Bull. Korean Math. Soc. **53** (2016), No. 1, pp. 205–214
http://dx.doi.org/10.4134/BKMS.2016.53.1.205

EMBEDDING OPEN RIEMANN SURFACES IN 4-DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. Any open Riemann surface has a conformal model in any orientable Riemannian manifold of dimension 4. Precisely, we will prove that, given any open Riemann surface, there is a conformally equivalent model in a prespecified orientable 4-dimensional Riemannian manifold. This result along with [5] now shows that an open Riemann surface admits conformal models in any Riemannian manifold of dimension ≥ 3.

1. Introduction

In 1999, the author [5] used Teichmüller theory to prove that, for given open Riemann surface $\mathbf{S}_0$, there exists a conformally equivalent model surface $\mathbf{S}$ in a prespecified orientable Riemannian manifold $\mathfrak{M}$ of dimension $\mathfrak{M} \geq 3$ except the partial proof for the embedding into 4-dimensional Riemannian manifold. In [5], the case of a 4-dimensional $\mathfrak{M}$ required the extra technical assumption that the normal bundle have a nowhere vanishing cross-section. In the present paper we remove this assumption and thus conclude, with [5], that an open Riemann surface now admits conformal embedding into *any* Riemannian manifold of dimension ≥ 3.

2. The main results

We will see in this paper that the methods used in the Ko's Embedding Theorems ([3, 4, 5, 6, 7, 8]) are even strong enough to prove this theorem for non-compact Riemann surfaces in 4-dimensional Riemannian manifold too.

$\mathfrak{C}^\infty$-embedded surfaces are called *classical surfaces* if they are viewed as Riemann surfaces whose conformal structure is given in the following natural way: the local coordinates are those which preserve angles and orientation.

Received January 26, 2015.

2010 *Mathematics Subject Classification.* 30F60.

Key words and phrases. embedding open Riemann surface, 4-dimensional Riemannian manifold, Teichmüller space, conformal deformation.

This paper was written as part of Konkuk University's research support program for its faculty on sabbatical leave in 2013.

In this paper, we follow carefully all the notations and arguments of [5], but the detailed expressions and computations may different because we define a new deformation function h, to prove:

Theorem 2.1 (Embedding Theorem)**.** *Assume that* $\mathbf{S}$ *is any Riemann surface,* $\mathfrak{C}^\infty$*-embedded in the orientable Riemannian manifold* $\mathfrak{M}$ *of dim* $\mathfrak{M} = 4$*. Let* $\mathbf{S}_0$ *be any Riemann surface structure on* $\mathbf{S}$*. Then* $\mathbf{S}_0$ *is conformally equivalent to a complete classical surface in* $\mathfrak{M}$*. A model can be constructed by deforming a given topologically equivalent complete Riemann surface* $\mathbf{S}$ *on each element in compact exhaustion of* $\mathbf{S}$ (*via the map* (4.2))*.*

We know that, in the above case, there are sections of the normal bundle of $\mathbf{S}$ in $\mathfrak{M}$ with isolated zeroes. (See Ko [6, Section 2.2] and the next paragraph for details on the sections of the normal bundle.) We consider this case only, otherwise the theorem is true [5].

Remark. It can be shown that if dim $\mathfrak{M} \neq 4$, then there always exists a nowhere vanishing section of the normal bundle N$\mathbf{S}$ of $\mathbf{S}$ in $\mathfrak{M}$ if $\mathbf{S}$ is compact. When dim $\mathfrak{M} = 4$, the nowhere vanishing section of the normal bundle N$\mathbf{S}$ exists if there are no obstructions. In this case the obstruction lies in the Euler class $e(\mathrm{N}\mathbf{S})$ of the normal bundle N$\mathbf{S}$. That is, if $e(\mathrm{N}\mathbf{S}) = 0$, then there is always such a section. For the proof see Ko [3, 6].

The argument now continues as in [5], Section 2. We need several supporting lemmas, especially Garsia's Continuity Lemma (Lemma 5.2 in [5]) and (revised) Brouwer's Fixed Point Lemma (Lemma 6.5 in [5]).

For the theory and the coordinate systems of Teichmüller space of a Riemann surface, we refer to [5], Section 4.

