k)) u_N(s(x,\theta_k)) \omega_k^{-\gamma,0}. \end{aligned} \tag{33}$$

Combing (32) and (33), yields

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta}} = (I_N^{\alpha,\beta} G\bar{u}_N - I_N^{\alpha,\beta} I(x), v_N)_{\omega^{\alpha,\beta}} + (I_N^{\alpha,\beta} g, v_N)_{\omega^{\alpha,\beta}}, \ \forall v_N \in P_N, \tag{34}$$

which gives rise to

$$\bar{u}_N = I_N^{\alpha,\beta} G\bar{u}_N - I_N^{\alpha,\beta} I(x) + I_N^{\alpha,\beta} g. \tag{35}$$

We first consider an auxiliary problem, i.e., we want to find $\hat{u}_N \in P_N$, such that

$$(\hat{u}_N, v_N)_{\omega^{\alpha,\beta},N} = (G\hat{u}_N, v_N)_{\omega^{\alpha,\beta},N} + (g, v_N)_{\omega^{\alpha,\beta},N}, \quad \forall v_N \in P_N. \tag{36}$$

In terms of the definition of $I_N^{\alpha,\beta}$, (36) can be written as

$$(\hat{u}_N, v_N)_{\omega^{\alpha,\beta}} = (I_N^{\alpha,\beta} G\hat{u}_N, v_N)_{\omega^{\alpha,\beta}} + (I_N^{\alpha,\beta} g, v_N)_{\omega^{\alpha,\beta}}, \quad \forall v_N \in P_N, \tag{37}$$

which is equivalent to

$$\hat{u}_N = I_N^{\alpha,\beta} G\hat{u}_N + I_N^{\alpha,\beta} g. \tag{38}$$

Lemma 4.2. *Suppose $\hat{u}_N$ is determined by (38), $-1 \leq \nu = \max(\alpha,\beta) \leq \min(0, \frac{1}{2} - \gamma)$ and $0 < \kappa < 1 - \gamma$, if the solution u of (2) satisfies $u \in H^{m,N}_{\omega^{\alpha,\beta}}(\Lambda)$,* *we have*

$$\|u-\hat u_N\|_\infty \le \begin{cases} CN^{\frac12-m}\log N|u|_{H^{m,N}_{\omega^c}}, & -1\le\nu<-\frac12,\\ CN^{\nu+1-m}|u|_{H^{m,N}_{\omega^c}}, & -\frac12\le\nu<\min(0,\frac12-\gamma).\end{cases}$$

$$\|u-\hat u_N\|_{\omega^{\alpha,\beta}} \le \begin{cases} CN^{-m}\left(|u|_{H^{m,N}_{\omega^c}}+\log N N^{\frac12-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right),\\ -1\le\nu<-\frac12,\\ CN^{-m}\left(|u|_{H^{m,N}_{\omega^c}}+N^{\nu+1-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right),\\ -\frac12\le\nu<\min(0,\gamma-\frac12).\end{cases} \tag{39}$$

Proof. Subtracting (38) from (3), yields

$$u-\hat u_N = Gu - I_N^{\alpha,\beta}Gu_N + g - I_N^{\alpha,\beta}g. \tag{40}$$

Set $\varepsilon = u-\hat u_N$, direct computation shows that

$$\begin{aligned} &Gu - I_N^{\alpha,\beta}G\hat u_N\\ &= Gu - I_N^{\alpha,\beta}Gu + (Gu-G\hat u_N) - \Big[(Gu-G\hat u_N) - I_N^{\alpha,\beta}(Gu-G\hat u_N)\Big]\\ &= (u-g) - I_N^{\alpha,\beta}(u-g) + G\varepsilon - [G\varepsilon - I_N^{\alpha,\beta}G\varepsilon]\\ &= I_N^{\alpha,\beta}u - u + I_N^{\alpha,\beta}g - g + G\varepsilon - [G\varepsilon - I_N^{\alpha,\beta}G\varepsilon]. \end{aligned} \tag{41}$$

