ON LORENTZ GCR SURFACES IN MINKOWSKI 3-SPACE

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Abstract. A generalized constant ratio surface (GCR surface) is defined by the property that the tangential component of the position vector is a principal direction at each point on the surface, see [8] for details. In this paper, by solving some differential equations, a complete classification of Lorentz GCR surfaces in the three-dimensional Minkowski space is presented. Moreover, it turns out that a flat Lorentz GCR surface is an open part of a cylinder, apart from a plane and a CMC Lorentz GCR surface is a surface of revolution.

1. Introduction

The concept of *constant slope surfaces* is introduced by Munteanu in [15], which are the surfaces whose normal makes a constant angle with the position vector. In particular, Munteanu gave a nice characterization of constant slope surfaces in Euclidean 3-space. Motivated by Munteanu's work, constant slope surfaces in Minkowski 3-space $\mathbb{R}_1^3$ were classified by the authors in [10, 11].

On the other hand, B. Y. Chen introduced in [3] the concept of *constant ratio submanifolds*, which is defined by the property that the ratio of the length of the tangential and normal components of its position vector is constant. Chen [3] also obtained the classification of constant ratio hypersurfaces in $\mathbb{R}^{n+1}$. Concerning the constant ratio surfaces and related concepts, refer also to [1, 13].

Note that a remarkable property of constant slope surfaces is that the tangential component of the position vector is a principal direction. Hence, as a generalization of constant slope surfaces, the first author and Munteanu [8] proposed to study the surfaces with the property that the tangential component of the position vector remains a principal direction in Euclidean space $\mathbb{R}^3$. They called theses surfaces as generalized constant ratio surfaces (in short GCR surfaces) in order to point out the connection with constant ratio surfaces defined by Chen [3].

Another closely related concept is the notion of constant angle surfaces, which is a class of surfaces whose tangent planes make a constant angle with a

fixed vector field of the ambient space. A natural extension of constant angle surfaces is the class of surfaces with a canonical principal direction, see the definitions in [4, 7]. In recent years, there are many classification results in different ambient spaces concerning the constant angle surfaces and surfaces with canonical principal directions, for instance, see [5, 6, 9, 12, 16, 17, 19].

Hence, to investigate the geometry and classification of GCR surfaces in Minkowski space is interesting and important. In this paper, we consider Lorentz GCR surfaces in Minkowski space $\mathbb{L}^3$. Precisely, we completely classify Lorentz GCR surfaces in $\mathbb{L}^3$, see Theorems 3.3 and 3.6. By the classification results, we find that some interesting Lorentz GCR surfaces of revolution in the 3-dimensional Minkowski space. At last, we give the characterizations of the flat Lorentz GCR surfaces and CMC Lorentz GCR surfaces in Minkowski 3-space.

2. Preliminaries

We denote by $\mathbb{L}^3$ the 3-dimensional Minkowski space with the Lorentz metric

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are vectors in $\mathbb{L}^3$.

Let $x : M \to \mathbb{L}^3$ be an isometric immersion of a spacelike or Lorentz surface M into $\mathbb{L}^3$. Denote the Levi-Civita connections of M and $\mathbb{L}^3$ by ∇ and $\tilde{\nabla}$, respectively. Let X and Y denote vector fields tangent to M and let ξ be a normal vector field. The Gauss and Weingarten formulae are given, respectively, by (cf. [2, 18])

$$(2.1) \qquad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \qquad \tilde{\nabla}_X \xi = -A_\xi X,$$

where h, A are the second fundamental form and the shape operator. It is well known that the second fundamental form h and the shape operator A are related by

$$(2.3) \qquad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are given respectively by

$$\langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle,$$

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

where R is the curvature tensor of the Levi-Civita connection on M.

The following fact is well-known:

An vector v in $\mathbb{L}^3$ has one of three Lorentz causal characters; it could be spacelike if $\langle v, v \rangle > 0$ or $v = 0$, timelike if $\langle v, v \rangle < 0$ and null (lightlike) if $\langle v, v \rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{L}^3$ is called spacelike, timelike, or null (lightlike), if all of its velocity vectors α' are respectively spacelike, timelike, or null (lightlike), for every $s \in I \subset \mathbb{R}$. We consider the timelike orientation as follows: a timelike vector $v = (x_1, x_2, x_3)$

is said to be future pointing (resp. past pointing) if and only if $x_3 > 0$ (resp. $x_3 < 0$). The pseudo-norm of the vector $x \in \mathbb{L}^3$ can be defined by

$$\|v\| = \sqrt{|\langle v, v \rangle|}.$$

The isometry group of $\mathbb{L}^3$ is the semi-direct product of the translations group and the orthogonal Lorentz group $O(1,2)$. With respect to the orthogonal group, there are three one-parameter subgroups of isometries of $\mathbb{L}^3$, that fix an axis (line), depending on the causal character of the axis. If the axis is spacelike it is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R} \text{ (hyperbolic group)}.$$

If the axis is timelike it is given by

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 < t < 2\pi \text{ (elliptic group)},$$

and if the axis is lightlike it is given by

$$\begin{pmatrix} \frac{t^2}{2} & 1 - \frac{t^2}{2} & t \\ t & -t & 1 \\ 1 + \frac{t^2}{2} & -\frac{t^2}{2} & t \end{pmatrix}, \quad t \in \mathbb{R} \text{ (parabolic group)}.$$

Due to the fact that the surfaces (spacelike or timelike) of revolution in $\mathbb{L}^3$ must be invariant by the action of one of the one-parameter subgroups of isometries cited above, one can obtain the parametrizations for the surfaces of revolution ([14]). By choosing the profile curve $\gamma(s)$ in the xz-plane, parametrized by $\gamma(s) = (f(s), 0, g(s))$, we have the following parametrizations if the axis is spacelike,

$$x_1(s, t) = (f(s), g(s) \cosh t, g(s) \sinh t), \quad g(s) \neq 0,$$

or

$$x_2(s, t) = f(s), g(s) \sinh t, g(s) \cosh t), \quad g(s) \neq 0$$

and if the axis is timelike,

$$x_2(s, t) = (f(s) \cos t, f(s) \sin t, g(s)), \quad f(s) \neq 0.$$

If the axis is lightlike, the profile curve is given by $\gamma(s) = (f(s), g(s), 0)$, where s is the arc length parameter, and the parametrization is given by

$$x_3(s, t) = \left(\big(f(s) - g(s)\big) \frac{t^2}{2} + g(s), \big(f(s) - g(s)\big) t, \big(f(s) - g(s)\big) \frac{t^2}{2} + f(s) \right).$$

We recall the definition of the angle between two vectors in Minkowski space [10]:

Definition 2.1. Let u and v be spacelike vectors in $\mathbb{L}^3$ that span a spacelike vector subspace. Then we have $|g(u,v)| \leq \|u\|\|v\|$ and hence, there is a unique real number $\phi \in [0, \pi/2]$ such that

$$|g(u,v)| = \|u\|\|v\| \cos \phi.$$

The real number ϕ is called the Lorentz spacelike angle between u and v.

