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ON THE STRUCTURE OF GRADED LIE TRIPLE SYSTEMS

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Abstract. We study the structure of an arbitrary graded Lie triple system $\mathfrak{T}$ with restrictions neither on the dimension nor the base field. We show that $\mathfrak{T}$ is of the form $\mathfrak{T} = U + \sum_j I_j$ with U a linear subspace of the 1-homogeneous component $\mathfrak{T}_1$ and any I_j a well described graded ideal of $\mathfrak{T}$, satisfying $[I_j, \mathfrak{T}, I_k] = 0$ if $j \neq k$. Under mild conditions, the simplicity of $\mathfrak{T}$ is characterized and it is shown that an arbitrary graded Lie triple system $\mathfrak{T}$ is the direct sum of the family of its minimal graded ideals, each one being a simple graded Lie triple system.

1. Introduction

The study of gradings on Lie algebras begins in the 1933 seminal Jordan's work [27], with the purpose of formalizing Quantum Mechanics. Since then, many papers describing different physic models by means of graded Lie type structures have appeared, being remarkable the interest on these objects in the last years. For instance, in the case of Lie algebras, we can cite many works related to theory of strings, to color supergravity, to Walsh functions, to electroweak interactions or to gauge models [1, 4, 9, 10, 17, 18, 20, 22, 25, 29, 34]. In the case of Lie superalgebras, we can also cite several works modelling continuous suppersymmetry transformations between bosons and fermions or conformal field theory [3, 5, 19, 26, 31]. Finally, as it is pointed out in [24], we note that Lie triple systems are well related to the theory of Quantum Mechanics with PT-symmetric Hamiltonians and Krein space-related models in general, by identifying this underlying structure in the recognizing of PT-like involutory structures in physical models. Lie triple systems also appear in the modelling of superconformal Cherm-Simons theories [30], being so of special interest the graded ones (see the recent papers [6, 11, 13, 21, 23, 28]).

In the reference [9] it is studied the structure of arbitrary graded Lie algebras, being extended to the framework of graded Lie superalgebras in [14]. Since

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graded Lie triple systems appear as the natural ternary extension of graded Lie algebras, we are interested in the present paper in studying the structure of graded Lie triple systems.

The paper is organized as follows. In § 2 we recall some basic results on the theory of Lie triple systems. In §3 we extend the techniques of connections in the support of the grading developed in [9, 14] to the framework of graded Lie triple systems $\mathfrak{T}$, so as to show in §4 that $\mathfrak{T}$ is of the form $\mathfrak{T} = U + \sum_j I_j$ with U a linear subspace of the 1-homogeneous space $\mathfrak{T}_1$ and any I_j a well described graded ideal of $\mathfrak{T}$, satisfying $[I_j, \mathfrak{T}, I_k] = 0$ if $j \neq k$. We would like to note that the works [9, 14] are developed for graded Lie algebras and superalgebras L with a symmetric support, that is, satisfying that if the homogeneous component $L_g \neq 0$, then $L_{g^{-1}} \neq 0$. In the present paper we also extend the connection in the support techniques introduced in the above papers to the non-necessarily symmetric support case. Hence, we are also extending the results of [9] to the case in which the support of the grading is non-symmetric.

In §5, and under mild conditions, the (graded) simplicity of $\mathfrak{T}$ is characterized and it is shown that an arbitrary graded Lie triple system $\mathfrak{T}$ is the direct sum of the family of its minimal graded ideals, each one being a simple graded Lie triple system.

Since any split Lie triple system T, that is, a Lie triple system which decomposes as $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda} T_\alpha)$ where T_α is the root space associated to the nonzero root $\alpha : H \to \mathbb{K}$, is a G-graded Lie triple system, G being the abelian free group generated by the set of nonzero roots Λ, we have that any split Lie triple system is a particular case of a graded Lie triple system. Hence, the present paper extends the results in [8, 12].

Finally, we note that throughout this paper, graded Lie triple systems $\mathfrak{T}$ are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

2. Preliminaries and basic definitions

Let $\mathfrak{T}$ be a linear space over an arbitrary base field $\mathbb{K}$. We say that $\mathfrak{T}$ is a *triple system* if it is endowed with a trilinear map

$$\langle \cdot,\cdot,\cdot\rangle : \mathfrak{T}\times\mathfrak{T}\times\mathfrak{T}\to\mathfrak{T},$$

called the *triple product* of $\mathfrak{T}$.

Definition 1. A triple system $\mathfrak{T}$ is called a *Lie triple system* if its triple product, denoted by $[\cdot,\cdot,\cdot]$, satisfies

(1) $[x, x, y] = 0$,
(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity),
(3) $[x, y, [a, b, c]] - [a, b, [x, y, c]] = [[x, y, a], b, c] + [a, [x, y, b], c]$

for any $x, y, z, a, b, c \in \mathfrak{T}$.

Observe that last identity means that for any $x, y \in \mathfrak{T}$, the left multiplication operator

$$\mathcal{L}(x, y) : \mathfrak{T}\to\mathfrak{T},\ z \mapsto [x, y, z]$$

acts as a derivation on $\mathfrak{T}$.

We recall that the *Annihilator* of a Lie triple system $\mathfrak{T}$, denoted by $\mathrm{Ann}(\mathfrak{T})$, is defined as the set of elements $x \in \mathfrak{T}$ such that $[x, \mathfrak{T}, \mathfrak{T}] = 0$.

A *two-graded* $\mathbb{K}$*-algebra* A is a $\mathbb{K}$-algebra which splits into the direct sum $A = A^0 \oplus A^1$ of linear subspaces (called the even and the odd part respectively) satisfying $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for any α, β in $\mathbb{Z}_2$. The *standard embedding algebra* of a Lie triple system $\mathfrak{T}$ is the two-graded Lie algebra $L = L^0 \oplus L^1$ where L^0 is the $\mathbb{K}$-span of $\{\mathcal{L}(x, y) : x, y \in T\}$, where $L^1 := \mathfrak{T}$ and where the product is given by

$$[(\mathcal{L}(x, y), z), (\mathcal{L}(u, v), w)]$$

$$:= (\mathcal{L}([u, v, y], x) - \mathcal{L}([u, v, x], y) + \mathcal{L}(z, w), [x, y, w] - [u, v, z]).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a Lie algebra and that the fact $[x, L^1] = 0$ for some $x \in L^0$ implies $x = 0$.