3. Outline of the proof

Let $\mathbf{S}$ be any Riemann surface $\mathfrak{C}^\infty$-embedded in the orientable Riemannian manifold $\mathfrak{M}$ of dim $\mathfrak{M} = 4$ and $\mathbf{S}_0$ be any Riemann surface structure on $\mathbf{S}$. We may assume that $\mathbf{S}$ and $\mathbf{S}_0$ are non-compact because the Embedding Theorem is known to be true, otherwise ([7, 8]). Then there exists a topological mapping $f : \mathbf{S}_0 \to \mathbf{S}$ by a consequence of the choice of $\mathbf{S}$ and $\mathbf{S}_0$. In terms of exhaustions, we do the following constructions. Since every Riemann surface admits a countable compact exhaustion by a subsurface, we may choose a regular exhaustion, that is, a sequence $\left\{\mathbf{S}_0^i\right\}$ on $\mathbf{S}_0$, of relatively compact regular subregions, such that $\overline{\mathbf{S}_0^i} \subset\subset \mathbf{S}_0^{i+1}$, $\cup \mathbf{S}_0^i = \mathbf{S}_0$ and $\partial \mathbf{S}_0^i$ consists of analytic arcs.

It is easy to show that $\mathbf{S}_0^i$ can be mapped by f_i topologically on a classical surface $\mathbf{S}_i$ such that $\partial \mathbf{S}_i$ consists of circles contained in ∂B_i, where, for $n =$ dim $\mathfrak{M}$,

$$B_i = \left\{ (x_1, x_2, \cdots, x_n) \;\middle|\; \sum_{j=1}^{n} x_j^2 \leq i^2 \right\}, \;\; \mathbf{S}_i \subset B_i, \;\; \mathbf{S}_{i+1} \cap B_i = \overline{\mathbf{S}_i}$$

with dim $B_i = 2 = \dim \mathbf{S} = \dim \mathbf{S}_0$, and $f_{i+1}|_{\mathbf{S}_0^i} = f_i$, $f = \lim f_i$ and $\mathbf{S} = \cup \mathbf{S}_i$ satisfy the above conditions ([11]).

We may assume that $\mathbf{S}_0^1$ is a disk. Let $p \in \mathbf{S}_0^1$ and $q \in \partial \mathbf{S}_0^1$ be distinguished points and put $p' = f(p)$, $q' = f(q)$ and $f(\mathbf{S}_0^i) = \mathbf{S}_i$. If $\mathbf{S}_0$ is simply connected, we introduce 4 distinguished points.

We will deform $\mathbf{S}$ in successive steps such that the i-th deformation ($i \geq 2$) takes place on $\mathbf{S}_i - \mathbf{S}_{i-1}$ only, and we will denote the resulting surface by $\mathbf{S}'$. Let $\mathbf{S}_i'$ be the part of $\mathbf{S}'$ corresponding to $\mathbf{S}_i$. We will show that $\mathbf{S}_0^i$ can be mapped conformally onto $\mathbf{S}_i'$ by a mapping f_i with the additional properties $f_i(p) = p',\ f_i(q) = q',\ i \geq 1$.

If this is accomplished, then we can derive our theorem that $\mathbf{S}_0$ and $\mathbf{S}$ are equivalent by [5, Lemma 3.1].

4. The existence of the functions f_i

Suppose $\mathbf{S}$ is any open Riemann surface, $\mathfrak{C}^\infty$-embedded in the orientable Riemannian manifold $\mathfrak{M}$ of dim $\mathfrak{M} = 4$. Let $\mathbf{S}_0$ be any Riemann surface structure on $\mathbf{S}$. Let $\{\mathbf{S}_0^i\}$ and $\{\mathbf{S}_i\}$ be exhaustions of $\mathbf{S}_0$ and $\mathbf{S}$ respectively. Assume that $\mathbf{S}_{i-1}$ is deformed into a surface $\mathbf{S}_{i-1}'$ such that conformal map $f_{i-1} : \mathbf{S}_0^{i-1} \to \mathbf{S}_{i-1}'$ with $f_{i-1}(p) = p'$ and $f_{i-1}(q) = q'$ exists. We are going to construct $\mathbf{S}_i'$ and f_i which is different from the function f_i in [5]. The existence of f_1 follows by Riemann's mapping theorem, the existence of $f_i,\ i \geq 2$, will be proved by induction.

Let $\mathbf{S}_i'' = (\mathbf{S}_i - \mathbf{S}_{i-1}) \cup \mathbf{S}_{i-1}'$. Let $\Gamma : \mathbf{S}_i'' \hookrightarrow \mathrm{N}\mathbf{S}_i'' \setminus \Gamma_0$ be a smooth section of the normal bundle $\mathrm{N}\mathbf{S}_i''$ of $\mathbf{S}_i''$ in $\mathfrak{M}$ which vanishes at some exceptional points, where Γ_0 is the zero section of $\mathrm{N}\mathbf{S}_i''$. Γ has a maximum length 1.

By Nash's Embedding Theorem ([10]), there is a $\mathfrak{C}^\infty$-isometric embedding $j : \mathfrak{M} \hookrightarrow \mathbb{R}^m$ for some sufficiently large m. This allows us to consider $\mathbf{S}$ and $\mathfrak{M}$ as subsets of $\mathbb{R}^m$.