Definition 2.2. Let u and v be spacelike vectors in $\mathbb{L}^3$ that span a timelike vector subspace. Then we have $|g(u,v)| > \|u\|\|v\|$ and hence, there is a unique positive real number ϕ such that

$$|g(u,v)| = \|u\|\|v\| \cosh \phi.$$

The real number ϕ is called the Lorentz timelike angle between u and v.

Definition 2.3. Let u and v be future pointing (past pointing) timelike vectors in $\mathbb{L}^3$. Then there is a unique non-negative real number ϕ such that

$$|g(u,v)| = \|u\|\|v\| \cosh \phi.$$

The real number ϕ is called the Lorentz timelike angle between u and v.

Definition 2.4. Let u be a spacelike vector and v a future pointing timelike vector in $\mathbb{L}^3$. Then there is a unique non-negative real number ϕ such that

$$|g(u,v)| = \|u\|\|v\| \sinh \phi.$$

The real number ϕ is called the Lorentz timelike angle between u and v.

3. Classification of Lorentz GCR surfaces

Let M be an orientable Lorentz surface in the 3-dimensional Minkowski space $\mathbb{L}^3$. For a generic point p on M immersed in $\mathbb{L}^3 \setminus \{0\}$, denote x by its position vector. The angle between two vectors in $\mathbb{L}^3$ is given by Definitions 2.1-2.4.

In the following, we will study the Lorentz surfaces M in $\mathbb{L}^3$ whose tangential component of the position vector is a principal direction on the surfaces, that is the generalized constant ratio surfaces, which are called GCR surface in short. For the Lorentz surface M in $\mathbb{L}^3$, since the normal vector ξ is always spacelike, we may assume ξ is unitary, i.e., $\langle \xi, \xi \rangle = 1$. Since one cannot define the angle between a lightlike vector and another casual vector in $\mathbb{L}^3$, we assume that x does not lie in the light cone. We distinguish the following two cases.

§1. Lorentz GCR surfaces lying in the spacelike cone

In this section we deal with the case: the immersion x lies in the spacelike cone. Then, there is a nonzero function μ such that $\langle x, x \rangle = \mu^2$.

We now introduce constants $c(\theta)$, $s(\theta)$ and ε, which are defined by

$$c(\theta) = \cos \theta, \quad s(\theta) = \sin \theta, \quad \varepsilon = 1$$

if the position vector x and the normal vector ξ span a spacelike vector space and by

$$c(\theta) = \cosh\theta, \quad s(\theta) = \sinh\theta, \quad \varepsilon = -1$$

if the position vector x and the normal vector ξ span a timelike vector space. It is easy to check that

$$c^2(\theta) + \varepsilon s^2(\theta) = 1,$$

and

$$X(c(\theta)) = -\varepsilon s(\theta)X(\theta),$$
$$X(s(\theta)) = c(\theta)X(\theta),$$

for any tangent vector field X.

Since the normal vector field ξ is spacelike, by Definitions 2.1 and 2.2 we can decompose x in the form

$$(3.1) \qquad \frac{x}{\mu} = U + c(\theta)\xi,$$

where U is the projection of x on the tangent plane of M. Hence we have

$$\langle U, U \rangle = \varepsilon s^2(\theta).$$

Now assume $\theta \neq 0$. Let $e_1 = \frac{U}{s(\theta)}$, which defines a unit (spacelike or time-like)tangent vector field on M and choose e_2 as a unit (timelike or spacelike) vector field on M orthogonal to e_1. In such a way, $\{e_1, e_2, \xi\}$ defines an oriented unit orthonormal basis for every point on M. The position vector x can be rewritten in the following form

$$(3.2) \qquad \frac{x}{\mu} = s(\theta)e_1 + c(\theta)\xi.$$

Since $\tilde{\nabla}_X x = X$ holds for every tangent vector field X to M, it follows from (3.2) that

$$(3.3) \qquad X = X(\mu)(s(\theta)e_1 + c(\theta)\xi) + \mu s(\theta)\tilde{\nabla}_X e_1 + \mu c(\theta)\tilde{\nabla}_X \xi.$$

Applying the Gauss and Weingarten formulae, and identifying the tangent and the normal parts give

$$(3.4) \qquad X = X(\mu)s(\theta)e_1 + \mu X(s(\theta))e_1 + \mu s(\theta)\nabla_X e_1 - \mu c(\theta)AX,$$

$$(3.5) \qquad c(\theta)X(\mu)\xi + \mu X(c(\theta))\xi + \mu s(\theta)h(X, e_1) = 0.$$

Taking the derivative with respect to tangent vector field X on both sides of the equality $\langle x, x \rangle = \mu^2$ yields

$$(3.6) \qquad \mu X(\mu) = \langle X, x \rangle.$$

Combining this with (3.2) gives

$$(3.7) \qquad X(\mu) = s(\theta)\langle X, e_1 \rangle.$$

Let $X = e_1$ and then $X = e_2$. The previous equation reduces to

$$(3.8) \qquad e_1(\mu) = \varepsilon s(\theta), \qquad e_2(\mu) = 0.$$

Substituting (3.7) into (3.5), and furthermore using (2.3) yield

$$(3.9) \qquad Ae_1 = (-\frac{c(\theta)}{\mu} + e_1(\theta))e_1 - e_2(\theta)e_2,$$

which implies the following result.