Definition 2. Let $\mathfrak{T}$ be a Lie triple system. It is said that $\mathfrak{T}$ is *graded* by means of an abelian group G if it decomposes as the direct sum of linear subspaces

$$\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g$$

where the homogeneous components satisfy $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k] \subset \mathfrak{T}_{ghk}$ for any $g, h, k \in G$ (denoting by juxtaposition the product in G). We call the *support* of the grading the set $\Sigma^1 := \{g \in G \setminus \{1\} : \mathfrak{T}_g \neq 0\}$.

The usual regularity conditions will be understood in the graded sense. That is, a *subtriple* of $\mathfrak{T}$ is a linear subspace S satisfying $[S, S, S] \subset S$ and such that splits as $S = \bigoplus_{g \in G} S_g$ with any $S_g = S \cap \mathfrak{T}_g$. A subtriple I of $\mathfrak{T}$ is an *ideal* if $[I, \mathfrak{T}, \mathfrak{T}] \subseteq I$, (which implies $[\mathfrak{T}, I, \mathfrak{T}] + [\mathfrak{T}, \mathfrak{T}, I] \subseteq I$). As an example, $\mathrm{Ann}(\mathfrak{T})$ is an ideal of $\mathfrak{T}$.

The graded Lie triple system $\mathfrak{T}$ will be called *simple* if $[\mathfrak{T}, \mathfrak{T}, \mathfrak{T}] \neq 0$ and its only ideals are $\{0\}$ and $\mathfrak{T}$. Finally, $\mathfrak{T}$ will be said *prime* if $[I, \mathfrak{T}, J] + [J, \mathfrak{T}, I] = 0$ with I, J ideals implies either $I = 0$ or $J = 0$.

Let $\mathfrak{L}$ be an arbitrary Lie algebra over $\mathbb{K}$. As usual, the term grading will always mean abelian group grading, that is, a decomposition in linear subspaces $\mathfrak{L} = \bigoplus_{g \in G} \mathfrak{L}_g$ where G is an abelian group and the homogeneous spaces satisfy $[\mathfrak{L}_g, \mathfrak{L}_h] \subset \mathfrak{L}_{gh}$. We also call the *support* of the grading the set $\{g \in G \setminus \{1\} : \mathfrak{L}_g \neq 0\}$.

Proposition 2.1. *Let $\mathfrak{T}$ be a G-graded Lie triple system and let $L = L^0 \oplus L^1$ be its standard embedding algebra. Then L^0 is a G-graded Lie algebra.*

Proof. Define $L^0_1 := \sum_{g \in G} [\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}]$ and $L^0_g := \sum_{h \in G} [\mathfrak{T}_h, \mathfrak{T}_{h^{-1}g}]$ for any $g \in G \setminus \{1\}$.

Clearly $L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g \subseteq L^0$. Conversely, since

$$\begin{aligned} L^0 = [\mathfrak{T},\mathfrak{T}] &= [\bigoplus_{g\in G}\mathfrak{T}_g, \bigoplus_{h\in G}\mathfrak{T}_h] \\ &\subseteq \sum_{g\in G}[\mathfrak{T}_g,\mathfrak{T}_{g^{-1}}] + \sum_{g,h\in G, h\neq g^{-1}}[\mathfrak{T}_g,\mathfrak{T}_h] \\ &\subseteq L^0_1 + \sum_{g,h\in G,h\neq g^{-1}} L^0_{gh} \\ &\subseteq L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g \end{aligned}$$

we get $L^0 = L^0_1 + \sum_{g\in G\setminus\{1\}} L^0_g$.

The direct character of the sum can be checked as follows. If $x \in L^0_g \cap (\sum_{h\in G\setminus\{g\}} L^0_h)$, then for any $q \in G$ and $y \in \mathfrak{T}_q$ we have $[x,y] \in \mathfrak{T}_{gq} \cap (\sum_{h\in G\setminus\{g\}} \mathfrak{T}_{hq})$ and so $[x,y]=0$. From here $[x,\mathfrak{T}]=0$ and so $x=0$. Hence we can write

$$L^0 = L^0_1 \oplus (\bigoplus_{g\in G\setminus\{1\}} L^0_g).$$

Finally, we have

$$[L^0_g, L^0_h] \subseteq L^0_{gh}$$

for any $g,h\in G$. Indeed,

$$\begin{aligned} [L^0_g, L^0_h] &= \sum_{k,l\in G} [[\mathfrak{T}_k,\mathfrak{T}_{k^{-1}g}],[\mathfrak{T}_l,\mathfrak{T}_{l^{-1}h}]] \\ &\subseteq [[\mathfrak{T}_k,\mathfrak{T}_{k^{-1}g},\mathfrak{T}_l],\mathfrak{T}_{l^{-1}h}] + [\mathfrak{T}_l,[\mathfrak{T}_k,\mathfrak{T}_{k^{-1}g},\mathfrak{T}_{l^{-1}h}]] \\ &\subseteq [\mathfrak{T}_{gl},\mathfrak{T}_{l^{-1}h}] + [\mathfrak{T}_l,\mathfrak{T}_{l^{-1}gh}] \\ &\subseteq L^0_{gh}. \end{aligned}$$

□

Observe that for any $g,h\in G$ we have

$$[\mathfrak{T}_g,\mathfrak{T}_h] \subset L^0_{gh}.$$

In the following, we shall denote by Σ^0 the support of the graded Lie algebra L^0.