We know that there are certain number of exceptional points z_j on $\mathbf{S}_i''$ where a section of the normal bundle $\mathrm{N}\mathbf{S}_i''$ vanishes, as we indicated in the Section 2. Fix a global coordinate $z \in \widetilde{\mathbf{S}_i''}$ such that $X : \widetilde{\mathbf{S}_i''} \to \mathbf{S}_i''$ is a conformal parametrization of $\mathbf{S}_i''$. Then the Riemann surface structure of $\mathbf{S}_i''$ may be viewed as induced by the metric $(dX)^2 = \lambda^2(z)|dz|^2$, where $\lambda(z)$ is a smooth real valued $(1,1)$-form.

Extend the function f_{i-1} to $\mathbf{S}_0^i$ such that the extended map $\xi : \mathbf{S}_0^i \to \mathbf{S}_i'' \subset \mathbb{R}^m$ is K-quasiconformal for a suitable K, $\mathfrak{C}^\infty$ except perhaps on $\partial \mathbf{S}_0^{i-1}$ and such that, for the complex dilatation $\mu(\xi) = \frac{\xi_{\bar z}}{\xi_z}$ of ξ,

$$\mu(\xi) = 0 \ \text{ on } \ \mathbf{S}_0^{i-1}, \quad \mu(\xi) = \frac{i}{2} \ \text{ on a disk } \ D \ \text{ in } \ \mathbf{S}_0^i - \mathbf{S}_0^{i-1}.$$

Such an extension is certainly possible (see Lehto and Virtanen [9, p. 92ff]).

By the previous constructions, $\mathbf{S}_0^{i-1} \to \mathbf{S}_{i-1}'$ is conformal. By the previous paragraph and the properties of the quasiconformal mappings, $\xi : \mathbf{S}_0^i \to \mathbf{S}_i''$ is a homeomorphism and so it has an inverse $g_0 : \mathbf{S}_i'' \to \mathbf{S}_0^i$ which is quasiconformal. Let $\mathcal{T}^{\#}(\mathbf{S}_i'')$ be a Teichmüller space of $\mathbf{S}_i''$, then it can be identified with the open unit ball $Q_1(\mathbf{S}_i'')$ in a normed linear vector space of symmetric holomorphic quadratic differentials on $\mathbf{S}_i''$.

Assume that $\omega \in \mathcal{T}^{\#}(\mathbf{S}_i'')$ is a local coordinate for a neighborhood of $[\mathrm{id}_{\mathbf{S}_i''}]$ in $\mathcal{T}^{\#}(\mathbf{S}_i'')$ provided $\|\omega\| = \iint_{\mathbf{S}_i''} |\omega| \leq 2\epsilon < 1$.

For later use, we define a surface $(\mathbf{S}_i'')_\omega$ as follows: For any $\omega = \phi_\omega(z)dz^2 \in \mathcal{T}^{\#}(\mathbf{S}_i'') \setminus \{0\}$, define a metric ds_ω^2 by

$$ds_\omega^2 := \lambda^2(z)\, |dz + \Psi_\omega(z) d\bar{z}|^2 \,, \tag{4.1}$$

where λ is a smooth real-valued $(1,1)$-form and

$$\Psi_\omega(z) = \begin{cases} \|\omega\| \frac{\overline{\phi_\omega(z)}}{|\phi_\omega(z)|} & \text{on} \quad \xi(D) \\ 0 & \text{outside} \quad \xi(D). \end{cases}$$

The metric (4.1) defines a new conformal structure on $\mathbf{S}_i''$ which will be denoted by $(\mathbf{S}_i'', ds_\omega^2) := (\mathbf{S}_i'')_\omega$.

If $g_\omega : \mathbf{S}_i'' \to (\mathbf{S}_i'')_\omega$ is a quasiconformal map and $[g_\omega] \in \mathcal{T}^{\#}(\mathbf{S}_i'')$, then we write $[g_\omega] = \omega$. Let $g_0 : \mathbf{S}_i'' \to \mathbf{S}_0^i$ be a homeomorphism so that $[g_0] \in \mathcal{T}^{\#}(\mathbf{S}_i'')$. Assume that $[g_0] = \omega_0 \in \mathcal{T}^{\#}(\mathbf{S}_i'')$ and denote by $B_\epsilon(\omega_0) \subset \mathcal{T}^{\#}(\mathbf{S}_i'')$ the set of elements in $\mathcal{T}^{\#}(\mathbf{S}_i'')$ with $\|\omega - \omega_0\| < \epsilon$.

Then Garsia's Continuity Lemma ([5], Lemma 5.2) and (revised) Brouwer's Fixed Point Lemma ([5], Lemma 6.5) follow.

