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A CRITERION FOR BOUNDED FUNCTIONS

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Abstract. We consider a sufficient condition for $w(z)$, analytic in $|z| < 1$, to be bounded in $|z| < 1$, where $w(0) = w'(0) = 0$. We apply it to the meromorphic starlike functions. Also, a certain Briot-Bouquet differential subordination is considered. Moreover, we prove that if $p(z)+zp'(z)\phi(p(z)) \prec h(z)$, then $p(z) \prec h(z)$, where $h(z) = [(1+z)(1-z)]^{\alpha}$, under some additional assumptions on $\phi(z)$.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and denote by $\mathcal{A}$ the class of analytic functions in $\mathbb{D}$ and usually normalized, i.e., $\mathcal{A} = \{f \in \mathcal{H} : f(0) = 0, f'(0) = 1\}$. We say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathbb{D}$, written $f \prec g$ if and only if there exists an analytic function $w \in \mathcal{H}$ such that $|w(z)| \le |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{D}$. Therefore $f \prec g$ in $\mathbb{D}$ implies $f(\mathbb{D}) \subset g(\mathbb{D})$. In particular if g is univalent in $\mathbb{D}$, then the Subordination Principle says that $f \prec g$ if and only if $f(0) = g(0)$ and $f(|z| < r) \subset g(|z| < r)$, for all $r \in (0, 1]$.

Let β, γ be complex numbers and let $p, h \in \mathcal{H}$, with $h(0) = p(0)$. The first-order differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{D})$$

is called the Briot-Bouquet differential subordination. A lot of the results on the Briot-Bouquet differential subordination are collected in [5, Ch.3]. It seems that among contained there cases was not considered the case $\gamma = 0$, $\beta = 1$ and

$$h(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2},$$

where $0 < \alpha < 1$. In this work we consider it.

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For integer $n \geq 0$, denote by Σ_n the class of meromorphic functions, defined in $\dot{\mathbb{U}} = \{z : 0 < |z| < 1\}$, which are of the form

$$F(z) = \frac{1}{z} + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

A function $F \in \Sigma_0$ is said to be starlike if it is univalent and the complement of $F(\dot{\mathbb{U}})$ is starlike with respect to the origin. Denote by Σ_0^* the class of such functions. If $F \in \Sigma_0$, then it is well-known that $F \in \Sigma_0^*$ if and only if

$$\mathfrak{Re}\left\{-\frac{zF'(z)}{F(z)}\right\} > 0$$

for $z \in \dot{\mathbb{U}}$. For $\alpha < 1$, let

$$\Sigma_{n,\alpha}^* = \left\{F \in \Sigma_n : \mathfrak{Re}\left\{-\frac{zF'(z)}{F(z)}\right\} > \alpha,\ z \in \dot{\mathbb{U}}\right\},$$

the class of meromorphic-starlike functions of order α. For $0 < \alpha \leq 1$, let

$$\Sigma_n^*(\alpha) = \left\{F \in \Sigma_n : \left|\arg\left\{-\frac{zF'(z)}{F(z)}\right\}\right| < \frac{\alpha\pi}{2},\ z \in \dot{\mathbb{U}}\right\}$$

the class of meromorphic-strongly starlike functions of order α.

Definition 1 ([5]). We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{D}} \setminus E(f)$, where

$$E(f) := \{\zeta : \zeta \in \partial\mathbb{D} \ \text{ and } \ \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that

$$f'(\zeta) \neq 0 \qquad (\zeta \in \partial(\mathbb{D}) \setminus E(f)).$$

Lemma 1.1 ([5]). *Let $q \in \mathcal{Q}$ with $q(0) = a$ and let*

$$p(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{D}$ with

$$p(z) \not\equiv a \quad \text{and} \quad n \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

If p is not subordinate to q, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{D} \setminus E(q),$$

for which

$$p(|z| < r_0) \subset q(\mathbb{D}),$$
$$p(z_0) = q(\zeta_0)$$

and

$$z_0 p'(z_0) = k\zeta_0 q'(\zeta_0)$$

for some $k \geq n$.