Proposition 3.1. *The projection vector U of the reduced position vector x/μ is a principal direction if and only if $e_2(\theta) = 0$ holds.*

From now on, we assume that M is a Lorentz GCR surface, namely the surface satisfying the condition $e_2(\theta) = 0$ and $e_1(\theta) \neq 0$. Therefore, there exists a smooth function ρ defined on M, such that

$$(3.10) \qquad\qquad\qquad Ae_2 = \rho e_2,$$

and (3.9) becomes

$$(3.11) \qquad\qquad Ae_1 = (-\frac{c(\theta)}{\mu} + e_1(\theta))e_1.$$

It is easy to calculate the Levi-Civita connection ∇.

Proposition 3.2. *The Levi-Civita connection ∇ of M is given by*

$$(3.12) \quad \nabla_{e_1}e_1 = \nabla_{e_1}e_2 = 0, \ \nabla_{e_2}e_1 = \frac{1 + \mu\rho c(\theta)}{\mu s(\theta)}e_2, \ \nabla_{e_2}e_2 = \frac{1 + \mu\rho c(\theta)}{\mu s(\theta)}e_1.$$

By (3.9)-(3.11), we derive from the Codazzi equation that

$$(3.13) \qquad e_1(\rho) + \big(\rho + \frac{c(\theta)}{\mu} - e_1(\theta)\big)\frac{1 + \rho\mu c(\theta)}{\mu s(\theta)} = 0.$$

We now state our main theorem in this subsection.

Theorem 3.3. *Let $x : M \to \mathbb{L}^3$ be a Lorentz surface immersed in the 3-dimensional Minkowski space $\mathbb{L}^3$. If the immersion x lies in the spacelike cone, then M is a GCR surface if and only if one of the following eight statements holds*:

    (1) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = s\big(\cos(\int \frac{\cot\theta}{s}ds), \sin(\int \frac{\cot\theta}{s}ds)\cosh t, \sin(\int \frac{\cot\theta}{s}ds)\sinh t\big),$$

       where θ is an angle function concerning the variable s;
    (2) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = s\big(\sin(\int \frac{\cot\theta}{s}ds), \cos(\int \frac{\cot\theta}{s}ds)\cosh t, \cos(\int \frac{\cot\theta}{s}ds)\sinh t\big),$$

       where θ is an angle function concerning the variable s;

(3) *the immersion $x(M)$ is a surface given by*

$$x(s,t) = s\left(\cos\left(\int \frac{\cot\theta}{s} ds\right) f(t) + \sin\left(\int \frac{\cot\theta}{s} ds\right) f(t) \times f'(t)\right),$$

where f is a timelike unit speed curve on $\mathbb{S}_1^2$ satisfying $(f, f', f'') \neq 0$ and θ is an angle function concerning the variable s;

(4) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = \frac{s}{2}\left(-e^{-\int \frac{1}{s}\coth\theta ds} + e^{\int \frac{1}{s}\coth\theta ds}(t^2 - 1), 2e^{\int \frac{1}{s}\coth\theta ds}t,\right.$$
$$\left.-e^{-\int \frac{1}{s}\coth\theta ds} + e^{\int \frac{1}{s}\coth\theta ds}(t^2 + 1)\right),$$

where θ is a positive angle function concerning the variable s;

(5) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = \frac{s}{2}\left(-e^{\frac{1}{s}\coth\theta ds} + e^{-\int \frac{1}{s}\coth\theta ds}(t^2 - 1), 2e^{-\int \frac{1}{s}\coth\theta ds}t,\right.$$
$$\left.-e^{\frac{1}{s}\coth\theta ds} + e^{-\int \frac{1}{s}\coth\theta ds}(t^2 + 1)\right),$$

where θ is a positive angle function concerning the variable s;

(6) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = s\left(\cosh\left(\int \frac{1}{s}\coth\theta ds\right)\cos t, \cosh\left(\int \frac{1}{s}\coth\theta ds\right)\sin t,\right.$$
$$\left.\sinh\left(\int \frac{1}{s}\coth\theta ds\right)\right),$$

where θ is a positive angle function concerning variable s;

(7) *the immersion $x(M)$ is a surface of revolution given by*

$$x(s,t) = s\left(\cosh\left(\int \frac{1}{s}\coth\theta ds\right), \sinh\left(\int \frac{1}{s}\coth\theta ds\right)\sinh t,\right.$$
$$\left.\sinh\left(\int \frac{1}{s}\coth\theta ds\right)\cosh t\right),$$

where θ is a positive angle function concerning the variable s;

(8) *the immersion $x(M)$ is given by*

$$x(s,t) = s\left(\cosh\left(\int \frac{1}{s}\coth\theta ds\right) f(t) + \sinh\left(\int \frac{1}{s}\coth\theta ds\right) f(t) \times f'(t)\right),$$

where f is a unit speed curve on $\mathbb{S}_1^2$ satisfying $(f, f', f'') \neq 0$ and θ is a positive angle function concerning the variable s.

Remark 3.4. If we choose θ to be a constant in Theorem 3.3, after a suitable translation of the variable s, the GCR surfaces are just the constant slope surfaces of Theorem 3.3 studied in [10].

Remark 3.5. Note that the profile curves of the surfaces of revolution (1) and (2) in Theorem 3.3 are given, respectively, by

$$\gamma(s) = s\big(\cos(\int \frac{\cot\theta}{s}ds), 0, \sin(\int \frac{\cot\theta}{s}ds)\big),$$

$$\gamma(s) = s\big(\sin(\int \frac{\cot\theta}{s}ds), 0, \cos(\int \frac{\cot\theta}{s}ds)\big),$$

the profile curves of the surfaces of revolution (4) and (5) in Theorem 3.3 are given, respectively, by

$$\gamma(s) = \frac{s}{2}\big(-e^{-\int \frac{\coth\theta}{s}ds} + e^{\int \frac{\coth\theta}{s}ds}, -e^{-\int \frac{\coth\theta}{s}ds} - e^{\int \frac{\coth\theta}{s}ds}, 0\big),$$

$$\gamma(s) = \frac{s}{2}\big(-e^{\int \frac{\coth\theta}{s}ds} + e^{-\int \frac{\coth\theta}{s}ds}, -e^{\int \frac{\coth\theta}{s}ds} - e^{-\int \frac{\coth\theta}{s}ds}, 0\big),$$

and the profile curves of the surfaces of revolution (6) and (7) in Theorem 3.3 are given by

$$\gamma(s) = s\big(\cosh(\int \frac{1}{s}\coth\theta ds), 0, \sinh(\int \frac{1}{s}\coth\theta ds)\big).$$

