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ELEMENTARY MATRIX REDUCTION OVER ZABAVSKY RINGS

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Abstract. We prove, in this note, that a Zabavsky ring R is an elementary divisor ring if and only if R is a Bézout ring. Many known results are thereby generalized to much wider class of rings, e.g. [4, Theorem 14], [7, Theorem 4], [9, Theorem 1.2.14], [11, Theorem 4] and [12, Theorem 7].

1. Introduction

Throughout this paper, all rings are commutative with an identity. A matrix A (not necessarily square) over a ring R admits diagonal reduction if there exist invertible matrices P and Q such that PAQ is a diagonal matrix (d_{ij}), for which d_{ii} is a divisor of $d_{(i+1)(i+1)}$ for each i. A ring R is called an elementary divisor ring provided that every matrix over R admits a diagonal reduction. A ring R is a Hermite ring if every 1×2 matrix over R admits a diagonal reduction. As is well known, a ring R is Hermite if and only if for all $a, b \in R$ there exist $a_1, b_1 \in R$ such that $a = a_1 d, b = b_1 d$ and $a_1 R + b_1 R = R$ ([9, Theorem 1.2.5]). A ring is a Bézout ring if every finitely generated ideal is principal. In 1956, Gillman and Henriksen gave an example of a Bézout ring (with zero divisors) that is not an elementary divisor ring. In fact, we have $\{$elementary divisor rings$\} \subsetneq \{$Hermite rings$\} \subsetneq \{$Bézout rings$\}$ (cf. [9]). An attractive problem is to investigate various conditions under which a Bézout ring is an elementary divisor ring.

We recall that an element $c \in R$ is adequate provided that for any $a \in R$ there exist some $r, s \in R$ such that (1) $c = rs$; (2) $rR + aR = R$; (3) $s'R + aR \neq R$ for each non-invertible divisor s' of s. Whether a ring with various adequate properties is an elementary divisor ring has been studied by many authors. A ring R is clean provided that every element in R is the sum of a unit and an idempotent. In [11, Theorem 4], Zabavsky and Bilavska proved an interesting result: every Bézout ring in which zero is adequate is a clean ring. Recently,

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Pihua claimed that every ring in which zero is adequate is semiregular [7, Theorem 4], and so such kind of ring is an elementary divisor ring [9, Theorem 2.5.2]. A Bézout ring is called an adequate ring provided that every nonzero element is adequate (cf. [10]). In his research of elementary divisor domains, Helmer proved that every adequate domain is an elementary divisor ring. After his work, Kaplansky showed that an adequate ring whose zero divisors are in the radical is an elementary divisor ring. Helmer also showed that an adequate ring is an elementary divisor ring if and only it is a Hermite ring. For general results about adequate conditions, we refer the reader to Zabavsky's book [9].

Recall that a ring R has stable range 1 if $aR+bR=R$ with $a,b\in R$ implies that there exists a $y\in R$ such that $a+by\in R$ is invertible. Such condition plays an important role in algebraic K-theory (cf. [2]). It includes many kind of rings, e.g., regular rings, semiregular rings, π-regular rings, local rings, clean rings, etc. Domsha and Vasiunyk combined this condition with adequate condition together. A ring R is called to have adequate range 1 if $aR+bR=R$ with $a,b\in R$ implies that there exists a $y\in R$ such that $a+by\in R$ is adequate. It was proved that every Bézout domain having adequate range 1 is an elementary divisor ring [4, Theorem 14].

In this note, we present a new type of rings over which every matrix admits an elementary diagonal reduction. We say that $c\in R$ is feckly adequate if for any $a\in R$ there exist some $r,s\in R$ such that (1) $c\equiv rs \pmod{J(R)}$; (2) $rR+aR=R$; (3) $s'R+aR\neq R$ for each non-invertible divisor s' of s. We call a ring R is a Zabavsky ring provided that $aR+bR=R$ implies that there exists a $y\in R$ such that $a+by\in R$ is feckly adequate. We prove, in this note, that a Zabavsky ring R is an elementary divisor ring if and only if R is a Bézout ring. Many known results are thereby generalized to much wider class of rings, e.g. [4, Theorem 14], [7, Theorem 4], [9, Theorem 1.2.14], [11, Theorem 4] and [12, Theorem 7].

We shall use $J(R)$ and $U(R)$ to denote the Jacobson radical of R and the set of all units in R, respectively. A ring R is called a domain if there is no any nonzero zero divisor of R.