Now we fix a map

$$h : \mathbf{S}_i'' \times \overline{B}_\epsilon(\omega_0) \to (-\epsilon, \epsilon)$$

so that h is a $\mathfrak{C}^\infty$-function with support on $\mathbf{S}_i - \mathbf{S}_{i-1}$ for each fixed ω. This h (which is different from the function h given in [5]) will be defined explicitly in Section 5.

Let the surface $(\mathbf{S}_i'')^\omega$ be the ϵ-normal deformation of $\mathbf{S}_i''$ defined by the map

$$(\mathfrak{S}_i'')^\omega := (\mathfrak{S}_i'')_{h(\cdot\,,\,\omega)} : \begin{array}{ccl} \mathbf{S}_i'' & \longrightarrow & \mathfrak{M} \subset \mathbb{R}^m \\ X(z) & \longmapsto & X(z) + h(X(z), \omega)\tilde{\Gamma}(X(z)) + O(h^2), \end{array} \tag{4.2}$$

where X is a local coordinate for $\mathbf{S}_i''$ and $\tilde{\Gamma}(X(z))$ is a unit tangent vector in $\mathbb{R}^m$ to the curve $\exp th(X(z))\Gamma(X(z))$ at the point $X(z)$. (For more details, see the Section 2 of [5].) Denote by $[(\mathfrak{S}_i'')^\omega] = [((\mathbf{S}_i'')^\omega, (\mathfrak{S}_i'')^\omega)]$ the conformal equivalence class of the surface $(\mathbf{S}_i'')^\omega$ as a marked surface $((\mathbf{S}_i'')^\omega, (\mathfrak{S}_i'')^\omega)$. We then define a map Ξ of $\overline{B}_\epsilon(\omega_0)$ to $\mathcal{T}^{\#}(\mathbf{S}_i'')$ by

$$\begin{array}{cccl} \Xi & : & \overline{B}_\epsilon(\omega_0) & \longrightarrow \mathcal{T}^{\#}(\mathbf{S}_i'') \\ & & \omega & \longmapsto [(\mathfrak{S}_i'')^\omega]. \end{array}$$

In addition, the function h will be so small that all the surfaces $(\mathbf{S}_i'')^\omega$ are embedded surfaces. Then as a consequence of (revised) Brouwer Fixed Point Lemma, we will have proved the existence of the conformal model if we can prove that, given $[g_0] = \omega_0$ and $\epsilon > 0$, for ω in the closed ball $\overline{B}_\epsilon(\omega_0) \subset Q_1(\mathbf{S}_i'')$

there is a family of deformations $(\mathbf{S}_i'')^\omega$ of $\mathbf{S}_i''$ depending on ω so that Lemma 6.3 (with the newly defined present deformation function h) of [5] is true.

By Lemma 6.3 of [5], Ξ satisfies the hypothesis of the (revised) Brouwer Fixed Point Lemma. Therefore there is a point $\omega_1 \in \overline{B}_\epsilon(\omega_0)$ so that

$$\Xi(\omega_1) = [(\mathfrak{S}_i'')^{\omega_1}] = \omega_0 = [g_0], \quad \text{where} \quad g_0 : \mathbf{S}_i'' \to \mathbf{S}_0^i.$$

This means that $\mathbf{S}_0^i$ can be mapped conformally onto the deformed surface $(\mathbf{S}_i'')^{\omega_1} = \mathbf{S}_i'$ by a conformal map, call it f_i, homotopic to $(\mathfrak{S}_i'')^\omega \circ \xi$ and satisfies the condition $f_i(p) = p'$, $f_i(q) = q'$.

5. The construction of the family $(\mathbf{S}_i'')^\omega$

As we said previously, we will deform $\mathbf{S}_i''$. Since the whole space we are considering here is $\mathbf{S}_i''$ (not $\mathbf{S}$), we take the fundamental domain P for $\mathbf{S}_i'' = (\mathbf{S}_i - \mathbf{S}_{i-1}) \cup \mathbf{S}_{i-1}'$ in $\widetilde{\mathbf{S}_i''}$. Let P_i (respectively P_{i-1}) be the fundamental domain for $\mathbf{S}_i$ (respectively $\mathbf{S}_{i-1}$). Then the fundamental domain for $\mathbf{S}_i - \mathbf{S}_{i-1}$ will be the domain $P_i - P_{i-1}$. We may assume that ∂P_i (respectively ∂P_{i-1}) has measure zero and hence $\partial(P_i - P_{i-1})$ has measure zero. We will construct a $\mathfrak{C}^\infty$ deformation function h non-vanishing only on $P_i - P_{i-1}$. In fact, we are assumed the deformation $\mathbf{S}_{i-1}'$ for $\mathbf{S}_{i-1}$ is already done, so the deformation function h for this part of $\mathbf{S}_i''$ needs to be zero. Therefore the deformation will actually take place only on $\mathbf{S}_i - \mathbf{S}_{i-1}$.