Example 1. Consider $\mathfrak{L} = \bigoplus_{g\in G}\mathfrak{L}_g$ a simple graded Lie algebra with support Σ, and the Lie triple system $\mathcal{T}(\mathfrak{L})$, where $\mathcal{T}(\mathfrak{L})$ agrees with $\mathfrak{L}$ as linear spaces, and the triple product is defined by $[x,y,z] := [[x,y],z]$. The standard embedding algebra of $\mathcal{T}(\mathfrak{L})$ is $\mathfrak{L}\oplus\mathfrak{L}$ with the product $[(x,y),(z,t)] = ([x,z]+[y,t],[x,t]+[y,z])$. It is straightforward to verify that $\mathcal{T}(\mathfrak{L}) = (0,\mathfrak{L})$ is a graded Lie triple system with support $\Sigma^1 = \Sigma$ and that its homogeneous spaces are $\mathcal{T}(\mathfrak{L})_g = (0,\mathfrak{L}_g)$ for any $g\in\Sigma$, and $\mathcal{T}(\mathfrak{L})_1 = (0,\mathfrak{L}_1)$. Observe that the supports of the graded Lie algebra $\mathfrak{L}^0 = (\mathfrak{L},0)$ and of the graded Lie triple system $\mathcal{T}(\mathfrak{L}) = (0,\mathfrak{L})$ agree.

3. Connections and gradings

From now on, $\mathfrak{T}$ denotes a graded Lie triple system with support Σ^1, and

$$\mathfrak{T} = \bigoplus_{g \in G} \mathfrak{T}_g = \mathfrak{T}_1 \oplus \big(\bigoplus_{g \in \Sigma^1} \mathfrak{T}_g \big)$$

the corresponding grading. Denote by $-\Sigma^i = \{-g : g \in \Sigma^i\}$, $i = 0, 1$.

Definition 3.1. Let g and h be two elements in Σ^1. We say that g is *connected* to h if there exist $g_1, g_2, \ldots, g_{2n+1} \in \pm\Sigma^1 \cup \{1\}$ such that

1. $\{g_1, g_1g_2g_3, \ldots, g_1g_2g_3 \cdots g_{2n}g_{2n+1}\} \subset \pm\Sigma^1$,
2. $\{g_1g_2, g_1g_2g_3g_4, \ldots, g_1g_2g_3 \cdots g_{2n}\} \subset \pm\Sigma^0$,
3. $g_1 = g$ and $g_1g_2g_3 \cdots g_{2n}g_{2n+1} \in \{h, h^{-1}\}$.

We also say that $\{g_1, \ldots, g_{2n+1}\}$ is a *connection* from g to h.

Proposition 3.1. *The relation $\sim$ in Σ^1, defined by $g \sim h$ if and only if g is connected to h is an equivalence relation.*

Proof. $\{g\}$ is a connection from g to itself and therefore $g \sim g$.

If $g \sim h$ and $\{g_1, \ldots, g_{2n+1}\}$ is a connection from g to h, then

$$\{g_1 \cdots g_{2n+1}, g_{2n+1}^{-1}, g_{2n}^{-1}, \ldots, g_2^{-1}\}$$

is a connection from h to g in case $g_1 \cdots g_{2n+1} = h$, and

$$\{g_1^{-1} \cdots g_{2n+1}^{-1}, g_{2n+1}, g_{2n}, \ldots, g_2\}$$

in case $g_1 \cdots g_{2n+1} = h^{-1}$. Therefore $h \sim g$.

Finally, suppose $g \sim h$ and $h \sim k$, and write $\{g_1, \ldots, g_{2n+1}\}$ for a connection from g to h and $\{h_1, \ldots, h_{2m+1}\}$ for a connection from h to k. If $m > 0$, then $\{g_1, \ldots, g_{2n+1}, h_2, \ldots, h_{2m+1}\}$ is a connection from g to k in case $g_1 \cdots g_{2n+1} = h$, and $\{g_1, \ldots, g_{2n+1}, h_2^{-1}, \ldots, h_{2m+1}^{-1}\}$ in case $g_1 \cdots g_{2n+1} = h^{-1}$. If $m = 0$, then $k \in \{h, h^{-1}\}$ and so $\{g_1, \ldots, g_{2n+1}\}$ is a connection from g to k. Therefore $g \sim k$ and $\sim$ is an equivalence relation. □

By Proposition 3.1 the connection relation is an equivalence relation in Σ^1 and so we can consider the quotient set

$$\Sigma^1/\sim = \{[g] : g \in \Sigma^1\},$$

becoming $[g]$ the set of elements in the support of the grading which are connected to g. By the definition of $\sim$, it is clear that if $h \in [g]$ and $h^{-1} \in \Sigma^1$, then $h^{-1} \in [g]$.

Our goal in this section is to associate an adequate subtriple $I_{[g]}$ to any $[g]$. Fix $g \in \Sigma^1$, we start by defining

$$\mathfrak{T}_{1,[g]} := \mathrm{span}_{\mathbb{K}} \big\{ [\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_{(hk)^{-1}}] : h \in [g], k \in [g] \cup \{1\} \big\} \subset \mathfrak{T}_1$$

and

$$V_{[g]} := \bigoplus_{h \in [g]} \mathfrak{T}_h.$$

Finally, we denote by $\mathfrak{T}_{[g]}$ the direct sum of the two subspaces above, that is,

$$\mathfrak{T}_{[g]} := \mathfrak{T}_{1,[g]} \oplus V_{[g]}.$$

Proposition 3.2. *For any $g \in \Sigma^1$, the graded linear subspace $\mathfrak{T}_{[g]}$ is a subtriple of $\mathfrak{T}$.*

Proof. We have to check that $\mathfrak{T}_{[g]}$ satisfies

$$[\mathfrak{T}_{[g]}, \mathfrak{T}_{[g]}, \mathfrak{T}_{[g]}] = [\mathfrak{T}_{1,[g]} \oplus V_{[g]}, \mathfrak{T}_{1,[g]} \oplus V_{[g]}, \mathfrak{T}_{1,[g]} \oplus V_{[g]}] \subset \mathfrak{T}_{[g]}.$$

Since $\mathfrak{T}_{1,[g]} \subset \mathfrak{T}_1$ we clearly have

$$[\mathfrak{T}_{1,[g]}, \mathfrak{T}_{1,[g]}, V_{[g]}] + [\mathfrak{T}_{1,[g]}, V_{[g]}, \mathfrak{T}_{1,[g]}] + [V_{[g]}, \mathfrak{T}_{1,[g]}, \mathfrak{T}_{1,[g]}] \subset V_{[g]}.$$