2. Nearly adequate rings

A Bézout ring R is called a nearly adequate ring provided that zero is feckly adequate in R. For the further use, we investigate the necessary and sufficient conditions under which a ring R is nearly adequate. A ring R is regular if for any $a\in R$ there exists a $b\in R$ such that $a=aba$. A ring R is π-regular if for any $a\in R$ there exists $n\in\mathbb{N}$ such that $a^n=a^nba^n$ for some $b\in R$. We prove, a Bézout ring R is nearly adequate if and only if $R/J(R)$ is regular if and only if $R/J(R)$ is π-regular. Examples of nearly adequate rings in which zero is not adequate are provided. We begin with:

Lemma 2.1. *Let R be a ring. Then $R/J(R)$ is π-regular if and only if for any $a \in R$, there exists an $n \in \mathbb{N}$, an element $e \in R$, a unit $u \in R$ and a $w \in J(R)$ such that $a^n = eu + w$ and $e - e^2 \in J(R)$.*

Proof. ($\Longrightarrow$) Let $a \in R$. Then $\overline{a^n} = \overline{a^n b a^n}$ for some $n \in \mathbb{N}$. Set $e = a^n b$ and $u = 1 - a^n b + a^n$. Then $\overline{e^2} = \overline{e} \in R/J(R)$ and $\left(\overline{u}\right)^{-1} = \overline{1 - a^n b + ba^n b}$ in $R/J(R)$. As units lift modulo $J(R)$, we see that $u \in U(R)$. Set $w := a^n - eu$. Then we obtain $a^n = eu + w$, where $e^2 - e, w \in J(R)$.

($\Longleftarrow$) For any $a \in R$, there exists an $n \in \mathbb{N}$, an element $e \in R$, a unit $u \in R$ and a $w \in J(R)$ such that $a^n = eu + w$ and $e - e^2 \in J(R)$. Hence, $\overline{a^n} = \overline{eu}$ in $R/J(R)$. Therefore $\overline{a^n} = \overline{a^n u^{-1} a^n}$, as required. $\square$

Lemma 2.2. *Let R be a Bézout ring. If $R/J(R)$ is π-regular, then R is nearly adequate.*

Proof. Suppose that $R/J(R)$ is π-regular. Let $a \in R$ be an arbitrary element. In light of Lemma 2.1, there exists an element $e \in R$, a unit $u \in R$ and a $w \in J(R)$ such that $a^n = eu + w(n \in \mathbb{N})$ and $e - e^2 \in J(R)$. Then $(1-e)e \in J(R)$. Clearly, $a^n u^{-1} + (1-e) = 1 + wu^{-1} \in U(R)$, and so $a^n u^{-1}\left(1 + wu^{-1}\right)^{-1} + (1-e)\left(1 + wu^{-1}\right)^{-1} = 1$. Hence, $(1-e)R + a^n R = R$. This implies that $(1-e)R + aR = R$. If s is a non-invertible divisor of e, then $e = ss'$ for some $s' \in R$. If $sR + aR = R$, then $sR + a^n R = R$. Thus, we can find some $x, y \in R$ such that $sx + a^n y = 1$. Hence, $sx + (eu + w)y = 1$, and so $s(x + s'uy) = 1 - wy \in U(R)$. This implies that s is invertible, an absurdity. Therefore $sR + aR \neq R$. Accordingly, R is nearly adequate. $\square$

Recall that a ring is feckly clean provided that for any $a \in R$ there exists an element $e \in R$ such that $a - e \in U(R)$ and $e - e^2 \in J(R)$. As is well known, a ring R is feckly clean if and only if $aR + bR = R$ with $a, b \in R$ implies that there are $x, y \in R$ such that $a|x, b|y, xy \in J(R)$ and $xR + yR = R$, if and only if $\mathrm{Max}(R)$ is zero-dimensional ([6, Theorem 3.13 and Proposition 3.12]). For more topological characterizations of such type of rings, we refer the reader to [3].

Lemma 2.3. *Every nearly adequate ring is feckly clean.*

Proof. Let R be a nearly adequate ring. Let $x \in R$. Then we have some $r, s \in R$ such that $rs \in J(R)$, where $rR + xR = R$ and $s'R + xR \neq R$ for any noninvertible divisor s' of s. We claim that $rR + sR = R$. If not, $rR + sR = hR \neq R$. Thus, $h \in R$ is a noninvertible divisor of s; hence that $hR + xR \neq R$. But h is a divisor of r, we get $hR + xR = R$, which is impossible. Write $rc + sd = 1$ in R. Then $(rc)^2 - rc = (rc)^2 - rc\left(rc + sd\right) = -(rs)(cd) \in J(R)$. Set $e = rc$. Then $e^2 - e \in J(R)$.