Proof of Theorem 3.3. Let $x(M)$ be a Lorentz GCR surface lying in the space-like cone in $\mathbb{L}^3$. From Proposition 3.2, we have

$$[e_1, e_2] = -\frac{1+\mu\rho c(\theta)}{\mu s(\theta)}e_2.$$

Hence there exists a nonzero smooth function λ on M such that $[\frac{1}{s(\theta)}e_1, \lambda e_2] = 0$ if and only if λ satisfies

(3.14) $$e_1(\lambda) - \lambda\frac{1+\mu\rho c(\theta)}{\mu s(\theta)} = 0.$$

Therefore, we may choose local coordinates (s, t) on M such that

(3.15) $$\frac{\partial}{\partial s} = \frac{1}{s(\theta)}e_1, \quad \frac{\partial}{\partial t} = \lambda e_2.$$

Then the metric tensor of M is given by

(3.16) $$g = \frac{\varepsilon}{s^2(\theta)}ds^2 - \varepsilon\lambda^2 dt^2.$$

In this case, the angle θ is a function depending only on the variable s. Consequently, (3.8) and (3.13) become

(3.17) $$\mu_s = \varepsilon, \qquad \mu_t = 0,$$

(3.18) $$\lambda_s = \lambda\frac{1+\mu\rho c(\theta)}{\mu s^2(\theta)}.$$

Solving (3.17) gives $\mu = \varepsilon s + c_0$. After making a translation of the variable s, we choose $c_0 = 0$.

Using the remark as above, expression (3.2) becomes

(3.19) $$x = \varepsilon s^2(\theta)sx_s + \varepsilon c(\theta)s\xi.$$

Furthermore, it follows from (3.9) and the Weingarten formula (2.2) that

$$(3.20) \qquad (\xi)_s = \big(\frac{\varepsilon c(\theta)}{s} - s(\theta)\theta'\big)\frac{\partial}{\partial s}.$$

Combining (3.19) with (3.20) yields

$$(3.21) \quad s^2(\theta)c(\theta)s^2 x_{ss} + (s(\theta)\theta's^2 - s^2(\theta)c(\theta)s)x_s + \varepsilon(c(\theta) - s(\theta)\theta's)x = 0.$$

We consider the following two cases:

Case A: $c(\theta) = \cos\theta$, $s(\theta) = \sin\theta$ and $\varepsilon = 1$.

In this case, equation (3.21) becomes

$$(3.22) \qquad \sin^2(\theta)\cos(\theta)s^2 x_{ss} + (\sin(\theta)\theta's^2 - \sin^2(\theta)\cos(\theta)s)x_s$$
$$+ (\cos(\theta) - \sin(\theta)\theta's)x = 0.$$

By putting $\Phi(s,t) = x/s$, the previous equation turns into

$$(3.23) \qquad s^2 \sin^2\theta \cos\theta \Phi_{ss} + (s\sin^2\theta\cos\theta + s^2\sin\theta\theta')\Phi_s + \cos^3\theta\Phi = 0.$$

Putting

$$(3.24) \qquad u(s) = \int \frac{\cot\theta}{s}ds,$$

then the differential equation (3.23) can be rewritten as

$$\Phi_{uu} + \Phi = 0.$$

Solving this equation, the position vector x can be expressed as

$$(3.25) \qquad x(s,t) = s\big(C_1(t)\cos u + C_2(t)\sin u\big),$$

where both $C_1(t)$ and $C_2(t)$ are vector-valued functions depending only on t.

Consequently, by using the condition $\langle x, x\rangle = s^2$ we get

$$(3.26) \qquad \langle C_1(t), C_1(t)\rangle = \langle C_2(t), C_2(t)\rangle = 1, \quad \langle C_1(t), C_2(t)\rangle = 0.$$

Moreover, from (3.15) and $\mu = s$, equation (3.13) becomes

$$(3.27) \qquad s^2\sin^2\theta\rho_s + (s\cos\theta\rho + 1)(s\rho + \cos\theta - \sin\theta\theta's) = 0.$$

If we put

$$(3.28) \qquad \beta = \frac{\rho s + \cos\theta}{\sin\theta},$$

then the previous equation turns into

$$(3.29) \qquad s\sin\theta\beta_s = -\cos\theta(\beta^2 + 1).$$

By integration, the solution of equation (3.29) is given by

$$(3.30) \qquad \beta = -\tan\Big(\int \frac{1}{s}\cot\theta ds + \phi(t)\Big),$$

where $\phi(t)$ is a smooth function. Combining this with (3.28), we get

$$(3.31) \qquad \rho = \frac{1}{s}\Big(-\sin\theta\tan(\int \frac{1}{s}\cot\theta ds + \phi(t)) - \cos\theta\Big).$$

Substituting these into (3.18) gives

$$(3.32) \qquad \frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{1}{s} \cot \theta \tan \Big(\int \frac{1}{s} \cot \theta ds + \varphi(t) \Big).$$

Solving this differential equation, we have

$$(3.33) \qquad \lambda = s \psi(t) \cos \Big(\int \frac{1}{s} \cot \theta ds + \varphi(t) \Big),$$

where ψ is a nonzero smooth function defined on M. In this case, the metric g is given by

$$(3.34) \qquad g = \frac{1}{\sin^2 \theta} ds^2 - s^2 \cos^2 \Big(\int \frac{1}{s} \cot \theta ds + \varphi(t) \Big) \psi^2(t) dt^2,$$

for some functions $\varphi(t)$ and $\psi(t)$. It follows from (3.25) and (3.34) that

$$(3.35) \qquad \langle C_1'(t), C_1'(t) \rangle = -\psi^2(t)(\cos \varphi(t))^2,$$
$$\langle C_2'(t), C_2'(t) \rangle = -\psi^2(t)(\sin \varphi(t))^2,$$

$$(3.36) \qquad \langle C_1'(t), C_2'(t) \rangle = \frac{1}{2} \psi^2(t) \sin(2\varphi(t)),$$
$$\langle C_1'(t), C_2(t) \rangle = \langle C_2'(t), C_1(t) \rangle = 0.$$