Now, observe that if $h \in [g]$ and $h^{-1} \in \Sigma^1$, then

$$[\mathfrak{T}_h, \mathfrak{T}_1, \mathfrak{T}_{h^{-1}}] + [\mathfrak{T}_{h^{-1}}, \mathfrak{T}_1, \mathfrak{T}_h] \subset \mathfrak{T}_{1,[g]} \tag{1}$$

and so

$$[\mathfrak{T}_1, \mathfrak{T}_h, \mathfrak{T}_{h^{-1}}] + [\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_1] \subset \mathfrak{T}_{1,[g]}. \tag{2}$$

Moreover, we also have

$$[[\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_1], \mathfrak{T}_1, \mathfrak{T}_1] \subset \mathfrak{T}_{1,[g]}. \tag{3}$$

Indeed, since

$$[[\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_1], \mathfrak{T}_1, \mathfrak{T}_1] \subseteq [[\mathfrak{T}_{h^{-1}}, \mathfrak{T}_1, \mathfrak{T}_h], \mathfrak{T}_1, \mathfrak{T}_1] + [[\mathfrak{T}_1, \mathfrak{T}_h, \mathfrak{T}_{h^{-1}}], \mathfrak{T}_1, \mathfrak{T}_1],$$

we can apply now identities in Definition 1 to get

$$[[\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_1], \mathfrak{T}_1, \mathfrak{T}_1] \subseteq [\mathfrak{T}_{h^{-1}}, \mathfrak{T}_1, \mathfrak{T}_h] + [\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_1] \subset \mathfrak{T}_{1,[g]}.$$

Taking into account Equations (2) and (3), it is straightforward to verify that

$$[\mathfrak{T}_{1,[g]}, \mathfrak{T}_{1,[g]}, \mathfrak{T}_{1,[g]}] \subset \mathfrak{T}_{1,[g]}.$$

We also have

$$[\mathfrak{T}_{1,[g]}, V_{[g]}, V_{[g]}] \subset \mathfrak{T}_{[g]}.$$

In fact, if $[\mathfrak{T}_1, \mathfrak{T}_h, \mathfrak{T}_k] \neq 0$ for some $h, k \in [g]$, then $h \in \Sigma^0$ and $hk \in \Sigma^1 \cup \{1\}$. From here, if $hk \neq 1$ and $\{g_1, \dots, g_{2n+1}\}$ is a connection from g to h, then $\{g_1, \dots, g_{2n+1}, 1, k\}$ is a connection from g to hk in case $g_1 \cdots g_{2n+1} = h$ and $\{g_1 \cdots g_{2n+1}, 1, k^{-1}\}$ in case $g_1 \cdots g_{2n+1} = h^{-1}$ being so $hk \in [g]$. If $hk = 1$ clearly $[\mathfrak{T}_1, \mathfrak{T}_h, \mathfrak{T}_{h^{-1}}] \subset \mathfrak{T}_{1,[g]}$. We have showed $[\mathfrak{T}_1, \mathfrak{T}_h, \mathfrak{T}_k] \subset \mathfrak{T}_{[g]}$ and so $[\mathfrak{T}_{1,[g]}, V_{[g]}, V_{[g]}] \subset \mathfrak{T}_{[g]}$. Consequently

$$[V_{[g]}, \mathfrak{T}_{1,[g]}, V_{[g]}] + [V_{[g]}, V_{[g]}, \mathfrak{T}_{1,[g]}] \subset \mathfrak{T}_{[g]}.$$

Finally, let us show

$$[V_{[g]}, V_{[g]}, V_{[g]}] \subset \mathfrak{T}_{[g]}.$$

Suppose then $[\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_l] \neq 0$ for some $h, k, l \in [g]$ being so $hk \in \Sigma^0 \cup \{1\}$ and $hkl \in \Sigma^1 \cup \{1\}$. If either $hk = 1$ or $hkl = 1$, then $[\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_l] = \mathfrak{T}_l \subset V_{[g]}$ or $[\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_l] \subset \mathfrak{T}_{1,[g]}$ respectively. Hence let us consider $hk \in \Sigma^0$ and $hkl \in \Sigma^1$, and take a connection $\{g_1, \dots, g_{2n+1}\}$ from g to h. We clearly have

$\{g_1, \ldots, g_{2n+1}, k, l\}$ is a connection from g to hkl in case $g_1 \cdots g_{2n+1} = h$ and $\{g_1 \cdots g_{2n+1}, k^{-1}, l^{-1}\}$ it is in case $g_1 \cdots g_{2n+1} = h^{-1}$. We have showed $hkl \in [g]$ and so $[\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_l] \subset V_{[g]}$, which concludes the proof. □

We call $\mathfrak{T}_{[g]}$ the *subtriple of* $\mathfrak{T}$ *associated* to $[g]$.

4. Decompositions

We begin this section by showing that for any $g \in \Sigma^1$, the subtriple $I_{[g]}$ is actually an ideal of $\mathfrak{T}$. We need to state some preliminary results.

Lemma 4.1. *The following assertions hold.*

1. *If* $g, h \in \Sigma^1$ *with* $gh \in \pm\Sigma^0 \cup \{1\}$, *then* $h \in [g]$.
2. *If* $g, h \in \Sigma^1$ *and* $g \in \pm\Sigma^0$ *with* $gh \in \pm\Sigma^1 \cup \{1\}$, *then* $h \in [g]$.
3. *If* $g, \overline{h} \in \Sigma^1$ *such that* $\overline{h} \notin [g]$, *then* $[\mathfrak{T}_g, \mathfrak{T}_{\overline{h}}] = [L^0_g, \mathfrak{T}_{\overline{h}}] = [L^0_g, L^0_{\overline{h}}] = 0$.

Proof. 1. If $gh = 1$, then $h = g^{-1}$ and so $h \sim g$. Suppose $gh \neq 1$. Since $gh \in \pm\Sigma^0$, we have $\{g, h, g^{-1}\}$ is a connection from g to h.