Claim I. $(x-e)R + rR = R$. If not, $(x-e)R + rR = tR \neq R$. Then $rR \subseteq tR$, and so $xR \subseteq tR$. This implies that $rR + xR \subseteq tR \neq R$, a contradiction.

Claim II. $(x-e)R+sR=R$. If not, $(x-e)R+sR=tR\neq R$. Then t is a noninvertible divisor of s, and so $tR+xR\neq R$. But $eR+sR=R$, and so $eR+tR=R$. Write $x-e=tw$ with $w\in R$. Then $e=x-tw$, and so $eR+tR\subseteq tR+xR\neq R$, which is impossible.

Therefore $(x-e)R+rsR=R$. Write $(x-e)p+(rs)q=1$ for some $p,q\in R$. As $rs\in J(R)$, we deduce that $(x-e)p=1-(rs)q\in U(R)$, and so $x-e\in U(R)$. This completes the proof. □

Lemma 2.4. *Let R be a nearly adequate ring. Then $J(R)=\{x\mid x-u\in U(R)$ for any $u\in U(R)\}$.*

Proof. Clearly, $J(R)\subseteq\{x\mid x-u\in U(R)$ for any $u\in U(R)\}$. Let $x\in R$ and $x-u\in U(R)$ for any $u\in U(R)$. Let $r\in R$. Then $xR+(1-xr)R=R$. Since R is nearly clean, by Lemma 2.3, $R/J(R)$ is clean, and then $R/J(R)$ has stable range 1 by [6, Theorem 3.10]. It follows that R has stable range 1. Thus, we have a $y\in R$ such that $u:=x+(1-xr)y\in U(R)$. Hence, $x-u=-(1-xr)y\in U(R)$, and then $1-xr\in U(R)$. Therefore $x\in J(R)$, as desired. □

We are now ready to prove:

Theorem 2.5. *Let R be a Bézout ring. Then the following are equivalent*:

(1) *R is nearly adequate.*
(2) *$R/J(R)$ is regular.*
(3) *$R/J(R)$ is π-regular.*

Proof. (1) ⇒ (2) Let $x\in R$. Then we have some $r,s\in R$ such that $rs\in J(R)$, where $rR+xR=R$ and $s'R+xR\neq R$ for any noninvertible divisor s' of s. As in the proof of Lemma 2.3, we see that $rR+sR=R$. Since $rR+xR=rR+sR=R$, we get $rR+sxR=R$. Write $rc+sxd=1$ in R. Set $e=rc$. Then $e^2-e=(rc)^2-rc=-(rc)(sxd)\in J(R)$. Let u be an arbitrary invertible element of R.

Claim I. $(u-ex)R+rR=R$. If not, $(u-ex)R+rR=tR\neq R$. Then $rR\subseteq tR$, and so $u\in eR+tR\subseteq rR+tR\subseteq tR$. This implies that $t\in U(R)$, a contradiction.

Claim II. $(u-ex)R+sR=R$. If not, $(u-ex)R+sR=tR\neq R$. Then t is a noninvertible divisor of s, and so $tR+xR\neq R$. It follows from $eR+sR=R$ that $eR+tR=R$. Write $u-ex=tw$ with $w\in R$. Then $u=ex+tw$, and so $uR\subseteq tR+xR\neq R$, an absurdity.

Finally, $(u-ex)R+rsR=R$. As $rs\in J(R)$, we get $u-ex\in U(R)$. This implies that $\big(x-(1-e)x\big)-u=ex-u\in U(R)$. In view of Lemma 2.3, R has stable range 1. It follows by Lemma 2.4 that $x-(1-e)x\in J(R)$. Clearly, $1-e=sxd\in xR$. Therefore $\overline{x}=\overline{x(sd)x}$ in $R/J(R)$, and so $R/J(R)$ is regular.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) In view of Lemma 2.2, R is nearly adequate, as asserted. □

Corollary 2.6. *Every nearly adequate ring is an elementary divisor ring.*

Proof. Let R be a nearly adequate ring. Then $R/J(R)$ is regular, by Theorem 2.5. In view of [5, Theorem 2.6], $R/J(R)$ is an elementary divisor ring. Therefore R is an elementary divisor ring, in terms of [9, Theorem 2.5.2]. $\square$

Corollary 2.7. *A ring R is nearly adequate if and only if*

(1) *R is a Bézout ring*;
(2) *Zero is adequate in $R/J(R)$.*

Proof. ($\Longrightarrow$) (1) is obvious. In view of Theorem 2.5, $R/J(R)$ is regular, proving (2), as every element in $R/J(R)$ is adequate.