5.1. The metric ds^2_ω

The metric of the ϵ-normal deformation $(\mathbf{S}_i'')^\omega$, defined by the map $(\mathfrak{S}_i'')^\omega$ in equation (4.2), of $\mathbf{S}_i''$ satisfies the equation

$$(5.1) \qquad \begin{aligned} (d(\mathfrak{S}_i'')^\omega)^2 &= (dX)^2 + (dh)^2 + o(h)|dz|^2 \\ &= \lambda^2(z)|dz|^2 + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy\right)^2 + o(h)|dz|^2. \end{aligned}$$

Let $\chi : (\mathbf{S}_i'')_\omega \to (\mathbf{S}_i'')^\omega$ be a mapping of $(\mathbf{S}_i'')_\omega$ onto $(\mathbf{S}_i'')^\omega$. Let ds^2_ω, given by (4.1), and $(d(\mathfrak{S}_i'')^\omega)^2$, be metrics for $(\mathbf{S}_i'')_\omega$ and $(\mathbf{S}_i'')^\omega$ respectively. We want to show that the dilatation K_χ (will be given in Lemma 5.1) of χ satisfies the hypotheses of Garsia's Continuity Lemma ([5], Lemma 5.2). It will be helpful to split ds^2_ω into the form given in (5.1). For fixed choice of global coordinate z in $\widetilde{\mathbf{S}_i''}$, ω uniquely defines $\phi_\omega(z)$. Let

$$(5.2) \qquad \Pi(\omega) = \left\{ z \in \widetilde{\mathbf{S}_i''} \ \middle|\ \Im\phi_\omega(z) \neq 0 \right\}.$$

The metric ds^2_ω is smooth on the set $\Pi(\omega)$. Let

$$(5.3) \qquad \gamma_\omega := (1 - \|\Psi_\omega\|_\infty)^{-2} = (1 - \|\omega\|)^{-2}.$$

Then define the following real valued functions on $\Pi(\omega)$.

$$\begin{aligned}\alpha_\omega^2 &:= 2\gamma_\omega(\|\Psi_\omega\|_\infty + \Re\Psi_\omega(z)) = 2\gamma_\omega(\|\omega\| + \Re\Psi_\omega(z)),\\ \beta_\omega^2 &:= 2\gamma_\omega(\|\Psi_\omega\|_\infty - \Re\Psi_\omega(z)) = 2\gamma_\omega(\|\omega\| - \Re\Psi_\omega(z)).\end{aligned} \tag{5.4}$$

On each connected component of the set $\Pi(\omega)$, choose continuous (real) branches of $\alpha_\omega, \beta_\omega$ so that

$$\operatorname{sign}(\alpha_\omega\beta_\omega) = \operatorname{sign}(\Im\Psi_\omega(z)) \quad \text{and} \quad \beta_\omega > 0.$$

Since $dz^2 = dx^2 - dy^2 + 2idxdy$ and $d\bar{z}^2 = dx^2 - dy^2 - 2idxdy$, we get

$$\gamma_\omega ds_\omega^2 = \lambda^2(z)\left(|dz|^2 + (\alpha_\omega dx + \beta_\omega dy)^2\right). \tag{5.5}$$

5.2. The deformation function h

To complete Theorem 2.1, we need to describe a function h on $\mathbf{S}_i''$ satisfying the following properties:

(1) h is $\mathfrak{C}^\infty$.
(2) $\|h\|_\infty < \epsilon$.
(3) $(dh)^2 \approx (\alpha_\omega dx + \beta_\omega dy)^2$ in view of equations (5.1) and (5.5).

We would like to define a function h satisfying condition 3 except on a sufficiently small set. But dh remains bounded on this set. Condition 3 suggests that we express $(dh)^2$ in terms of α_ω and β_ω. On general Riemann surfaces, α_ω and β_ω must be non-constant functions of z. The definition of h will come as a solution of a differential equation in which α_ω, β_ω and their derivatives appear as coefficients. In order to get a $\mathfrak{C}^\infty$ solution, we need α_ω, β_ω to be smooth on all of P, that is on $\widetilde{\mathbf{S}_i''}$. Also they, together with their derivatives, must change as little as possible. For h to be well-defined on $\mathbf{S}_i''$, it is convenient that it be zero on P_{i-1} and in a neighborhood of the edges of $P_i - P_{i-1}$ (and neighborhoods of exceptional points) but remains smooth.

In this section, we will eventually construct the deformation function h in terms of $\lambda(z)$, $\alpha_\omega(z)$, $\beta_\omega(z)$ and some large number N.