If $\cos \varphi(t) = 0$, then (3.35) and (3.36) yield

$$(3.37) \qquad \langle C_1'(t), C_1'(t) \rangle = \langle C_1'(t), C_2'(t) \rangle = 0, \ \langle C_2'(t), C_2'(t) \rangle = -\psi^2(t).$$

Thus, from (3.16), (3.36) and (3.37) we obtain $C_1'(t) = 0$. This shows that C_1 is a constant spacelike vector in $\mathbb{L}^3$. Without loss of generality, we assume that $C_1 = (1, 0, 0)$. After making a change of the t-coordinate, we can assume $\langle C_2'(t), C_2'(t) \rangle = -1$. Combining these with (3.16) gives

$$C_2(t) = (0, \cosh t, \sinh t).$$

Substituting these into (3.25) and by (3.24), we obtain case (1) in Theorem 3.3.

If $\sin \varphi(t) = 0$, then (3.35) and (3.36) yield

$$(3.38) \qquad \langle C_2'(t), C_2'(t) \rangle = \langle C_1'(t), C_2'(t) \rangle = 0, \ \langle C_2'(t), C_2'(t) \rangle = -\psi^2(t).$$

Thus, from (3.16), (3.36) and (3.37) we obtain $C_2'(t) = 0$. This shows that C_2 is a constant spacelike vector in $\mathbb{L}^3$. Without loss of generality, we assume that $C_2 = (1, 0, 0)$. After making a change of the t-coordinate, we can assume $\langle C_1'(t), C_1'(t) \rangle = -1$. Combining these with (3.16) gives

$$C_1(t) = (0, \cosh t, \sinh t).$$

Substituting these into (3.25) and by (3.24), we obtain case (2) in Theorem 3.3.

Suppose $\sin \varphi(t) \cos \varphi(t) \neq 0$. After a change of the t-coordinate, we may assume $\langle C_1'(t), C_1'(t) \rangle = -1$. Since $\langle C_2(t), C_1(t) \rangle = \langle C_2(t), C_1'(t) \rangle = 0$, it follows that $C_2 = \pm C_1 \times C_1'$. We assume $C_2 = C_1 \times C_1'$. Since $\langle C_1'(t), C_2'(t) \rangle \neq 0$,

we have $\langle C_1'(t), C_1(t) \times C_1''(t) \rangle \neq 0$. Put $f(t) = C_1(t)$. Combining these with (3.25) yields case (3) in Theorem 3.3.

Case B: $c(\theta) = \cosh\theta$, $s(\theta) = \sinh\theta$ and $\varepsilon = -1$.

In this case, equation (3.21) becomes

$$(3.39) \qquad \sinh^2(\theta)\cosh(\theta)s^2 x_{ss} + (\sinh(\theta)\theta's^2 - \sinh^2(\theta)\cosh(\theta)s)x_s$$
$$- (\cosh(\theta) + \sinh(\theta)\theta's)x = 0.$$

Putting $\Phi(s,t) = x/s$, the previous equation turns into

$$(3.40) \quad s^2\sinh^2\theta\cosh\theta\Phi_{ss} + (s\sinh^2\theta\cosh\theta + s^2\sinh\theta\theta')\Phi_s - \cosh^3\theta\Phi = 0.$$

Putting

$$(3.41) \qquad\qquad u(s) = e^{2\int \frac{1}{s}\coth\theta ds},$$

then the differential equation (3.40) can be rewritten as

$$4u^2\Phi_{uu} + 4u\Phi_u - \Phi = 0.$$

Solving this equation, we find that the position vector x can be expressed as

$$(3.42) \qquad\qquad x(s,t) = s\Big(C_1(t)\sqrt{u} + C_2(t)\frac{1}{\sqrt{u}}\Big),$$

where both $C_1(t)$ and $C_2(t)$ are vector-valued functions depending only on t in $\mathbb{R}^3$.

Consequently, by using the condition $\langle x, x \rangle = s^2$ we get

$$(3.43) \qquad \langle C_1(t), C_1(t) \rangle = \langle C_2(t), C_2(t) \rangle = 0, \quad \langle C_1(t), C_2(t) \rangle = \frac{1}{2}.$$

Moreover, from (3.16) and $\mu = -s$, equation (3.13) becomes

$$(3.44) \qquad s^2\sinh^2\theta\rho_s + (1 - s\cosh\theta\rho)(-s\rho + \cosh\theta + \sinh\theta\theta's) = 0.$$

If we put

$$(3.45) \qquad\qquad \beta = \frac{\rho s - \cosh\theta}{\sinh\theta},$$

then the previous equation turns into

$$(3.46) \qquad\qquad s\sinh\theta\beta_s = \cosh\theta(1 - \beta^2).$$

Case B.1: If $\beta_s = 0$, which yields $\beta = \pm 1$. In this case,

$$\rho = \frac{1}{s}(\cosh\theta \pm \sinh\theta),$$

and (3.18) becomes

$$(3.47) \qquad\qquad \frac{\lambda_s}{\lambda} = \frac{\sinh\theta \pm \cosh\theta}{s\sinh\theta}.$$

Solving (3.47) gives

$$(3.48) \qquad\qquad \lambda = se^{\pm\int\frac{1}{s}\coth\theta ds}\phi(t),$$

where ϕ is a nonzero smooth function defined on M. After a change of the t-coordinate, we can assume $\phi(t) = 1$. Thus the metric (3.34) takes the form

$$(3.49) \qquad g = -\frac{1}{\sinh^2\theta}ds^2 + s^2 e^{\pm 2\int \frac{1}{s}\coth\theta ds}dt^2.$$