Lemma 1.1 is a generalization of Jack's lemma [3]. To prove the main results, we also need the following generalization of Nunokawa's lemma, [6], [7], see also [2].

Lemma 1.2 ([9]). *Let $p(z)$ be of the form*

$$p(z) = 1 + \sum_{n=m\geq 1}^{\infty} a_n z^n, \quad a_m \neq 0, \quad (z \in \mathbb{D}), \tag{1}$$

with $p(z) \neq 0$ in $\mathbb{D}$. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \pi\alpha/2 \quad \text{in} \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \pi\alpha/2$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq m\left(a^2+1\right)/(2a) \quad \text{when} \quad \arg\{p(z_0)\} = \pi\alpha/2 \tag{2}$$

and

$$k \leq -m\left(a^2+1\right)/(2a) \quad \text{when} \quad \arg\{p(z_0)\} = -\pi\alpha/2, \tag{3}$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \ a > 0.$$

2. Main result

Theorem 2.1. *Let $w(z)$ be analytic in $\mathbb{D}$ with $w(0) = w'(0) = 0$ and suppose that*

$$\left| w(z) - \frac{zw'(z)}{w(z)} \right| < \sqrt{\frac{1 - \mathfrak{Re}\{z\}}{1 + \mathfrak{Re}\{z\}}} \quad (z \in \mathbb{D}). \tag{4}$$

Then we have

$$|w(z)| < 1 \quad (z \in \mathbb{D}).$$

Proof. If there exists a point z_0, $|z_0| < 1$, such that

$$|w(z)| < 1 \quad (|z| < |z_0|)$$

and

$$w(z_0) = e^{i\theta},$$

then from Lemma 1.1, we have

$$\frac{z_0 w'(z_0)}{w(z_0)} = k \geq 2.$$

Then it follows that

$$\begin{aligned} \left| w(z_0) - \frac{z_0 w'(z_0)}{w(z_0)} \right|^2 &= \left| e^{i\theta} - \frac{z_0 w'(z_0)}{e^{i\theta}} \frac{e^{i\theta}}{1 + e^{i\theta}} \right|^2 \\ &= (\cos\theta - k/2)^2 + \sin^2\theta \left(1 - \frac{k}{2(1+\cos\theta)} \right)^2 \end{aligned}$$

$$= \varphi(k), \text{ say.}$$

Then we have

$$\begin{aligned}\varphi'(k) &= -(\cos\theta - k/2) - \frac{\sin^2\theta}{1+\cos\theta}\left(1 - \frac{k}{1+\cos\theta}\right) \\ &= \frac{k}{1+\cos\theta} - 1 > 0,\end{aligned}$$

and

$$\begin{aligned}\varphi(2) &= (\cos\theta - 1)^2 + \left(1 - \frac{1}{1+\cos\theta}\right)^2 \sin^2\theta \\ &= \frac{2\cos^2\theta}{1+\cos\theta} - 2\cos\theta + 1 \\ &= \frac{1-\cos\theta}{1+\cos\theta} \\ &= \frac{1-\mathfrak{Re}\{z_0\}}{1+\mathfrak{Re}\{z_0\}}.\end{aligned}$$

Therefore, there exists a point z_0, $|z_0| < 1$, such that

$$\left|w(z_0) - \frac{z_0 w'(z_0)}{w(z_0)}\right|^2 \geq \frac{1-\mathfrak{Re}\{z_0\}}{1+\mathfrak{Re}\{z_0\}}$$

for all $k \geq 2$. It contradicts (4) and it completes the proof. □

Corollary 2.2. *Let*

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$$

be analytic in $0 < |z| < 1$ *and suppose that for* $0 < \alpha < \alpha_0$

$$\left|-\frac{zF''(z)}{F'(z)} - 2\right| < \sqrt{\frac{1-\mathfrak{Re}\,\{z\}}{1+\mathfrak{Re}\,\{z\}}} \quad (z \in \mathbb{D}).$$

Then

$$\left|-\frac{zF'(z)}{F(z)} - 1\right| < 1 \quad (z \in \mathbb{D}),$$

it follows that $F(z)$ *is meromorphic-starlike in* $\mathbb{D}$.

For another sufficient condition for strongly starlikeness, we refer to the recent paper [8].