($\Longleftarrow$) For any $x \in R$, we can find some $r, s \in R$ such that $\overline{0} = \overline{rs}$, where $\overline{r}\big(R/J(R)\big) + \overline{x}\big(R/J(R)\big) = R/J(R)$ and $\overline{s'}\big(R/J(R)\big) + \overline{x}\big(R/J(R)\big) \neq R/J(R)$ for any noninvertible divisor s' of s. Thus, $rs \in J(R)$. Further, $rR + xR = R$ and $s'R + xR \neq R$ for any noninvertible divisor s' of s. Therefore R is nearly adequate. $\square$

We note that " $\Rightarrow$ " in Corollary 2.7 can not be proved in a direct route by the definitions. Let $R = \mathbb{Z}[\alpha]$, where $\alpha^2 = 1$. Choose $J = (1+\alpha), s' = 5 - 3\alpha, s = 3 + \alpha \in R$. Then $\overline{s'} = 2$ is a noninvertible divisor of $\overline{s} = 4$ in R/J, while s' is not a noninvertible divisor of s in R.

Recall that every idempotent lifts modulo a right ideal I of R provided that if $x - x^2 \in I$, then there exists an idempotent $e \in R$ such that $x - e \in I$. We have:

Corollary 2.8. *Let R be a Bézout ring. Then zero is adequate in R if and only if*

(1) *R is nearly adequate*;
(2) *Every idempotent lifts modulo $J(R)$.*

Proof. ($\Longrightarrow$) (1) is obvious. In view of [11, Theorem 4], R is clean, proving (2), by the Nicholson Theorem (i.e., a ring is clean if and only if every idempotent lifts modulo its any right ideal).

($\Longleftarrow$) In view of Theorem 2.5, $R/J(R)$ is regular. Since every idempotent lifts modulo $J(R)$, as in the proof of Lemma 2.2, zero is adequate in R. $\square$

A ring R is semiregular if $R/J(R)$ is regular and idempotents lift modulo $J(R)$. We now derive:

Corollary 2.9 ([7, Theorem 4])**.** *Let R be a Bézout ring. Then zero is adequate in R if and only if R is semiregular.*

Proof. This is obvious, by Corollary 2.8 and Theorem 2.5. $\square$

From Corollary 2.9, we observe that the difference between nearly adequate rings and adequate rings is just to forget the lifting of idempotents. Further, we claim that a Bézout ring R is a nearly adequate ring if and only if $R/J(R)$ is nearly adequate.

Example 2.10. Every finite Bézout ring is nearly adequate.

Proof. Since every finite ring is π-regular, we complete the proof, by Theorem 2.5. $\square$

Example 2.11. Let $R = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0, 3 \nmid n, 5 \nmid n\}$. Then R is nearly adequate, while zero is not adequate in R.

Proof. Let $I = \frac{a}{b}R + \frac{c}{d}R$, where $\frac{a}{b}, \frac{c}{d} \in R$. As $\mathbb{Z}$ is a principal ideal domain, we can find some $p \in \mathbb{Z}$ such that $a\mathbb{Z} + c\mathbb{Z} = p\mathbb{Z}$. One easily checks that $I = pR$. Thus, R is a Bézout ring. As in the proof of [1, Example 17], R has only two maximal ideals $3R$ and $5R$. Since $3R + 5R = R$, by Chinese Remainder Theorem, we deduce that $R/J(R) \cong R/3R \times R/5R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$. As $\mathbb{Z}_3$ and $\mathbb{Z}_5$ are regular rings, $R/J(R)$ is regular. It follows by Theorem 2.5 that R is nearly adequate. As in the proof of [1, Example 17], R is not clean. In light of [11, Theorem 4], zero is not adequate in R, as desired. $\square$

Example 2.12. Let F be a field, and let $R = F[[x, y]]$. Let $S = R - (x)\bigcup(y)$. Then R_S is nearly adequate, but zero is not adequate in R_S.