For this purpose, we need to extend the functions α_ω and β_ω on whole of P since they are not defined on $P \setminus \Pi(\omega)$. This work has been done in [5], Section 3.2. And in [5], Sections 7.2.1 to 7.2.3, we set them as $\tilde{\alpha}_\omega$ and $\tilde{\beta}_\omega$, and, for a compact subset F in $Q_1(\mathbf{S}_i'')$ not containing 0, constructed several other auxiliary functions on $\Delta \times F$ such as the real-valued continuous functions $\mu_\eta(z,\omega)$ and $u(z,\omega)$, maximum value u_0 of $|u(z,\omega)|$, $\gamma(z,\omega) = e^{u_0 - u(z,\omega)}$, an exact function $\varrho(z,\omega) = \gamma(z,\omega)(\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)$ and a differentiable function $k(z,\omega)$ with $|k| \le k_0$ satisfying $\varrho = dk$.

Next we define a function to take care of some (as noted in the Section 2) fixed exceptional points on $\mathbf{S}_i''$ (actually, we work on exceptional points on $\mathbf{S}_i - \mathbf{S}_{i-1}$ since we already deformed a surface $\mathbf{S}_{i-1}$) where the section Γ of the normal bundle $\mathrm{N}\mathbf{S}_i''$ vanishes. Let, for $j = 1, \dots, r$, z_j be fixed exceptional points on $P_i - P_{i-1}$ and $E_j := E_{\delta_j}(z_j)$ be a small neighborhood of z_j so that the area $E \cap (P_i - P_{i-1}) < l_E \cdot \eta$, where $E = \cup_{j=1}^r E_j$ and l_E is a small constant($< \frac{1}{4}$)

depending on E. We define a real-valued $\mathfrak{C}^\infty$-function (Beltrami differential on $\widetilde{\mathbf{S}_i''}$) $\upsilon(x,y)$ ($\|\upsilon\|_\infty < 1$) on P so that its support lies in the complement of the set E. (This can be done, using a theorem of Bers, by defining a $\mathfrak{C}^\infty$-function (Beltrami differential $\upsilon_j(x,y)$ ($\|\upsilon_j\|_\infty < 1$)) having a support on a complement of each E_j and multiplying them all. See [1] for more information.) Let I_j be a small neighborhood containing $\overline{E_j}$ (closure of E_j), $j = 1, \dots, r$, with the area $I \cap (P_i - P_{i-1}) < l_I \cdot \eta$, where $I = \cup_{j=1}^r I_j$ and l_I is a small constant($< \frac{1}{4}$) depending on $I \supset \overline{E}$. We define a function $\mho(x,y)$ on P using $\upsilon(x,y)$ as follow so that it is $\mathfrak{C}^\infty$ on P:

$$\mho(x,y) = \begin{cases} 0 & \text{for } (x,y) \in E \cup [\widetilde{\mathbf{S}_i''} - (P_i - P_{i-1})] \\ 1 & \text{for } (x,y) \in P_i - P_{i-1} - I \quad (\text{where } I \supset \overline{E}). \end{cases} \tag{5.6}$$

On $I - E$, we may define any $\mathfrak{C}^\infty$-function (since it does not really matter which form we use as you may see in Lemma 5.2 as long as its sup norm is less than 1 and it is expressed) in terms of $\upsilon(x,y)$ so that $\mho$ is $\mathfrak{C}^\infty$ and $\|\mho\|_\infty \le 1$ on whole P, that is, on whole $\mathbf{S}_i''$.

As a final auxiliary function, we define a real-valued function $\nu_\eta(x)$ for $\eta < \frac{1}{16}$ as follow.

(1) $|\dot{\nu_\eta}(x)| \le 1$,

(2) $\nu_\eta(x) = \begin{cases} x & \text{for } \eta - 1 \le x \le 1 - \eta \\ 2 - x & \text{for } 1 + \eta \le x \le 3 - \eta, \end{cases}$

(3) $\nu_\eta(x+4) = \nu_\eta(x)$.

Let F be a compact subset in $Q_1(\mathbf{S}_i'')$ which does not contain 0.

For $(x,y) \in \widetilde{\mathbf{S}_i''}$ (i.e., P) and $\omega \in F$, define h by

$$h(x,y,\omega,N) = \frac{1}{N}\lambda(x,y)\mho(x,y)\mu_\eta(x,y,\omega)\frac{1}{\gamma(z,\omega)} \cdot \nu_\eta(N \cdot k(x,y,\omega)), \tag{5.7}$$

where N (which will be determined at the end of this section) is a sufficiently large natural number depending on F and ϵ (which will be determined in the Section 5.4 and it will guarantee the existence of ϵ-normal deformation surface $(\mathbf{S}_i'')^\omega$. Refer to Theorem 2.1 of [6], Section 2). Then for each N, h is a $\mathfrak{C}^\infty$-function on $\widetilde{\mathbf{S}_i''}$ having support on $P_i - P_{i-1}$ and continuous on $\widetilde{\mathbf{S}_i''} \times F$ and we have

$$dh^2 = \lambda^2 \cdot \mho^2 \cdot \mu_\eta^2 \cdot \dot{\nu_\eta}^2(Nk)\frac{1}{\gamma^2(z,\omega)}(dk)^2 + o(\frac{1}{N})|dz|^2.$$