The insertion of (41) into (40) yields

$$\varepsilon = \int_{-1}^x (x-s)^{-\gamma}k(x,s)\varepsilon(s)ds + J_1 + J_2, \tag{42}$$

where

$$J_1 = u - I_N^{\alpha,\beta}u, \quad J_2 = I_N^{\alpha,\beta}G\varepsilon - G\varepsilon.$$

It follows from Gronwall inequality that

$$\|\varepsilon(x)\|_\infty \le C\left(\|J_1\|_\infty + \|J_2\|_\infty\right). \tag{43}$$

Due to Lemma 3.4,

$$\|J_1\|_\infty = \|u - I_N^{\alpha,\beta}u\|_\infty \le \begin{cases} CN^{\frac12-m}\log N|u|_{H^{m,N}_{\omega^c}}, & -1\le\nu<-\frac12,\\ CN^{\nu+1-m}|u|_{H^{m,N}_{\omega^c}}, & \text{otherwise.}\end{cases} \tag{44}$$

By virtue of Lemma 3.7, Lamme 3.8 and Lemma 3.6, we have

(45)

$$\begin{aligned} \| J_2 \|_\infty &= \| (I_N^{\alpha,\beta}-I)G\varepsilon \|_\infty\\ &= \| (I_N^{\alpha,\beta}-I)(G\varepsilon - \mathcal{T}_N G\varepsilon) \|_\infty\\ &\le \| I_N^{\alpha,\beta}(G\varepsilon - \mathcal{T}_N G\varepsilon) \|_\infty + \| G\varepsilon - \mathcal{T}_N G\varepsilon \|_\infty\\ &\le \begin{cases} \mathcal{O}(\log N) \| G\varepsilon - \mathcal{T}_N G\varepsilon \|_\infty, & -1<\nu\le-\frac12,\\ \mathcal{O}(N^{\nu+\frac12}) \| G\varepsilon - \mathcal{T}_N G\varepsilon \|_\infty, & \text{otherwise,}\end{cases} \end{aligned}$$

$$\le \begin{cases} C\log N N^{-\kappa} \parallel G\varepsilon \parallel_{0,\kappa}, & -1<\nu\le -\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}-\kappa} \parallel G\varepsilon \parallel_{0,\kappa}, & -\frac{1}{2}\le\nu<\min(0,\frac{1}{2}-\gamma), \end{cases} \quad 0<\kappa<1-\gamma,$$

$$\le \begin{cases} C\log N N^{-\kappa} \parallel \varepsilon \parallel_{\infty}, & -1<\nu\le -\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}-\kappa} \parallel \varepsilon \parallel_{\infty}, & -\frac{1}{2}\le\nu<\min(0,\frac{1}{2}-\gamma), \end{cases} \quad 0<\kappa<1-\gamma.$$

Combing (43), (44) and (45) we obtain, when N is large enough,

$$\|u-\hat{u}_N\|_\infty \le \begin{cases} CN^{\frac{1}{2}-m}\log N|u|_{H^{m,N}_{\omega^c}}, & -1\le\nu<-\frac{1}{2}, \\ CN^{\nu+1-m}|u|_{H^{m,N}_{\omega^c}}, & -\frac{1}{2}\le\nu<\min(0,\frac{1}{2}-\gamma). \end{cases}$$

Now we investigate the $\|\cdot\|_{\omega^{\alpha,\beta}}$-error estimate. It follows from (42) and the Gronwall inequality that

$$\|\varepsilon(x)\|_{\omega^{\alpha,\beta}} \le C\left(\|J_1\|_{\omega^{\alpha,\beta}}+\|J_2\|_{\omega^{\alpha,\beta}}\right). \tag{46}$$

By Lemma 3.4,

$$\|J_1\|_{\omega^{\alpha,\beta}} = \|u-I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}. \tag{47}$$