Proof. As $F[[x, y]]/(x) \cong F[[y]]$ is an integral domain, we see that (x) is a prime ideal of $F[[x, y]]$. Likewise, (y) is a prime ideal of R. Set $S = R - (x)\bigcup(y)$. Then S is a multiplicative closed subset of R. Let P be a maximal ideal of R_S. Then we can find an ideal Q of R such that $P = Q_S$ such that $Q\bigcap S = \emptyset$. Hence, $Q \subseteq (x)\bigcup(y)$. Assume that $Q \nsubseteq (x)$ and $Q \nsubseteq (y)$. Then we can find some $b \in Q$, but $b \notin (x)$. Likewise, we have some $c \in Q$, but $c \notin (y)$. Set $a = b + c$. Then $a \in Q$, but $a \notin (x)\bigcup(y)$. This gives a contradiction. Hence, $Q \subseteq (x)$ or $Q \subseteq (y)$. It follows that $Q_S \subseteq (x)_S$ or $Q_S \subseteq (y)_S$. By the maximality of P, we get $P = (x)_S$ or $(y)_S$. Thus, R_S has exactly two maximal ideals $(x)_S$ and $(y)_S$. Accordingly, $R_S/J(R_S) \cong R_S/(x)_S \times R_S/(y)_S$ is regular. Obviously, R_S is a PID, and then it is a Bézout ring. In light of Theorem 2.5, R_S is nearly adequate. Obviously, R_S is indecomposable. This implies that R_S is not clean; otherwise, it is local by [1, Theorem 3], which is impossible. Therefore zero is not adequate in R_S, by [11, Theorem 4]. $\square$

3. Elementary matrix reduction

As is well known, an adequate ring is an elementary divisor ring if and only if it is a Hermite ring ([9, Theorem 1.2.14]). The aim of this section is to extend this result to any Zabavsky rings.

Lemma 3.1. *Let R be a Bézout ring. If $a \in R$ is feckly adequate, then R/aR is nearly adequate.*

Proof. Let $\overline{b} \in R/aR$. Then there exist $r, s \in R$ such that $a \equiv rs \pmod{J(R)}$, $rR + bR = R$ and $s'R + bR \neq R$ for any noninvertible divisor s' of s. Hence, $\overline{a} \equiv \overline{rs} \big(\text{mod } J(R/aR)\big)$, i.e., $\overline{rs} \in J(R/aR)$. Clearly, $\overline{r}(R/aR) + \overline{b}(R/aR) = R/aR$. Let $\overline{t} \in R/aR$ be a noninvertible divisor of $\overline{s}$. Then t is a divisor of $s + ak$ for some $k \in R$. Write $s + ak = t\beta$ for some $\beta \in R$. Then $s + rsk = t\beta + w$ for a $w \in J(R)$, and so $s(1 + rk) = t\beta + w$. If $sR + tR = R$, then $sp + tq = 1$

for some $p, q \in R$. It follows that $s(1+rk)p + t(1+rk)q = 1 + rk$, and so $(t\beta + w)p + t(1+rk)q = 1 + rk$. As $w \in J(R)$, we get $r(-k)(1-wp)^{-1} + t(\beta p + (1+rk)q)(1-wp)^{-1} = 1$. This implies that $rR + tR = R$; hence, $(rs)R + tR = R$. As $a - rs \in J(R)$, we see that $aR + tR = R$, and then $\overline{t} \in R/aR$ is invertible, a contradiction. Therefore $sR + tR \neq R$. Since R is a Bézout ring, we have a noninvertible $u \in R$ such that $sR + tR = uR$. We infer that u is a noninvertible divisor of s. Hence, $uR + bR \neq R$. This proves that $\overline{u}(R/aR) + \overline{b}(R/aR) \neq R/aR$; otherwise, there exist $x, y, z \in R$ such that $ux + by = 1 + az$. This implies that $ux + by = 1 + w'z + rsz = 1 + w'z + ucrz$ for $c \in R$ and $w' \in J(R)$, because $a - rs \in J(R)$. Hence, $u(x - crz)(1+w'z)^{-1} + by(1+w'z)^{-1} = 1$, a contradiction. Thus $\overline{t}(R/aR) + \overline{b}(R/aR) \neq R/aR$, and so the result is proved. □