Except on a small set A of $P_i - P_{i-1}$ (in fact, on the set $\widetilde{\mathbf{S}_i''} - (P_i - P_{i-1})$, we have $h = 0$, so that $dh^2 = 0$), this reduces to

$$dh^2 = \lambda^2(x,y)(\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)^2 + o(\frac{1}{N})|dz|^2, \tag{5.8}$$

where A is given by

$$A = \left\{(x,y) \in P_i - P_{i-1} \mid \mu_\eta^2 \cdot \mho^2 \cdot \dot{\nu_\eta}^2(N \cdot k(x,y,\omega)) \ne 1\right\}. \tag{5.9}$$

Let A_1 be the set

$A_1 = \{(x,y) \in P_i - P_{i-1} \,|\, \mu_\eta^2(x,y,\omega) \neq 1\} \equiv \{(x,y) \in P_i - P_{i-1} \,|\, \mu_\eta(x,y,\omega) \neq 1\}$ (since $0 \leq \mu_\eta(x,y,\omega) \leq 1$), then A_1 has an area ([5], Section 7.2.2)

$$\operatorname{area} A_1 < \frac{\eta}{2}.$$

Let

$$A_2 = \{(x,y) \in P_i - P_{i-1} \,|\, \mho^2(x,y) \neq 1\}$$

and

$$A_3 = \{(x,y) \in P_i - P_{i-1} \mid \dot{\nu_\eta}^2(Nk(x,y,\omega)) \neq 1\},$$

then it becomes $A = A_1 \cup A_2 \cup A_3$. Since we know that

$$\operatorname{area} A_2 = (P_i - P_{i-1}) \cap I < l_I \cdot \eta,$$

we only need to compute the area of the set A_3. But A_3 becomes

$$A_3 = \left\{ (x,y) \in P_i - P_{i-1} \mid \; \left| k(x,y,\omega) - \frac{1}{N} \right| \leq \frac{\eta}{N} \left(\operatorname{mod} \frac{2}{N} \right) \right\}.$$

A_3 has an area ([5], Section 7.2.3)

$$\operatorname{area}(A_3) = \int_{\Phi_\omega(A_3)} |\det(D\Phi_\omega^{-1})| \, dxdy \leq \frac{1}{\sigma} \operatorname{area}(\Phi_\omega(A_3)) < l_F \cdot \eta \tag{5.10}$$

if $l_F > \frac{4k_0}{\sigma}$, where $e^{u_0 - u} \cdot \tilde{\beta}_\omega \geq \sigma$ for all $(x,y,\omega) \in (P_i - P_{i-1}) \times F$.

So finally we obtain

$$\operatorname{area} A = \operatorname{area}(A_1 \cup A_2 \cup A_3) < \left(\frac{1}{2} + l_I + l_F \right) \eta. \tag{5.11}$$

Here we take $N > 4(l_F\sigma - 4k_0) + \frac{1}{\epsilon} \max_{z \in P} |\lambda(z)|$, so that for this N the inequality (5.10) is true.

5.3. Comparison of the metrics $(d(\mathfrak{S}_i'')^\omega)^2$ and ds_ω^2

Recall that the deformed surface $(\mathbf{S}_i'')^\omega$ is defined by (4.2). Then for K_χ^2, we will get:

Lemma 5.1. *Assume that $h(x,y,\omega,N)$ is given by the formula (5.7) and that the supremum and the infimum are taken over all directions at a point z. Then the metric of the deformed surface $(\mathbf{S}_i'')^\omega := (\mathbf{S}_i'')_{h(\cdot,\omega)}$, defined by the map $(\mathfrak{S}_i'')^\omega(x,y)$ as given in the equation (4.2), satisfies the relations:*

(1) $\left(\sup\left((d(\mathfrak{S}_i'')^{\omega_m})^2/(d(\mathfrak{S}_i'')^\omega)^2\right)\right) / \left(\inf\left((d(\mathfrak{S}_i'')^{\omega_m})^2/(d(\mathfrak{S}_i'')^\omega)^2\right)\right) \longrightarrow 1$ *as* $\omega_m \to \omega$.