2 We can argue similarly with the connection $\{g, 1, (gh)^{-1}\}$.

3. Consequence of 1 and 2. □

Lemma 4.2. *If* $g, \overline{h} \in \Sigma^1$ *are not connected, then* $[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_{\overline{h}}] = 0$.

Proof. If $[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] = 0$ it is clear. Suppose then $[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] \neq 0$ and

$$[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_{\overline{h}}] \neq 0.$$

We have either $[\mathfrak{T}_{g^{-1}}, \mathfrak{T}_{\overline{h}}, \mathfrak{T}_g] \neq 0$ or $[\mathfrak{T}_{\overline{h}}, \mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] \neq 0$ what contradicts in any case Lemma 4.1-3. From here $[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_{\overline{h}}] = 0$. □

Lemma 4.3. *For any* $g_0 \in \Sigma^1$, *if* $g \in [g_0]$ *and* $h, k \in \Sigma^1 \cup \{1\}$ *the following assertions hold.*

1. *If* $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k] \neq 0$, *then* $h, k, ghk \in [g_0] \cup \{1\}$.
2. *If* $[\mathfrak{T}_h, \mathfrak{T}_g, \mathfrak{T}_k] \neq 0$, *then* $h, k, hgk \in [g_0] \cup \{1\}$.
3. *If* $[\mathfrak{T}_h, \mathfrak{T}_k, \mathfrak{T}_g] \neq 0$, *then* $h, k, hgk \in [g_0] \cup \{1\}$.

Proof. 1. The fact $[\mathfrak{T}_g, \mathfrak{T}_h] \neq 0$ implies by Lemma 4.1 that $h \sim g$ in case $h \neq 1$. From here, $h \in [g_0] \cup \{1\}$. To show $k, ghk \in [g_0] \cup \{1\}$ we will distinguish two possibilities. In the first one, suppose that $ghk = 1$ and so $ghk \in [g_0] \cup \{1\}$. If we had $k \neq 1$, then $gh \in \Sigma^0$. As $(gh)^{-1} = k$, then $\{g, h, 1\}$ would be a connection from g to k and we conclude $k \in [g_0] \cup \{1\}$.

In the second one, suppose $ghk \neq 1$. If $gh \neq 1$, then $gh \in \Sigma^0$ and so $\{g, h, k\}$ is a connection from g to ghk. Hence $ghk \in [g_0]$. In the case $k \neq 1$, we have $\{g, h, (ghk)^{-1}\}$, is a connection from g to k. So $k \in [g_0]$. Finally, if $gh = 1$, then necessarily $k \in [g_0]$. Indeed, if k is not connected to g, we would have by Lemma 4.2 that $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k] = [\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_k] = 0$, a contradiction. From here $ghk = k \in [g_0]$.

2. and 3. are direct consequences of item 1. □

Lemma 4.4. *For any* $g_0 \in \Sigma^1$*, if* $g, k \in [g_0]$*,* $h \in [g_0] \cup \{1\}$ *with* $ghk = 1$ *and* $l, m \in \Sigma^1 \cup \{1\}$ *the following assertions hold.*

1. *If* $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_l, \mathfrak{T}_m] \neq 0$*, then* $l, m, lm \in [g_0] \cup \{1\}$*.*
2. *If* $[\mathfrak{T}_l, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_m] \neq 0$*, then* $l, m, lm \in [g_0] \cup \{1\}$*.*
3. *If* $[\mathfrak{T}_l, \mathfrak{T}_m, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k]] \neq 0$*, then* $l, m, lm \in [g_0] \cup \{1\}$*.*

Proof. 1. Since

$$0 \neq [[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_l, \mathfrak{T}_m]$$
$$\subset [\mathfrak{T}_g, \mathfrak{T}_h, [\mathfrak{T}_k, \mathfrak{T}_l, \mathfrak{T}_m]] + [\mathfrak{T}_k, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_l], \mathfrak{T}_m] + [\mathfrak{T}_k, \mathfrak{T}_l, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_m]],$$

some of the above three summands is nonzero.

Suppose $[\mathfrak{T}_g, \mathfrak{T}_h, [\mathfrak{T}_k, \mathfrak{T}_l, \mathfrak{T}_m]] \neq 0$. Since $k \neq 1$ and $[\mathfrak{T}_k, \mathfrak{T}_l, \mathfrak{T}_m] \neq 0$, Lemma 4.3-1 shows $l, m, klm \in [g_0] \cup \{1\}$. Now, if $klm = 1$, then $lm = k^{-1} \in [g_0]$. If $klm \neq 1$, taking into account $0 \neq [\mathfrak{T}_g, \mathfrak{T}_h, [\mathfrak{T}_k, \mathfrak{T}_l, \mathfrak{T}_m]] \subset [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{klm}]$, Lemma 4.3-3 and the fact that $g \in [g_0]$ give us that $ghklm = lm \in [g_0] \cup \{1\}$. Therefore $l, m, lm \in [g_0] \cup \{1\}$.

We can argue similarly if either $[\mathfrak{T}_k, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_l], \mathfrak{T}_m] \neq 0$ or

$$[\mathfrak{T}_k, \mathfrak{T}_l, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_m]] \neq 0$$

to get $l, m, lm \in [g_0] \cup \{1\}$.

2. and 3. are direct consequences of item 1. □

Lemma 4.5. *For any* $g_0 \in \Sigma^1$*, if* $g, k \in [g_0]$*,* $h \in [g_0] \cup \{1\}$ *with* $ghk = 1$ *and* $\overline{h} \notin [g_0]$ *the following assertions hold.*

1. $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_{\overline{h}}] = 0$.
2. $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], L^0_{\overline{h}}] = 0$.
3. $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_1, \mathfrak{T}_{\overline{h}}] = 0$.