Lemma 3.2. *A ring R is an elementary divisor ring if and only if*

(1) *R is a Hermite ring;*
(2) *Every matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $aR + bR + cR = R$ admits elementary diagonal reduction.*

Proof. One easily checks that $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}$. Therefore the result follows, by [6, Theorem 1.1] and [8, Theorem 2.5]. □

Lemma 3.3. *If $(b + ar)R + cR = R$ with $a, b, c, r \in R$, then $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $aR + bR + cR = R$ admits elementary diagonal reduction.*

Proof. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Since $(b + ar)R + cR = R$, we have $B := A\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+ar \\ 0 & c \end{pmatrix}$. It suffices to prove B admits a diagonal reduction. Write $(b+ar)x + cy = 1$ for some $x, y \in R$. Then the matrix $\begin{pmatrix} x & y \\ -c & b+ar \end{pmatrix}$ is invertible, and we see that

$$\begin{pmatrix} x & y \\ -c & b+ar \end{pmatrix} B \begin{pmatrix} 1 & \\ -ax & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -ac \end{pmatrix},$$

as desired. □

Theorem 3.4. *Let R be a Zabavsky ring. Then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. ($\Longrightarrow$) This is obvious.

($\Longleftarrow$) Step I. Suppose that $aR + bR + cR = R$ with $a, b, c \in R$. Write $ax + by + cz = 1$ with $x, y, z \in R$. By hypothesis, there exist $k \in R$ such that $w := a + byk + czk \in R$ is feckly adequate. In view of Lemma 3.1, R/wR is nearly adequate. It follows by Theorem 2.5 that $R/wR/J(R/wR)$ is regular, and so it has stable range 1. We infer that R/wR has stable range 1. Clearly, $(a + byk + czk)x + by(1 - kx) + cz(1 - kx) = 1$. Thus, $\overline{by(1-kx) + cz(1-kx)} = \overline{1}$ in R/wR. Thus, we can find $h \in R$ such that $\overline{b + cz(1-kx)h} \in U(R/wR)$. It follows that $\big(b + cz(1-kx)h\big)R + \big(a + byk + czk\big)R = R$. Hence,

$$\big(b + cz(1-kx)h\big)R + \big(a + (b + cz(1-kx)h)yk + czk(1 - (1-kx)hy)\big)R = R.$$

Therefore,

$$\big(b+cz(1-kx)h\big)R+\big(a+czk(1-(1-kx)hy)\big)R=R.$$

Thus, R has stable range 2. According to [9, Theorem 2.1.2], R is a Hermite ring.

Step II. Let $A=\left(\begin{smallmatrix}a' & 0\\ b' & c'\end{smallmatrix}\right)$ with $a'R+b'R+c'R=R$. Then there exist $x,y,z\in R$ such that $a'x+b'y+c'z=1$. Since R is a Zabavsky ring, we can find some $s\in R$ such that $w:=b'+a'xs+c'zs\in R$ is feckly adequate. Hence,

$$\begin{pmatrix}1 & 0\\ xs & 1\end{pmatrix}A\begin{pmatrix}1 & 0\\ zs & 1\end{pmatrix}=\begin{pmatrix}a' & 0\\ w & c'\end{pmatrix}.$$

Since R is a Hermite ring, there exists some $Q=(q_{ij})\in GL_2(R)$ such that $(w,c')Q=(0,c)$ for a $c\in R$. This implies that $\left(\begin{smallmatrix}a' & 0\\ w & c'\end{smallmatrix}\right)Q=\left(\begin{smallmatrix}a & b\\ 0 & c\end{smallmatrix}\right)$. Clearly, $wR\subseteq cR$. Additionally, we see that $aR+bR+cR=R$. Since $w\in R$ is feckly adequate, R/wR is nearly adequate by Lemma 3.1. In view of Theorem 2.5, $(R/wR)/J(R/wR)$ is regular. For any $\overline{\alpha}\in R/wR$, we can find some $\beta\in R$ such that $\big(\alpha-\alpha\beta\alpha\big)+wR\in J(R/wR)$. It follows that $\big(\alpha-\alpha\beta\alpha\big)+cR\in J(R/cR)$. Thus, $(R/cR)/J(R/cR)$ is regular; hence, $(R/cR)/J(R/cR)$ has stable range 1. This implies that R/cR has stable range 1.