Theorem 2.3. *Let* $p(z)$ *of the form*

$$p(z) = 1 + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{D}). \tag{5}$$

Suppose that $\alpha \in (0,1]$ *and*

$$p(z) + \frac{zp'(z)}{p(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \quad (z \in \mathbb{D}). \tag{6}$$

Then we have

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}).$$

Proof. If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \pi\alpha/2 \quad (|z| < |z_0|)$$

and

$$|\arg\{p(z_0)\}| = \pi\alpha/2, \quad p(z_0) = (\pm ia)^{\alpha},$$

then from Nunokawa's Lemma 1.2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{a^2+1}{2a} \geq 1, \text{ when } \arg\{p(z_0)\} = \pi\alpha/2$$

and

$$k \leq -\frac{a^2+1}{2a} \leq -1, \text{ when } \arg\{p(z_0)\} = -\pi\alpha/2.$$

For the case $\arg\{p(z_0)\} = \alpha\pi/2$, we have

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = (ia)^{\alpha} + i\alpha k. \tag{7}$$

Let us put $z = e^{i\theta}$ in the right hand side of (6).

$$\left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} = \left(\frac{i\sin\theta}{1-\cos\theta}\right)^{\alpha} + \frac{i\alpha}{\sin\theta}. \tag{8}$$

It is easy to see that it is possible to find θ_0 such that for given $a > 0$

$$a = \frac{\sin\theta_0}{1-\cos\theta_0}.$$

Then

$$\alpha k > \frac{\alpha(a^2+1)}{2a} = \frac{\alpha}{\sin\theta_0}, \tag{9}$$

and hence from (7), (8) and (9) we get that

$$p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}$$

lies outside the image of the unit disc under the function

$$\left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2},$$

which is convex in the direction of the imaginary axis. It contradicts (5). For the case $\arg\{p(z_0)\} = -\alpha\pi/2$, in the same way as before, we also can obtain a contradiction (5), which completes the proof. □

For $\alpha = 1$ Theorem 2.3 becomes the result in [5, p. 140]. For $0 < \alpha < 1$ Theorem 2.3 is an extension of Theorem 3.2i [5, p. 97].

Corollary 2.4. *Let*

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$

be analytic in $\mathbb{D}$ *and suppose that*

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} + \frac{2\alpha z}{1-z^2} \quad (z \in \mathbb{D}).$$

Then we have

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{D}),$$

it follows that $f(z)$ *is strongly starlike of order* α *in* $\mathbb{D}$.

Theorem 2.5. *Let* $h(z) = \{(1+z)/(1-z)\}^{\alpha}$, $\alpha \in (0, 1]$, *and* $p(z)$ *are analytic in* $\mathbb{D}$ *with* $h(0) = p(0) = 1$. *Assume also that* $\phi(p(z))$ *is analytic in* $\mathbb{D}$, *moreover* $\mathfrak{Re}\{\phi(h(z))\} \geq 0$ *in* $\mathbb{D}$. *If*

$$p(z) + zp'(z)\phi(p(z)) \prec h(z) \quad (z \in \mathbb{D}), \tag{10}$$

then

$$p(z) \prec h(z) \quad (z \in \mathbb{D}).$$

Proof. If there exists a point $z_0 = e^{i\theta_0}$, $|z_0| < 1$, such that

$$p(z) \neq h(e^{i\theta}) \text{ for all } \theta,\ 0 \leq \theta < 2\pi \text{ and } |z| < |z_0|$$

and

$$p(z_0) = h(e^{i\theta_0}),$$

then from the hypothesis of the theorem, we have the following picture, for the case $\arg\{p(z_0)\} = \alpha\pi/2 < 0$.