Proof. 1. We have

$$[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_{\overline{h}}, \mathfrak{T}]$$
$$\subset [\mathfrak{T}_g, \mathfrak{T}_h, [\mathfrak{T}_k, \mathfrak{T}_{\overline{h}}, \mathfrak{T}]] + [\mathfrak{T}_k, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{\overline{h}}], \mathfrak{T}] + [\mathfrak{T}_k, \mathfrak{T}_{\overline{h}}, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}]]. \tag{4}$$

Consider the first and third summands in (4). Since $k \neq 1$, then Lemma 4.1-3 gives us $[\mathfrak{T}_g, \mathfrak{T}_h, [\mathfrak{T}_k, \mathfrak{T}_{\overline{h}}, \mathfrak{T}]] = [\mathfrak{T}_k, \mathfrak{T}_{\overline{h}}, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}]] = 0$. Consider now the second summand in (4). As $gh \neq 1$, then

$$[\mathfrak{T}_k, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{\overline{h}}], \mathfrak{T}] = 0$$

by Lemma 4.1-3.

We have showed $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_{\overline{h}}, \mathfrak{T}] = 0$ and consequently

$$[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_{\overline{h}}] = 0.$$

2. By applying Lemma 4.1-3 we have

$$[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], L^0_{\overline{h}}] \subseteq [L^0_{\overline{h}}, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k]]$$
$$\subseteq [[L^0_{\overline{h}}, \mathfrak{T}_g], \mathfrak{T}_h, \mathfrak{T}_k] + [\mathfrak{T}_g, [L^0_{\overline{h}}, \mathfrak{T}_h], \mathfrak{T}_k] + [\mathfrak{T}_g, \mathfrak{T}_h, [L^0_{\overline{h}}, \mathfrak{T}_k]] = 0.$$

3. Suppose $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_1, \mathfrak{T}_{\overline{h}}] \neq 0$. By Lemma 4.4-1 we would have $\overline{h} \in [g_0]$,

a contradiction. From here $[[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k], \mathfrak{T}_1, \mathfrak{T}_{\overline{h}}] = 0$. □

Theorem 4.1. *The following assertions hold.*

1. *For any $g_0 \in \Sigma^1$, the subtriple $\mathfrak{T}_{[g_0]} = \mathfrak{T}_{1,[g_0]} \oplus V_{[g_0]}$ of $\mathfrak{T}$ associated to $[g_0]$ is an ideal of $\mathfrak{T}$.*
2. *If $\mathfrak{T}$ is simple, then Σ^1 has all of its elements connected and*

$$\mathfrak{T}_1 = \sum_{g\in\Sigma^1, h\in\Sigma^1\cup\{1\}} [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}].$$

Proof. 1. Taking into account

$$\mathfrak{T}_{1,[g_0]} := \mathrm{span}_{\mathbb{K}}\{[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] : g \in [g_0], h \in [g_0] \cup \{1\}\} \subseteq \mathfrak{T}_1 \tag{5}$$

we have by Equations (1), (2) and (3) that $[\mathfrak{T}_{1,[g_0]}, \mathfrak{T}_1, \mathfrak{T}_1] \subseteq \mathfrak{T}_{1,[g_0]}$. Equation (5) together with Lemma 4.4 imply $[\mathfrak{T}_{1,[g_0]}, \mathfrak{T}_1, \mathfrak{T}_g] + [\mathfrak{T}_{1,[g_0]}, \mathfrak{T}_g, \mathfrak{T}_1] + [\mathfrak{T}_{1,[g_0]}, \mathfrak{T}_g, \mathfrak{T}_h] \subset \mathfrak{T}_{[g_0]}$ for any $g, h \in \Sigma^1$. From here,

$$[\mathfrak{T}_{1,[g_0]}, \mathfrak{T}, \mathfrak{T}] = [\mathfrak{T}_{1,[g_0]}, \mathfrak{T}_1 \oplus (\bigoplus_{g\in\Sigma^1_G} \mathfrak{T}_g), \mathfrak{T}_1 \oplus (\bigoplus_{h\in\Sigma^1} \mathfrak{T}_h)] \subset \mathfrak{T}_{[g_0]}. \tag{6}$$

Since $V_{[g_0]} := \bigoplus_{g\in[g_0]} \mathfrak{T}_g$, we have by Lemma 4.3 and Equation (5) that

$$[\bigoplus_{g\in[g_0]} \mathfrak{T}_g, \mathfrak{T}_1, \mathfrak{T}_1] + [\bigoplus_{g\in[g_0]} \mathfrak{T}_g, \mathfrak{T}_1, \mathfrak{T}_h] + [\bigoplus_{g\in[g_0]} \mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_1] + [\bigoplus_{g\in[g_0]} \mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k] \subset \mathfrak{T}_{[g_0]}$$

for any $h, k \in \Sigma^1$. So

$$[V_{[g_0]}, \mathfrak{T}, \mathfrak{T}] = [\bigoplus_{g\in[g_0]} \mathfrak{T}_g, \mathfrak{T}_1 \oplus (\bigoplus_{h\in\Sigma^1_G} \mathfrak{T}_h), \mathfrak{T}_1 \oplus (\bigoplus_{k\in\Sigma^1_G} \mathfrak{T}_k)] \subset \mathfrak{T}_{[g_0]}. \tag{7}$$

From Equations (6) and (7) we have

$$[\mathfrak{T}_{[g_0]}, \mathfrak{T}, \mathfrak{T}] = [\mathfrak{T}_{1,[g_0]} \oplus V_{[g_0]}, \mathfrak{T}, \mathfrak{T}] \subset \mathfrak{T}_{[g_0]}$$

and so$\mathfrak{T}_{[g_0]}$ is an ideal of $\mathfrak{T}$.

2. The simplicity of $\mathfrak{T}$ implies $\mathfrak{T}_{[g_0]} = \mathfrak{T}$. From here $[g_0] = \Sigma^1$ and $\mathfrak{T}_1 = \sum_{g\in\Sigma^1, h\in\Sigma^1\cup\{1\}} [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}]$. □

Theorem 4.2. *For a linear complement $\mathcal{U}$ of* $\mathrm{span}_{\mathbb{K}}\{[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] : g \in \Sigma^1, h \in \Sigma^1 \cup \{1\}\}$ *in $\mathfrak{T}_1$, we have*

$$\mathfrak{T} = \mathcal{U} + \sum_{[g]\in\Sigma^1/\sim} I_{[g]},$$

where any $I_{[g]}$ is one of the ideals described in Theorem 4.1, which also satisfy $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ if $[g] \neq [h]$.