Case "$+$". On one hand, by (3.49), expression (3.42) implies

$$(3.50) \qquad \langle C_1'(t), C_1'(t)\rangle = 1, \ \langle C_1'(t), C_2'(t)\rangle = \langle C_2'(t), C_2'(t)\rangle = 0,$$

$$(3.51) \qquad \langle C_1(t), C_2'(t)\rangle = \langle C_2(t), C_1'(t)\rangle = 0.$$

In this case, $\{C_1(t), C_2(t), C_1'(t)\}$ forms a pseudo-orthonormal frame in $\mathbb{L}^3$. equation (3.26) implies that $\langle C_2'(t), C_2(t)\rangle = 0$. Therefore, we have $C_2'(t) = 0$, which shows that $C_2(t)$ is a constant null vector in $\mathbb{L}^3$. Without loss of generality, we may choose

$$C_2(t) = -\frac{1}{2}(1, 0, 1).$$

It follows from (3.26) and the first equation of (3.51) that

$$(3.52) \qquad C_1(t) = \frac{1}{2}(t^2 - 1, 2t, t^2 + 1).$$

Substituting these into (3.42) gives case (4) in Theorem 3.6.

Case "$-$". Similarly, we obtain that $C_1(t)$ is a constant null vector. Put

$$C_1(t) = -\frac{1}{2}(1, 0, 1),$$

then we have

$$(3.53) \qquad C_2(t) = \frac{1}{2}(t^2 - 1, 2t, t^2 + 1).$$

Substituting these into (3.42) gives case (5).

Case B.2: If $\beta_s \neq 0$, the solution of the equation (3.46) is given by

$$\beta = \frac{u(s)\phi(t) - 1}{u(s)\phi(t) + 1},$$

where $\phi(t)$ is a nonzero function defined on M. Hence, the function ρ is given by

$$\rho = \frac{1}{s}\Big(\frac{u(s)\phi(t) - 1}{u(s)\phi(t) + 1}\sinh\theta + \cosh\theta\Big).$$

Substituting this into (3.18) and by $\mu = -s$ that

$$(3.54) \qquad \frac{\lambda_s}{\lambda} = \frac{1}{s} + \frac{1}{s}\frac{\cosh\theta}{\sinh\theta}\frac{u(s)\phi(t) - 1}{u(s)\phi(t) + 1}.$$

By integration one has

$$(3.55) \qquad \lambda = s\frac{u(s)\phi(t) + 1}{\sqrt{u(s)\phi(t)}}\psi(t)$$

for another non-zero function ψ depending only on t. In this case, the metric is given by

$$(3.56) \qquad g = -\frac{1}{\sinh^2 \theta} ds^2 + s^2 \frac{(u(s)\phi(t)+1)^2}{u(s)\phi(t)} \psi^2(t) dt^2.$$

If we put

$$(3.57) \qquad \tilde{C}_1(t) = C_1(t) + C_2(t), \quad \tilde{C}_2(t) = C_1(t) - C_2(t),$$

then $\tilde{C}_1(t)$, $\tilde{C}_2(t)$ satisfies

$$(3.58) \qquad \langle \tilde{C}_1(t), \tilde{C}_1(t) \rangle = -\langle \tilde{C}_2(t), \tilde{C}_2(t) \rangle = 1, \quad \langle \tilde{C}_1(t), \tilde{C}_2(t) \rangle = 0,$$

and (3.42) becomes

$$(3.59) \qquad x(s,t) = s\left(\cosh(\int \frac{1}{s} \coth\theta ds) \tilde{C}_1(t) + \sinh(\int \frac{1}{s} \coth\theta ds) \tilde{C}_2(t) \right).$$

It follows from (3.56) and (3.59) that

$$(3.60) \qquad \langle \tilde{C}'_2(t)), \tilde{C}'_2(t) \rangle = \frac{(\phi(t)-1)^2}{\phi(t)} \psi^2(t),$$

$$\langle \tilde{C}'_1(t), \tilde{C}'_2(t) \rangle = \frac{(\phi(t)-1)(\phi(t)+1)}{\phi(t)} \psi^2(t),$$

$$(3.61) \qquad \langle \tilde{C}'_1(t), \tilde{C}'_1(t) \rangle = \frac{(\phi(t)+1)^2}{\phi(t)} \psi^2(t),$$

$$\langle \tilde{C}'_1(t), \tilde{C}'_2(t) \rangle = \langle \tilde{C}'_2(t), \tilde{C}'_1(t) \rangle = 0.$$

If $\phi(t) = 1$, then (3.60) becomes

$$(3.62) \qquad \langle \tilde{C}'_2(t), \tilde{C}'_2(t) \rangle = \langle \tilde{C}'_1(t), \tilde{C}'_2(t) \rangle = 0.$$

Thus, from (3.58), (3.61) and (3.62) we obtain $\tilde{C}'_2(t) = 0$. This shows that $\tilde{C}_2$ is a timelike constant vector in $\mathbb{L}^3$. Without loss of generality, we assume that $\tilde{C}_2 = (0,0,1)$. After making a change of the t-coordinate, we can assume $\langle \tilde{C}'_1(t), \tilde{C}'_1(t) \rangle = 1$. Combining these with (3.58) gives

$$\tilde{C}_1(t) = (\cos t, \sin t, 0).$$

Substituting these into (3.59), we obtain case (6) in Theorem 3.6.

If $\phi(t) = -1$, then from (3.60) and (3.61) that

$$(3.63) \qquad \langle \tilde{C}'_1(t), \tilde{C}'_1(t) \rangle = \langle \tilde{C}'_1(t), \tilde{C}'_2(t) \rangle = 0.$$

Thus, from (3.58), (3.61) and (3.63) we obtain $\tilde{C}'_1(t) = 0$. This shows that $\tilde{C}_1$ is a spacelike constant vector in $\mathbb{L}^3$. Without loss of generality, we assume that $\tilde{C}_1 = (1,0,0)$. After making a change of the t-coordinate, we can assume $\langle \tilde{C}'_2(t), \tilde{C}'_2(t) \rangle = 1$. Combining these with (3.58) gives

$$\tilde{C}_2(t) = (0, \sinh t, \cosh t).$$

Substituting these into (3.59), we obtain case (7) in Theorem 3.6.