Then from Nunokawa's Lemma 1.2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$\alpha\pi/2 = \arg\{p(z_0)\}$$

and

$$k \geq 1, \text{ when } \arg\{p(z_0)\} = \pi\alpha/2 > 0$$

while

$$k \leq -1, \text{ when } \arg\{p(z_0)\} = \pi\alpha/2 < 0.$$

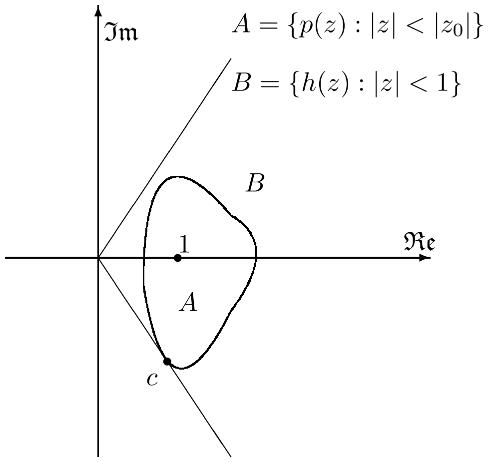

FIGURE 1. $c = p(z_0) = h(e^{i\theta_0})$.

For the case $\arg\{p(z_0)\} = \alpha\pi/2 > 0$, $1 \leq k$, we have

$$\begin{aligned} &\arg\{p(z_0) + z_0 p'(z_0)\phi(p(z_0))\} \\ &= \arg\left\{p(z_0)\left(1 + \frac{z_0 p'(z_0)}{p(z_0)}\phi(p(z_0))\right)\right\} \\ &= \arg\left\{h(e^{i\theta_0})\right\} + \arg\left\{1 + i\alpha k\phi\left(h(e^{i\theta_0})\right)\right\} \\ &> \arg\left\{h(e^{i\theta_0}\right\}. \end{aligned}$$

This contradicts the hypothesis (10) and for the case $\arg\{p(z_0)\} = \alpha\pi/2 < 0$, $k \leq -1$, applying the same method as the above, we have

$$\begin{aligned} &\arg\{p(z_0) + z_0 p'(z_0)\phi(p(z_0))\} \\ &= \arg\left\{h(e^{i\theta_0})\right\} + \arg\left\{1 + i\alpha k\phi\left(h(e^{i\theta_0})\right)\right\} \\ &< \arg\left\{h(e^{i\theta_0}\right\}. \end{aligned}$$

This is also a contradiction and therefore, it completes the proof. □

Recall here the well known theorem due to Hallenbeck and Ruscheweyh [4].

Theorem A ([4])**.** *Let the function h be analytic and convex univalent in $\mathbb{D}$ with $h(0) = a$. Let also $p(z) = a + b_n z^n + b_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{D}$. If*

$$p(z) + \frac{zp'(z)}{c} \prec h(z), \quad z \in \mathbb{D} \tag{11}$$

for $\mathfrak{Re}\{c\} \geq 0,\ c \neq 0,$ *then*

$$p(z) \prec q_n(z) \prec h(z), \quad z \in \mathbb{D},$$

where $q_n(z) = \frac{c}{nz^{c/n}} \int_0^z t^{c/n-1} h(t)\mathrm{d}t$. *Moreover, the function* $q_n(z)$ *is convex univalent and is the best dominant of* $p \prec q_n$ *in the sense that if* $p \prec q$, *then* $q_n \prec q$.

An another generalization of Theorem A we refer to [5, p. 70]. However, Theorem 2.5 cannot be written in the form presented in [5, p. 70] because $h(z) = \{(1+z)/(1-z)\}^{\alpha}$ is not convex. Therefore, Theorem 2.5 is a new extension of Theorem A. For another generalization of Theorem A in this direction we refer to [5, p. 70] and to [1], [10]. For $\phi(z) = 1/(\beta z + \gamma)$ Theorem 2.5 becomes the following corollary, with the Briot-Bouquet differential subordination, for related result we refer to [5, p. 81].

Corollary 2.6. *Let* $h(z) = \{(1+z)(1-z)\}^{\alpha}$, $\alpha \in (0,1]$, *and* $p(z)$ *are analytic in* $\mathbb{D}$ *with* $h(0) = p(0) = 1$. *Let* $\phi(z) = 1/(\beta z + \gamma)$. *Assume also that* $\phi(p(z))$ *is analytic in* $\mathbb{D}$, *moreover* $\mathfrak{Re}\{\phi(\beta h(z) + \gamma)\} \geq 0$ *in* $\mathbb{D}$. *If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{D}), \tag{12}$$

then

$$p(z) \prec h(z) \quad (z \in \mathbb{D}).$$

For $\phi(z) = z^{1/\alpha}$ Theorem 2.5 becomes the following corollary.