It follows from Lemma 3.5, Lemma 3.7 and Lemma 3.8 that

$$\begin{aligned} \parallel J_2 \parallel_{\omega^{\alpha,\beta}} &= \parallel (I_N^{\alpha,\beta}-I)G\varepsilon \parallel_{\omega^{\alpha,\beta}} \\ &= \parallel (I_N^{\alpha,\beta}-I)(G\varepsilon-\mathcal{T}_N G\varepsilon) \parallel_{\omega^{\alpha,\beta}} \\ &\le \parallel I_N^{\alpha,\beta}(G\varepsilon-\mathcal{T}_N G\varepsilon) \parallel_{\omega^{\alpha,\beta}} + \parallel G\varepsilon-\mathcal{T}_N G\varepsilon \parallel_{\omega^{\alpha,\beta}} \\ &\le C \parallel G\varepsilon-\mathcal{T}_N G\varepsilon \parallel_\infty \\ &\le CN^{-\kappa} \parallel \varepsilon \parallel_\infty, \quad 1<\kappa<1-\gamma. \end{aligned} \tag{48}$$

Combing (46), (47) and (48) we obtain, when N is large enough,

$$\|u-\hat{u}_N\|_{\omega^{\alpha,\beta}} \le \begin{cases} CN^{-m}\left(|u|_{H^{m,N}_{\omega^c}} + \log N N^{\frac{1}{2}-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right), \\ -1\le\nu<-\frac{1}{2}, \\ CN^{-m}\left(|u|_{H^{m,N}_{\omega^c}} + N^{\nu+1-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right), \\ -\frac{1}{2}\le\nu<\min(0,\gamma-\frac{1}{2}). \end{cases}$$

This completes the proof of the lemma. □

Theorem 4.3. *Suppose that the solution u of (2) satisfies $u\in H^{m,N}_{\omega^{\alpha,\beta}}(\Lambda)$, $-1\le\nu=\max(\alpha,\beta)\le\min(0,\frac{1}{2}-\gamma)$ and $0<\kappa<1-\gamma$, for the pseudo spectral Jacobi-Galerkin solution $\bar{u}_N$, such that (11) holds, we have*

$$\begin{aligned} &\|u-\bar{u}_N\|_\infty \\ &\le \begin{cases} CN^{-m}\log N\left[N^{\frac{1}{2}}|u|_{H^{m,N}_{\omega^c}}+K^*\|u\|_\infty\right], & -1\le\nu<-\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}-m}\left[N^{\frac{1}{2}}|u|_{H^{m,N}_{\omega^c}}+K^*\|u\|_\infty\right], & -\frac{1}{2}\le\nu\le\min(0,\frac{1}{2}-\gamma), \end{cases} \end{aligned} \tag{49}$$

$\|u-\bar{u}_N\|_{\omega^{\alpha,\beta}}$

$$\leq \begin{cases} CN^{-m}\left[\left(1+\log N N^{-\kappa}\right)K^*\|u\|_\infty + |u|_{H^{m,N}_{\omega^c}} + \log N N^{\frac{1}{2}-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right], \\ -1 \leq \nu < -\frac{1}{2}, \\ CN^{-m}\left[\left(1+N^{\nu+\frac{1}{2}-\kappa}\right)K^*\|u\|_\infty + |u|_{H^{m,N}_{\omega^c}} + N^{\nu+1-\kappa}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}\right], \\ -\frac{1}{2} \leq \nu \leq \min(0, \frac{1}{2}-\gamma), \end{cases}$$

where $K^* = \max_{x\in(-1,1)} |k(x, s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}}$.