Suppose $\phi(t) \neq \pm 1$. After making a change of the t-coordinate, we may assume $\langle \tilde{C}'_1(t), \tilde{C}'_1(t) \rangle = 1$. Since $\langle \tilde{C}_2(t), \tilde{C}_1(t) \rangle = \langle \tilde{C}_2(t), \tilde{C}'_1(t) \rangle = 0$, it follows that $\tilde{C}_2 = \pm \tilde{C}_1 \times \tilde{C}'_1$. We assume $\tilde{C}_2 = \tilde{C}_1 \times \tilde{C}'_1$. Since $\langle \tilde{C}'_1(t), \tilde{C}'_2(t) \rangle \neq 0$, we have $\langle \tilde{C}'_1(t), \tilde{C}_1(t) \times \tilde{C}''_1(t) \rangle \neq 0$. Denote by $f(t) = \tilde{C}_1(t)$. Then the immersion (3.59) reduces to case (8) in Theorem 3.6.

Conversely, it is easy to verify that each of the eight types of Lorentz surfaces is a GCR surface in $\mathbb{L}^3$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

§2. Lorentz GCR surfaces lying in the timelike cone

In this subsection we deal with the remaining case: the immersion x lies in the timelike cone. Then, there is a nonzero function μ such that $\langle x, x \rangle = -\mu^2$ and by Definition 2.3 we can decompose x in the form

$$(3.64) \qquad\qquad \frac{x}{\mu} = \cosh\theta e_1 + \sinh\theta\xi,$$

where e_1 is a unit spacelike tangent vector field on M.

Theorem 3.6. *Let $x : M \to \mathbb{L}^3$ be a Lorentz surface immersed in the 3-dimensional Minkowski space $\mathbb{L}^3$. If x lies in the timelike cone, then M is a GCR surface if and only if one of the following five statements holds*:

(1) *the immersion $x(M)$ is a surface given by*

$$x(s,t) = \frac{s}{2}\Big(e^{-\int \frac{1}{s}\tanh\theta ds} + e^{\int \frac{1}{s}\tanh\theta ds}(t^2 - 1), 2e^{\int \frac{1}{s}\tanh\theta ds}t,$$
$$e^{-\int \frac{1}{s}\tanh\theta ds} + e^{\int \frac{1}{s}\tanh\theta ds}(t^2 + 1)\Big),$$

where θ is an angle function concerning the variable s;

(2) *the immersion $x(M)$ is a surface given by*

$$x(s,t) = \frac{s}{2}\Big(e^{\int \frac{1}{s}\tanh\theta ds} + e^{-\int \frac{1}{s}\tanh\theta ds}(t^2 - 1), 2e^{-\int \frac{1}{s}\tanh\theta ds}t,$$
$$e^{\int \frac{1}{s}\tanh\theta ds} + e^{-\int \frac{1}{s}\tanh\theta ds}(t^2 + 1)\Big),$$

where θ is an angle function concerning the variable s;

(3) *the immersion $x(M)$ is a surface given by*

$$x(s,t) = s\Big(\sinh(\int \frac{1}{s}\tanh\theta ds), \cosh(\int \frac{1}{s}\tanh\theta ds)\sinh t,$$
$$\cosh(\int \frac{1}{s}\tanh\theta ds)\cosh t\Big),$$

where θ is an angle function concerning the variable s;

(4) *the immersion $x(M)$ is a surface given by*

$$x(s,t) = s\Big(\sinh(\int \frac{1}{s}\tanh\theta ds)\sin t, \sinh(\int \frac{1}{s}\tanh\theta ds)\cos t,$$
$$\cosh(\int \frac{1}{s}\tanh\theta ds)\Big),$$

where θ is an angle function concerning the variable s;

(5) *the immersion x(M) is given by*

$$x(s,t) = s\big(\cosh(\int \frac{1}{s}\tanh\theta ds)f(t) + \sinh(\int \frac{1}{s}\tanh\theta ds)f(t) \times f'(t)\big),$$

where θ is an angle function concerning the variable s and f is a unit speed curve on $\mathbb{H}^2$ satisfying $\langle f'', f''\rangle \neq -\langle f, f''\rangle^2$.

Remark 3.7. The proof is extremely similar to the case that the position vector x lying in the spacelike cone and the normal vector ξ span a timelike vector space as discussed above. Hence, we only state our main theorem in this subsection.

Remark 3.8. If we choose θ to be a constant in Theorem 3.6, after a suitable translation of the variable s, the GCR surfaces are just the constant slope surfaces of Theorem 3.4 in [10].

4. Some characterizations of Lorentz GCR surfaces in $\mathbb{L}^3$

In this section,we will study the Lorentz GCR surfaces with some additional assumptions in $\mathbb{L}^3$. We mainly give some characterization theorems for the flat GCR surfaces or the GCR surfaces with constant mean curvature.

Theorem 4.1. *The flat Lorentz GCR surface immersed in $\mathbb{L}^3$ is either an open part of plane or cylinder.*

Proof. If M be GCR surface lying in the spacelike cone immersed in $\mathbb{L}^3$. Since M is flat, from (3.10) and (3.11), we have

$$(4.1) \qquad \rho(e_1(\theta) - \frac{c(\theta)}{\mu}) = 0.$$

On the other hand, by (3.13) we conclude that $e_1(\theta) = \frac{c(\theta)}{\mu}$. Since $\mu = \varepsilon s$, it follows that

$$(4.2) \qquad \theta' = \frac{c(\theta)}{\varepsilon s(\theta)s}.$$

By integration, we get

$$(4.3) \qquad \theta = \int \frac{c(\theta)}{\varepsilon s(\theta)s}ds.$$

On the other hand, by solving the differential equation (4.2), we get

$$(4.4) \qquad sc(\theta) = d,$$

where d is constant.

If $c(\theta) = \cos\theta$, $s(\theta) = \sin\theta$ and $\varepsilon = 1$, equations (4.3) and (4.4) become

$$(4.5) \qquad \theta = \int \frac{\cot\theta}{s}ds, \qquad \cos\theta = \frac{d}{s}.$$

In this case, by Remark 3.5, the profile curves of the revolution of the surface (1) and (2) in Theorem 3.3 are

$$(4.6) \qquad \gamma(s) = \big(d, \quad 0, \quad \pm\sqrt{s^2 - d^2} \big),$$

$$(4.7) \qquad \gamma(s) = \big(\pm\sqrt{s^2 - d^2}, \quad 0, \quad d \big),$$

which are lines parallel to the coordinate axis. So the revolution of the surface is either an open part of plane or an open part of cylinder.