Corollary 2.7. *Let* $h(z) = \{(1+z)/(1-z)\}^{\alpha}$, $\alpha \in (0,1]$, *and* $p(z),\ p^{1/\alpha}(z)$ *are analytic in* $\mathbb{D}$ *with* $h(0) = p(0) = 1$. *If*

$$p(z) + zp'(z)p^{1/\alpha}(z) \prec h(z) \quad (z \in \mathbb{D}), \tag{13}$$

then

$$p(z) \prec h(z) \quad (z \in \mathbb{D}).$$

For $\alpha = 1/2$ Corollary 2.7 becomes the following corollary.

Corollary 2.8. *Let* $h(z) = \sqrt{(1+z)/(1-z)}$ *and* $p(z)$ *are analytic in* $\mathbb{D}$ *with* $h(0) = p(0) = 1$. *If*

$$p(z) + zp'(z)p^2(z) \prec h(z) \quad (z \in \mathbb{D}), \tag{14}$$

then

$$p(z) \prec h(z) \quad (z \in \mathbb{D}).$$

Theorem 2.9. *Let* $p(z)$ *be analytic in* $|z| < 1$, *with* $p(0) = -1$ *and suppose that*

$$\frac{zp'(z)}{p(z)} - p(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}). \tag{15}$$

Then we have

$$-p(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}). \tag{16}$$

Proof. If

$$-p(z) \not\prec \frac{1+z}{1-z} \quad \text{for} \quad |z|<1,$$

then by Lemma 1.1, there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{D} \setminus \{-1\},$$

for which

$$\mathfrak{Re}\,\{-p(|z|<r_0)\} > 0, \tag{17}$$

$$-p(z_0) = \frac{1+\zeta_0}{1-\zeta_0} \tag{18}$$

and

$$-z_0 p'(z_0) = k\zeta_0 \frac{2}{(1-\zeta_0)^2} \tag{19}$$

for some $k \geq 1$. By (18) and (19) we have

$$\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) = \frac{2k\zeta_0}{1-\zeta_0^2} + \frac{1+\zeta_0}{1-\zeta_0}, \tag{20}$$

furthermore,

$$\begin{aligned}
\mathfrak{Re}\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} &= \mathfrak{Re}\left\{\frac{2k\zeta_0}{1-\zeta_0^2}\right\} + \mathfrak{Re}\left\{\frac{1+\zeta_0}{1-\zeta_0}\right\} \\
&= \mathfrak{Re}\left\{\frac{ki}{\mathfrak{Im}\,\{\zeta_0\}}\right\} + \mathfrak{Re}\left\{\frac{i\mathfrak{Im}\,\{\zeta_0\}}{1-\mathfrak{Re}\,\{\zeta_0\}}\right\} \\
&= 0.
\end{aligned}$$

This contradicts hypothesis (15) and it completes the proof. □

Applying the above theorem, we have the following result.

Theorem 2.10. *Let*

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

be analytic in $0<|z|<1$ *and suppose that*

$$\mathfrak{Re}\left\{1 + \frac{zF''(z)}{F'(z)} - 2\frac{zF'(z)}{F(z)}\right\} > 0 \quad (|z|<1). \tag{21}$$

Then we have

$$\mathfrak{Re}\left\{-\frac{zF'(z)}{F(z)}\right\} > 0 \quad (z \in \mathbb{D}). \tag{22}$$

Proof. Let us put

$$p(z) = \frac{zF'(z)}{F(z)} \qquad p(0) = -1.$$

Then we have

$$p(z)F(z) = zF'(z),$$

hence

$$\frac{zp'(z)}{p(z)} + \frac{zF'(z)}{F(z)} = 1 + \frac{zF''(z)}{F'(z)},$$

or

$$\frac{zp'(z)}{p(z)} - p(z) = 1 + \frac{zF''(z)}{F'(z)} - 2\frac{zF'(z)}{F(z)}.$$

By (21) it has positive real part and hence by Theorem 2.9 we get (21). □

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