Proof. Now subtracting (35) from (38) leads to

$$\bar{u}_N - \hat{u}_N = I_N^{\alpha,\beta}(G\bar{u}_N - G\hat{u}_N) - I_N^{\alpha,\beta}I(x), \tag{50}$$

which can be simplified as, by setting $E = \bar{u}_N - \hat{u}_N$

$$\begin{aligned} E &= I_N^{\alpha,\beta}GE - I_N^{\alpha,\beta}I(x) \\ &= GE - GE + I_N^{\alpha,\beta}GE - I_N^{\alpha,\beta}I(x) \\ &= GE + Q_1 - I_N^{\alpha,\beta}I(x) \end{aligned} \tag{51}$$

with $Q_1 = I_N^{\alpha,\beta}GE - GE$. It follows from the Gronwall inequality that

$$\|E\|_\infty = \|Q_1\|_\infty + \|I_N^{\alpha,\beta}I(x)\|_\infty. \tag{52}$$

Similarly to (45), we have

$$\begin{aligned} \|Q_1\|_\infty &= \|I_N^{\alpha,\beta}GE - GE\|_\infty \\ &\leq \begin{cases} CN^{-\kappa}\log N\|E\|_\infty, & -1 \leq \mu < -\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}-\kappa}\|E\|_\infty, & -\frac{1}{2} \leq \nu < \min(0, \frac{1}{2}-\gamma), \end{cases} \end{aligned}$$

with $0 < \kappa < 1-\gamma$.

Using Lemma 3.3 and Lemma 3.6, we have

$$\|I_N^{\alpha,\beta}I(x)\|_\infty \leq \begin{cases} C\log N \max\limits_{x\in(-1,1)} I(x), & -1 \leq \nu < -\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}} \max\limits_{x\in(-1,1)} I(x), & -\frac{1}{2} \leq \nu < 0, \end{cases}$$

$$\leq \begin{cases} CN^{-m}\log N \max_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \|\bar{u}_N\|_{\omega^{-\gamma,0}}, & -1 \leq \nu < -\frac{1}{2}, \\ CN^{-m}N^{\nu+\frac{1}{2}} \max\limits_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \|\bar{u}_N\|_{\omega^{-\gamma,0}}, & -\frac{1}{2} \leq \nu < 0, \end{cases}$$

$$\leq \begin{cases} CN^{-m}\log N \max\limits_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \|\bar{u}_N\|_\infty, & -1 \leq \nu < -\frac{1}{2}, \\ CN^{-m}N^{\nu+\frac{1}{2}} \max\limits_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \|\bar{u}_N\|_\infty, & -\frac{1}{2} \leq \nu < 0, \end{cases}$$

$$\leq \begin{cases} CN^{-m}\log N \max\limits_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \left(\|u\|_\infty + \|E\|_\infty\right), & -1 \leq \nu < -\frac{1}{2}, \\ CN^{-m}N^{\nu+\frac{1}{2}} \max\limits_{x\in(-1,1)} |k(x,s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}} \left(\|u\|_\infty + \|E\|_\infty\right), & -\frac{1}{2} \leq \nu < 0. \end{cases}$$

Set $K^* = \max_{x\in(-1,1)} |k(x, s(x,\cdot))|_{H^{m;N}_{\omega^{-\gamma,0}}(\Lambda)}$, we now obtain the estimate E by using (52)

$$\|E\|_\infty \le \begin{cases} CN^{-m}\log N K^*\|u\|_\infty, & -1 \le \nu < -\frac{1}{2}, \\ CN^{\nu+\frac{1}{2}-m}K^*\|u\|_\infty, & -\frac{1}{2} \le \nu < \min(0, \frac{1}{2}-\gamma). \end{cases} \tag{53}$$

Next, we will give the error estimates in $\|\cdot\|_{\omega^{\alpha,\beta}}$. It follows from (51) and the Gronwall inequality that

$$\|E\|_{\omega^{\alpha,\beta}} = \|Q_1\|_{\omega^{\alpha,\beta}} + \|I_N^{\alpha,\beta} I(x)\|_{\omega^{\alpha,\beta}}. \tag{54}$$

$\|Q_1\|_{\omega^{\alpha,\beta}}$ can be established in a similar way as (48),

$$\|Q_1\|_{\omega^{\alpha,\beta}} = \|I_N^{\alpha,\beta}GE - GE\|_{\omega^{\alpha,\beta}} \le CN^{-\kappa}\|E\|_\infty, \quad 0 < \kappa < 1-\gamma.$$