For the Lorentz GCR surface (3) in Theorem 3.3, by (4.5), the immersion becomes

$$x(s,t) = s\big(\cos\theta f(t) + \sin\theta f(t) \times f'(t) \big)$$
$$(4.8) \qquad = d \cdot f(t) \pm \sqrt{s^2 - d^2} f(t) \times f'(t),$$

where f is a unit speed curve on $\mathbb{S}_1^2$. By rechoosing the parameter, (4.8) becomes

$$(4.9) \qquad x(s,t) = d \cdot f(t) + s f(t) \times f'(t).$$

Here, M is ruled surface with the gauss curvature $K = 0$, so the ruled surface is developable and hence $(f, f', f'') = 0$, which is a contradiction.

If $c(\theta) = \cosh\theta$, $s(\theta) = \sinh\theta$ and $\varepsilon = -1$. equations (4.3) and (4.4) become

$$(4.10) \qquad \theta = -\int \frac{\coth\theta}{s} ds, \qquad \cosh\theta = \frac{d}{s}.$$

In this case, by Remark 3.5, the profile curves of the revolution of the surface (4) and (5) in Theorem 3.3 are

$$(4.11) \qquad \gamma(s) = \big(\pm\sqrt{d^2 - s^2}), \quad -d, \quad 0 \big),$$

and the profile curves of the surfaces of revolution (6) and (7) in Theorem 3.3 are

$$(4.12) \qquad \gamma(s) = \big(d, \quad 0, \quad \pm\sqrt{d^2 - s^2}) \big),$$

which are lines parallel to the coordinate axis. So the revolution of the surface is an open part of plane or cylinder.

For the Lorentz GCR surface (8) in Theorem 3.3, by (4.5), the immersion becomes

$$x(s,t) = s\big(\cosh\theta f(t) - \sinh\theta f(t) \times f'(t) \big)$$
$$(4.13) \qquad = d \cdot f(t) \pm \sqrt{d^2 - s^2} f(t) \times f'(t),$$

where f is a unit speed curve on $\mathbb{S}_1^2$. By rechoosing the parameter, (4.13) becomes

$$(4.14) \qquad x(s,t) = d \cdot f(t) + s f(t) \times f'(t)$$

Here, M is ruled surface with the gauss curvature $K = 0$, so the ruled surface is developable and hence $(f, f', f'') = 0$, which is a contradiction.

A similar discussion on Theorem 3.6 leads to the same conclusion.

Consequently, we can conclude the immersion is either an open part of plane or cylinder. □

Theorem 4.2. *The Lorentz GCR surfaces with constant mean curvature immersed in $\mathbb{L}^3$ are surfaces of revolution.*

Proof. In Theorem 3.3, we classify the Lorentz GCR surfaces in $\mathbb{L}^3$. It is easy to see that (1)-(2) and (4)-(7) are surfaces of revolution.

For the case (3), its tangent plane is spanned by

$$x_s = \frac{1}{\sin\theta}\Big[\sin\big(\theta - \int \frac{\cot\theta}{s}ds\big)f(t) + \cos\big(\theta - \int \frac{\cot}{s}ds\big)f(t) \times f'(t)\Big].$$

$$x_t = s\Big[\cos\big(\int \frac{\cot\theta}{s}ds\big)f'(t) + \sin\big(\int \frac{\cot\theta}{s}ds\big)f(t) \times f''(t)\Big]$$

$$= s\Big[\cos\big(\int \frac{\cot\theta}{s}ds\big) + k_g(t)\sin\big(\int \frac{\cot\theta}{s}ds\big)\Big]f'(t),$$

where $k_g(t) = \langle f'' \times f, f'\rangle$.

Since f is a unit speed curve on $\mathbb{S}_1^2$, we have

$$f''(t) = f(t) + k_g(t)f(t) \times f'(t)$$

Hence $k_g(t)$ represents the geodesic curvature of the curve f.

We compute the unit normal to the surface

$$n = \sin\big(\theta - \int \frac{\cot\theta}{s}ds\big)f(t) \times f'(t) + \cos\big(\theta - \int \frac{\cot\theta}{s}ds\big)f(t),$$

the first fundamental form of M

$$I = \frac{1}{\sin^2\theta}ds^2 - s^2\Big[\cos\big(\int \frac{\cot\theta}{s}\big)ds + k_g(t)\sin\big(\int \frac{\cot\theta}{s}\big)ds\Big]dt^2,$$

and the second fundamental form of M

$$II = -\frac{1}{\sinh\theta}\big(\frac{\coth\theta}{s} + \theta'\big)ds^2 + \varrho dt^2,$$

where

$$\varrho = -s\Big[\cosh\big(\int \frac{\coth\theta}{s}\big)ds - k_g(t)\sinh\big(\int \frac{\coth\theta}{s}\big)ds\Big]$$

$$\cdot \Big[\cosh\big(\int \frac{\coth\theta}{s}ds + \theta\big) - k_g(t)\sinh\big(\int \frac{\coth\theta}{s}ds + \theta\big)\Big].$$

It follows that the mean curvature of the surface is obtained from

$$2sH = -(\cosh\theta + \theta'\sinh\theta \cdot s) - \frac{\cosh\big(\int \frac{\coth\theta}{s}ds\big) - k_g(t)\sinh\big(\int \frac{\coth\theta}{s}ds\big)}{\cosh\big(\int \frac{\coth\theta}{s}ds + \theta\big) - k_g(t)\sinh\big(\int \frac{\coth\theta}{s}ds + \theta\big)}.$$

When H is a constant, it follows that k_g is constant. Thus, f (as a curve in $\mathbb{L}^3$) is a planar curve of constant curvature $k = \sqrt{1 - k_g^2}$. Hence f is a circle on $\mathbb{H}^2$. Therefore, a GCR surface with constant mean curvature should be a surface of revolution.

For the case (8) and the cases in Theorem 3.6, a similar method can yield the same conclusion as well.   □

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