(2) $$K_\chi^2 = \left(\sup\left((d(\mathfrak{S}_i'')^\omega)^2/ds_\omega^2\right)\right) / \left(\inf\left((d(\mathfrak{S}_i'')^\omega)^2/ds_\omega^2\right)\right) \leq \begin{cases} 1 + c_1(\eta;N) & \text{on } P - A,\ \omega \in F \\ 4\gamma_\omega + c_2(\eta;N) & \text{on } A,\ \omega \in F \end{cases}$$

if $\omega \in F$, where the constant c_1 can be made arbitrarily small for each fixed η and for sufficiently large N, c_2 is some constant which is not necessarily small. The area of A is given in (5.11).

Remark. On P_{i-1}, since $h = 0$ and $ds_\omega^2 = \lambda^2(z)|dz|^2$, $(d(\mathfrak{S}_i'')^\omega)^2 = \lambda^2(z)|dz|^2$ so that (1) and (2) of Lemma 5.1 follows immediately. Therefore we need to consider all computations on $P_i - P_{i-1}$ only.

In view of the above Remarks, to prove Lemma 5.1, we have to consider the following lemmas.

Lemma 5.2. *Given h as in equation* (5.7)*, and $\gamma_\omega ds_\omega^2$ as in* (5.5)*,*

$$|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds_\omega^2| \leq \begin{cases} R(\eta; N)ds_\omega^2 & \text{on } P_i - P_{i-1} - A,\ \omega \in F \\ \tilde{R}(\eta; N)ds_\omega^2 & \text{on } A,\ \omega \in F, \end{cases}$$

where area A is given in (5.11)*. The inequalities are valid for $N > N_F + \frac{1}{\epsilon} \cdot \max_{z\in P}|\lambda(z)|$, where $N_F > 4(l_F\sigma - 4k_0)$ with l_F (given in* (5.10)*) a constant depending on the compact set F. For each fixed η, $R(\eta; N)$ can be made small as $N \to \infty$ and $\tilde{R}(\eta; N)$ is some constant which is bounded as a function of N.*

Proof. Use the equations (5.8) and (5.5) to obtain

$$\begin{aligned} &|dh^2 + \lambda^2|dz|^2 - \gamma_\omega ds_\omega^2| \\ &= \left|\lambda^2 \cdot (\tilde{\alpha}_\omega dx + \tilde{\beta}_\omega dy)^2(\mho^2\mu_\eta^2\dot{\nu}_\eta^2 - 1) + o(\frac{1}{N})|dz|^2\right|. \end{aligned} \tag{5.12}$$

On $P_i - P_{i-1} - A$, we have $\mu_\eta^2 = \dot{\nu}_\eta^2 = 1$ and $\mho^2(x, y) = 1$ (see the above equation (5.9) and property (2) of the function μ_η in [5], Section 7.2.2), so the right hand side of the equation (5.12) becomes

$$\left|o(\frac{1}{N})|dz|^2\right| \leq R(\eta; N)ds_\omega^2$$

for some small constant $R(\eta; N)$.

On A, since $\mu_\eta^2 \cdot \dot{\nu}_\eta^2 \neq 1$ or $\mho^2(x, y) \neq 1$, the right hand side (RHS′) of the equation (5.12) becomes

$$\text{RHS}' \leq \left|\gamma_\omega \left[(\mho^2 \cdot \mu_\eta^2 \cdot \dot{\nu}_\eta^2 - 1) + o(\frac{1}{N})\right] ds_\omega^2\right| \leq \tilde{R}(\eta; N)ds_\omega^2 \tag{5.13}$$

for some constant $\tilde{R}(\eta; N)$ which is not necessarily very small. □

Lemma 5.3. *Given $h(x, y, \omega, N)$ as in the equation* (5.7)*, the metric of the deformed surface $(\mathbf{S}_i'')^\omega$ defined by the equation* (4.2) *satisfies the inequality*

$$\left|(d(\mathfrak{S}_i'')^\omega)^2 - dh^2 - dX^2\right| \leq c(\eta; N)ds_\omega^2$$

for each fixed η and $\omega \in F$, where $c(\eta; N) \to 0$ as $N \to \infty$.

Proof. Apply the proof of Lemma 7.43 of [5] to the h given in equation (5.7). □

To prove Lemma 5.1, we now apply Lemmas 5.2 and 5.3 using the same arguments, with h given in (5.7), in the proof of Lemma 7.38 of [5].

5.4. Final words

Thus far we have checked every condition we need in the hypotheses of Garsia's Continuity lemma for some compact set F in $\mathcal{T}^{\#}(\mathbf{S}_i'')$. Therefore if we take $\epsilon = \frac{1}{2}\min\{1 - \|\omega_0\|, \|\omega_0\|\}$ and $F = \overline{B_\epsilon(\omega_0)} \subset \mathcal{T}^{\#}(\mathbf{S}_i'') \setminus \{0\}$, then we may now complete the process in the Section 4.

Acknowledgement. I appreciate referees for their helpful comments and careful reading.

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