Using Lemma 3.3 and Lemma 3.5, we have

$$\begin{aligned} \|I_N^{\alpha,\beta} I(x)\|_{\omega^{\alpha,\beta}} \le C\|I(x)\|_{\omega^{\alpha,\beta}} &\le CN^{-m}K^*\|\hat{u}_N\|_{\omega^{-\gamma,0}} \\ &\le CN^{-m}K^*\left(\|u\|_\infty + \|E\|_\infty\right). \end{aligned}$$

By the convergence result in (53) ($m = 1$), we have

$$\|E\|_\infty \le \|u\|_\infty$$

for sufficiently large N. So that

$$\|I_N^{\alpha,\beta} I(x)\|_{\omega^{\alpha,\beta}} \le CN^{-m}K^*\|u\|_\infty.$$

We obtain, when N is large enough,

(55)

$$\|E\|_{\omega^{\alpha,\beta}} \le \begin{cases} CN^{-m}\left(1+\log N N^{-\kappa}\right)K^*\|u\|_\infty, & -1 \le \nu < -\frac{1}{2}, \\ CN^{-m}\left(1+N^{\nu+\frac{1}{2}-\kappa}\right)K^*\|u\|_\infty, & -\frac{1}{2} \le \nu < \min(0, \frac{1}{2}-\gamma). \end{cases}$$

Finally, it follows from triangular inequality,

$$\begin{aligned} \|u - \bar{u}_N\|_\infty &\le \|u - \hat{u}_N\|_\infty + \|\hat{u}_N - \bar{u}_N\|_\infty, \\ \|u - \bar{u}_N\|_{\omega^{\alpha,\beta}} &\le \|u - \hat{u}_N\|_{\omega^{\alpha,\beta}} + \|\hat{u}_N - \bar{u}_N\|_{\omega^{\alpha,\beta}}, \end{aligned}$$

as well as Lemma 4.2, (53) and (55), we can obtain the desired estimated (49). □

5. Numerical experiments

We give numerical examples to confirm our analysis. To examine the accuracy of the results, $\|\cdot\|_\infty$ and $\|\cdot\|_{\omega^{\alpha,\beta}}$ errors are employed to assess the efficiency of the method. All the calculations are supported by the software Matlab.

Example 5.1. We consider the following the linear weakly singular Volterra integral equation (1) with

$$y(t) = f(t) - \int_0^t (t-\tau)^{-\gamma} y(\tau)d\tau, \quad 0 \le t \le 6, \quad 0 < \gamma < 1,$$

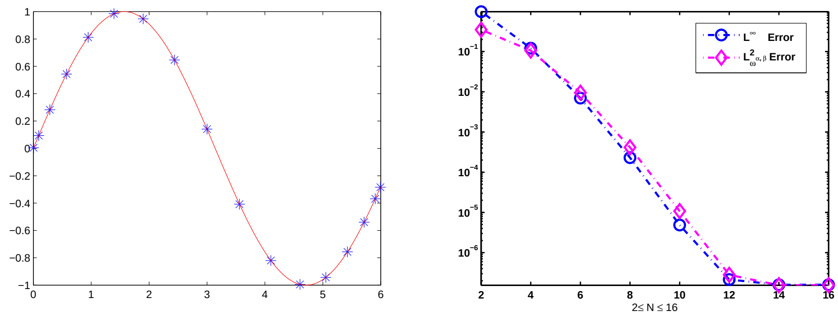

FIGURE 1. Example 5.1: Comparison between approximate solution of pseudo-spectral Jacobi-Galerkin method and exact solution of $y(t) = t^{-\gamma}\sin(t)$ with $\gamma = 0.8$ (left). The errors versus the number of collocation points in L^∞ and weighted L^2 norms (right).

with

$$f(t) = \frac{\sin(t)}{t^\gamma} + \pi^{\frac{1}{2}} t^{\frac{1}{2}-\gamma}\Gamma(1-\gamma)B(\frac{1}{2}-\gamma, \frac{t}{2})\sin(\frac{t}{2}),$$

where $B(\cdot,\cdot)$ is the Bessel function defined by

$$B(\theta,\vartheta) = \left(\frac{\vartheta}{2}\right)^\theta \sum_{k=0}^{\infty} \frac{\left(-\frac{\vartheta^2}{4}\right)^k}{k!\Gamma(\theta+k+1)},$$

The corresponding exact solution is given by $y(t) = t^{-\gamma}\sin(t)$.