Clearly, $\overline{a}(R/cR)+\overline{b}(R/cR)=R/cR$. Then we can find some $r\in R$ such that $\overline{b+ar}\in R/cR$ is invertible. Hence, $\overline{(b+ar)d}=\overline{1}$, and then $(b+ar)d+cp=1$ for some $p\in R$. Therefore $(b+ar)R+cR=R$. In light of Lemma 3.3, $\left(\begin{smallmatrix}a & b\\ 0 & c\end{smallmatrix}\right)$ admits elementary diagonal reduction, and therefore so does A, as asserted. □

Corollary 3.5. *Let R be a Bézout ring. If every $a\notin J(R)$ is feckly adequate, then R is an elementary divisor ring.*

Proof. Suppose that $pR+qR=R$ with $p,q\in R$. If $p\in J(R)$, then $q\in U(R)$. Hence, $p+q\in U(R)$, and so $p+q\in R$ is feckly adequate. If $p\notin J(R)$, then $p+q\cdot 0\in R$ is feckly adequate. Therefore R is a Zabavsky ring, and then we obtain the result, by Theorem 3.4. □

As an immediate consequence of Corollary 3.5, we conclude that every Bézout ring in which every nonzero element is feckly adequate is an elementary divisor ring. This generalizes [12, Theorem 7] as well.

Example 3.6. Let $R=\{a+bx\mid a\in\mathbb{Z},b\in\mathbb{Q},x^2=0\}$. Then R is a Bézout ring in which every $a\notin J(R)$ is feckly adequate.

Proof. Let $J=(a_1+b_1x)R+(a_2+b_2x)R$. Set $I=\{\alpha\in\mathbb{Z}\mid \alpha+\beta x\in J$ for some $\beta\in\mathbb{Q}\}$. Since $\mathbb{Z}$ is a principal ideal domain, we have some $p\in\mathbb{Z}$ and $q\in\mathbb{Q}$ such that $a_1\mathbb{Z}+a_2\mathbb{Z}=p\mathbb{Z}$ and $b_1\mathbb{Z}+b_2\mathbb{Z}=q\mathbb{Z}$. If $I\neq 0$, then $J=pR$. If $I=0$, then $J=(qx)R$. Thus, R is a Bézout ring. Clearly, $J(R)=x\mathbb{Q}$. Let $f(x)=y+bx\notin J(R)$, and let $h(x)=z+cx\in R$. Then $y\neq 0$. Since $\mathbb{Z}$ is a principal ideal domain, it is adequate. Thus, there exist $s,t\in R$ such that $y=st,(s,z)=1$, and that $(t',z)\neq 1$ for any non-unit divisor t' of t. If $(s,t)\neq 1$, then we have a nonunit $d\in R$ such that $(s,t)=d$. Hence, $(d,z)\neq 1$, and then $(s, z) \neq 1$, an absurdity. Therefore $(s, t) = 1$, and so we can find some $e, d \in R$ such that $se + dt = b$. One easily checks that $f(x) = \big(s + dx\big)\big(t + ex\big)$. Set $s(x) = s + dx$ and $t(x) = t + ex$. Then $f(x) = s(x)t(x)$. Clearly, we can find some $k, l \in \mathbb{Z}$ such that $ks + lz = 1$. Hence, $1 - \big(ks(x) + lh(x)\big) \in J(R)$. Thus, $ks(x) + lh(x) \in U(R)$. This shows that $\big(s(x), h(x)\big) = 1$. If $t'(x) = m + fx$ is a nonunit divisor of $t(x)$, then m is a nonunit divisor of t. By hypothesis, $(m, z) \neq 1$. This implies that $\big(t'(x), h(x)\big) \neq 1$. Thus, $f(x) \in R$ is adequate, and so $f(x)$ is feckly adequate. In this case, R is not nearly adequate. $\square$

Following Domsha and Vasiunyk, a ring R has adequate range 1 provided that $aR + bR = R$ implies that there exists a $y \in R$ such that $a + by \in R$ is adequate ([4]). For instance, every VNL ring (i.e., for any $a \in R$, either a or $1 - a$ is regular) has adequate range 1 ([4, Theorem 11 and Theorem 12]). We now extend [4, Theorem 14] to Bézout rings (maybe with zero divisors).

Corollary 3.7. *If R has adequate range 1, then R is an elementary divisor ring if and only if R is a Bézout ring.*

Proof. As every adequate element in a ring is feckly adequate, if R has adequate range 1, then it is a Zabavsky ring. Therefore we complete the proof, by Theorem 3.4. $\square$

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