Proof. By Proposition 3.1, we can consider the quotient set $\Sigma^1/\sim := \{[g] : g \in \Sigma^1\}$. We have $I_{[g]}$ is well defined and by Theorem 4.1-1 an ideal of $\mathfrak{T}$. Therefore

$$\mathfrak{T} = \mathcal{U} + \sum_{[g]\in\Sigma^1/\sim} I_{[g]}.$$

The assertion $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ if $[g] \neq [h]$ is consequence of writing $[I_{[g]}, \mathfrak{T}, I_{[h]}] = [\mathfrak{T}_{1,[g]} \oplus V_{[g]}, \mathfrak{T}_1 \oplus (\bigoplus_{k\in\Sigma^1} \mathfrak{T}_k), \mathfrak{T}_{1,[h]} \oplus V_{[h]}]$ and applying Lemma 4.3 and Lemma 4.4 taking into account $[g] \neq [h]$. $\square$

Observe that the fact $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ if $[g] \neq [h]$ implies

$$[I_{[g]}, I_{[h]}, \mathfrak{T}] = [[I_{[g]}, I_{[h]}], \mathfrak{T}] = 0.$$

As any element in $[I_{[g]}, I_{[h]}] \subset L^0$ is a linear mapping from $\mathfrak{T}$ onto itself we conclude

$$[I_{[g]}, I_{[h]}] = 0$$

if $[g] \neq [h]$.

Definition 3. We will say that $\mathfrak{T}_1$ is *tight* if $\mathfrak{T}_1 = \mathrm{span}_{\mathbb{K}}\{[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] : g \in \Sigma^1, h \in \Sigma^1 \cup \{1\}\}$.

Corollary 4.1. *If* $\mathrm{Ann}(\mathfrak{T}) = 0$ *and* $\mathfrak{T}_1$ *is tight, then* $\mathfrak{T}$ *is the direct sum of the ideals given in Theorem 4.1-1,*

$$\mathfrak{T} = \bigoplus_{[g]\in\Sigma^1/\sim} I_{[g]}.$$

Proof. From the fact $\mathfrak{T}_1$ is tight we clearly have

$$\mathfrak{T} = \sum_{[g]\in\Sigma^1/\sim} I_{[g]}.$$

To finish, we show the direct character of the sum. Given

$$x \in I_{[g]} \cap \Big(\sum_{[h]\in(\Sigma^1/\sim)\setminus[g]} I_{[h]} \Big)$$

we have from the fact $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ that

$$[x, \mathfrak{T}, I_{[g]}] + \Big[x, \mathfrak{T}, \sum_{[h]\in(\Sigma^1/\sim)\setminus[g]} I_{[h]}\Big] = 0.$$

It implies $[x, \mathfrak{T}, \mathfrak{T}] = 0$, that is, $x \in \mathrm{Ann}(\mathfrak{T}) = 0$. Thus $x = 0$. $\square$

5. The simple components

The study of the structure of graded Lie triple systems has been reduced to consider those satisfying the property saying that the support of the grading has all of its elements connected. Knowing whether or not such systems are simple is a natural question. Under mild conditions we give an affirmative answer to this question. We recall that Σ^i, $i = 0, 1$, is called *symmetric* if $g \in \Sigma^i$ implies $g^{-1} \in \Sigma^i$. From now on we will suppose Σ^i is symmetric for $i = 0, 1$.

We begin by introducing the concepts of Σ^1-multiplicativity and maximal length for graded Lie triple systems in a similar way than for graded Lie algebras, graded Lie superalgebras, graded Leibniz algebras, split Leibniz superalgebras, split Lie triple systems and split 3-Lie algebras (see [9, 12, 13, 14, 15, 16]).

Definition 5.1. It is said that a graded Lie triple system $\mathfrak{T}$ is Σ^1-*multiplicative* if given $g, h, k \in \Sigma^1 \cup \{1\}$, such that $gh \in \Sigma^0$ and $ghk \in \Sigma^1$, then $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_k] \neq 0$.

Definition 5.2. A graded Lie triple system $\mathfrak{T}$ is called of *maximal length* if $\dim \mathfrak{T}_g = 1$ for any $g \in \Sigma^1$.

Let us see some examples of Σ^1-multiplicative and of maximal length graded Lie triple systems:

Recall that a graded Lie algebra $\mathfrak{L}$ gives rise in a natural way to a graded Lie triple system (see Example 1). Now, if we take as $\mathfrak{L}$ a simple separable $\mathfrak{L}^*$-algebra [32], or a simple locally finite split Lie algebra over a field of characteristic zero, [2, 32], it is well known that any of such an algebras satisfies that if $\alpha, \beta, \alpha+\beta \in \Lambda$, where Λ denotes the set of nonzero roots, then $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] = \mathfrak{L}_{\alpha+\beta}$, (see [33, Proposition I.7 (v) and Theorem III.19]), and that $\dim \mathfrak{L}_\alpha = 1$. From here, we obtain that the Lie triple system $\mathcal{T}(\mathfrak{L})$ is Σ^1-multiplicative and of maximal length (see §1). We also can take as $\mathfrak{L}$ the split Lie algebras considered in [7] and the graded Lie algebras studied in [9], which also give rise to a Σ-multiplicative and of maximal length graded Lie triple system $\mathcal{T}(\mathfrak{L})$. Further examples are the Lie triple systems considered in [11, 30].

Since our next goal is to characterize the simplicity of a graded Lie triple system $\mathfrak{T}$ in terms of connections in its support, and taking into account $\mathrm{Ann}(\mathfrak{T})$ is an ideal of $\mathfrak{T}$ and Theorem 4.1-2, we are going to center on graded Lie triple systems satisfying $\mathrm{Ann}(\mathfrak{T}) = 0$ and with $\mathfrak{T}_1$ tight.