We have reported the obtained numerical results of pseudo-spectral Jacobi-Galerkin for $N = 16$ and $\gamma = 0.8$ in Figure 1 (left). We can see that the numerical result of pseudo-spectral Jacobi-Galerkin approximation solutions are in good agreement with exact solution $y(t) = t^{-\gamma}\sin(t)$. Figure 1 (right) illustrates L^∞ and weighted L^2_ω errors of pseudo-spectral Jacobi-Galerkin method versus the number N of the steps. Clearly, these figures show the exponential rate of convergence predicted by the proposed method.

In practice, many Volterra equations are usually nonlinear. However, the nonlinearity adds rather little to the difficulty of obtaining a numerical solution. The methods described above remain applicable. Below we will provide a numerical example using the spectral technique proposed in this work.

Example 5.2. Consider the following the nonlinear weakly singular Volterra integral equation (1) with

$$y(t) = f(t) - \int_0^t (t-\tau)^{-\gamma}\tan(y(t))d\tau, \quad 0 \le t \le 2, \quad 0 < \gamma < 1,$$

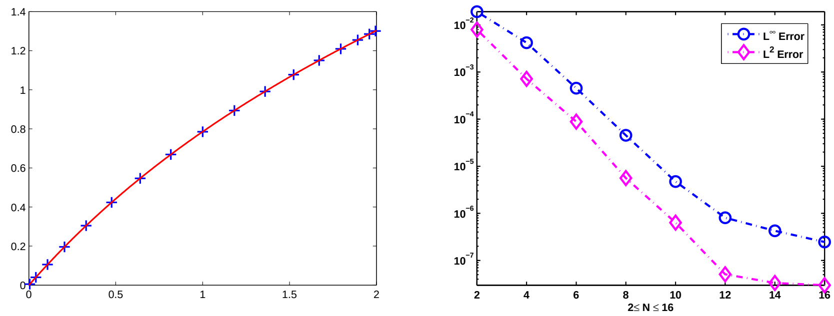

FIGURE 2. Example 5.2: Comparison between approximate solution of pseudo-spectral Jacobi-Galerkin method and exact solution of $y(t) = a\tan(t^{1-\gamma})$ with $\gamma = 0.4$ (left). The errors versus the number of collocation points in L^∞ and weighted L^2 norms (right).

with

$$f(t) = t^\gamma \arctan(t^{1-\gamma}) + \sqrt{\pi} t^\gamma \left(\frac{t}{2}\right)^{2-2\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(\frac{3}{2}-\gamma)}.$$

This example has a smooth solution $y(t) = \arctan(t^{1-\gamma})$.

This is a nonlinear problem. The numerical scheme leads to a nonlinear system, and a proper solver for the nonlinear system (e.g., Newton method) should be used. Figure 2 (left) presents the approximate and exact solution with $\gamma = 0.4$, which are found in excellent agreement. Next, Figure 2 (right) illustrates the L^∞ and weighted L^2_ω errors of the pseudo-spectral-Galerkin method. These results indicate that the spectral accuracy is obtained for this problem.

6. Conclusions and future work

This work has been concerned with the spectral and pseudo-spectral Jacobi-Galerkin analysis of the Volterra integral equations with weakly singular kernel. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of spectral approximations decay exponentially in both infinity and weighted norms, which is a desired feature for a spectral method.

Although in this work our convergence theory does not cover the nonlinear case, the methods described above remain applicable, it will be possible to extend the results of this paper to nonlinear case which will be the subject of our future work.

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