Lemma 5.1. *Let $\mathfrak{T}$ be a Σ^1-multiplicative graded Lie triple system with $\mathrm{Ann}(\mathfrak{T}) = 0$ and $\mathfrak{T}_1$ tight. If for any $g \in \Sigma^1$ we have $\dim L_g^0 \leq 1$, then any ideal I of $\mathfrak{T}$ such that $I \subset \mathfrak{T}_1$ satisfies $I = \{0\}$.*

Proof. Suppose there exists a nonzero ideal I of $\mathfrak{T}$ such that $I \subset \mathfrak{T}_1$. Given $g \in \Sigma^1$, as $[I, \mathfrak{T}_1, \mathfrak{T}_g] + [I, \mathfrak{T}_g, \mathfrak{T}_1] \subset \mathfrak{T}_g \cap \mathfrak{T}_1$ then $[I, \mathfrak{T}_1, \mathfrak{T}_g] = [I, \mathfrak{T}_g, \mathfrak{T}_1] = 0$. If we take $h \in \Sigma^1$ with $gh \neq 1$ we also have $[I, \mathfrak{T}_g, \mathfrak{T}_h] \subset \mathfrak{T}_{gh} \cap \mathfrak{T}_1 = 0$. Now, if $[I, \mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] \neq 0$ for some $g \in \Sigma^1$, then there exist $t_g \in \mathfrak{T}_g$, $t_{g^{-1}} \in \mathfrak{T}_{g^{-1}}$ and $t_1 \in I$ such that $[t_1, t_g, t_{g^{-1}}] \neq 0$. Hence $0 \neq [t_1, t_g] \in L_g^0$ and so necessarily $\dim L_g^0 = 1$. The Σ^1-multiplicativity of $\mathfrak{T}$ (consider $1, g, 1 \in \Sigma^1 \cup \{1\}$) and the fact $\dim L_g^0 = 1$ give us the existence of $0 \neq t_1' \in \mathfrak{T}_1$ such that $0 \neq [t_1, t_g, t_1'] \in \mathfrak{T}_g$. As $t_1 \in I$, we conclude $0 \neq t_g' := [t_1, t_g, t_1'] \in I \subset \mathfrak{T}_1$, a contradiction. From here $[I, \mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] = 0$. Finally, we can see that $[I, \mathfrak{T}_1, \mathfrak{T}_1] = 0$. Indeed, we have by the above

$$[I, \mathfrak{T}_1, [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}]] \subseteq [[I, \mathfrak{T}_1, \mathfrak{T}_g], \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] + [\mathfrak{T}_g, [I, \mathfrak{T}_1, \mathfrak{T}_h], \mathfrak{T}_{(gh)^{-1}}]$$
$$+ [\mathfrak{T}_g, \mathfrak{T}_h, [I, \mathfrak{T}_1, \mathfrak{T}_{(gh)^{-1}}]] = 0.$$

We have showed $I \subset \mathrm{Ann}(\mathfrak{T}) = 0$, a contradiction. Hence $I = \{0\}$. $\square$

From now on $\mathfrak{T}$ will be a graded Lie triple system of maximal length satisfying the hypothesis of Lemma 5.1 and with all of the elements in its support connected. If we consider a nonzero ideal I of $\mathfrak{T}$, then we can find $0 \neq x \in I$ such that $x = t_1 + \sum_{j=1}^m t_{g_j} \in I$, with $t_1 \in \mathfrak{T}_1 \cap I$, any $t_{g_j} \in \mathfrak{T}_{g_j} \cap I$ with $g_j \neq 1$, $g_j \neq g_k$ if $j \neq k$ and satisfying some $t_{g_j} \neq 0$ by the above lemma. Let us choose such an $x \in I$, and fix any g_{j_0}, $j_0 \in \{1, \ldots, m\}$, such that $t_{g_{j_0}} \neq 0$. Since $0 \neq t_{g_{j_0}} \in I$ and $\dim \mathfrak{T}_{g_{j_0}} = 1$, we get

$$0 \neq \mathfrak{T}_{g_{j_0}} \subset I. \tag{8}$$

Given any $h \in \Sigma^1$ with $h \notin \{g_{j_0}, g_{j_0}^{-1}\}$, as g_{j_0} and h are connected, the Σ^1-multiplicativity and maximal length of $\mathfrak{T}$ give us a connection $\{g_1, \ldots, g_{2r+1}\}$ from g_{j_0} to h such that

$$g_1 = g_{j_0}, g_1 g_2 g_3, \ldots, g_1 g_2 \cdots g_{2r+1} \in \Sigma^1,$$

$$g_1 g_2, g_1 g_2 g_3 g_4, \ldots, g_1 g_2 \cdots g_{2r} \in \Sigma^0,$$

and

$$g_1 g_2 \cdots g_{2r+1} \in \{h, h^{-1}\},$$

with

$$[\mathfrak{T}_{g_1}, \mathfrak{T}_{g_2}, \mathfrak{T}_{g_3}] = \mathfrak{T}_{g_1 g_2 g_3}, [[\mathfrak{T}_{g_1}, \mathfrak{T}_{g_2}, \mathfrak{T}_{g_3}], \mathfrak{T}_{g_4}, \mathfrak{T}_{g_5}] = \mathfrak{T}_{g_1 g_2 g_3 g_4 g_5},$$
$$\vdots$$
$$[[\ldots[[\mathfrak{T}_{g_1}, \mathfrak{T}_{g_2}, \mathfrak{T}_{g_3}], \mathfrak{T}_{g_4}, \mathfrak{T}_{g_5}], \ldots], \mathfrak{T}_{g_{2r}}, \mathfrak{T}_{g_{2r+1}}] \in \{\mathfrak{T}_h, \mathfrak{T}_{h^{-1}}\}.$$

From here, Equation (8) allows us to assert that

$$\text{either } 0 \neq \mathfrak{T}_h \subset I \text{ or } 0 \neq \mathfrak{T}_{h^{-1}} \subset I \text{ for any } h \in \Sigma^1 \tag{9}$$

and so $[\mathfrak{T}_h, \mathfrak{T}, \mathfrak{T}_{h^{-1}}] \subset I$. Observe that if $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] \neq 0$ with $h \notin \{1, g^{-1}\}$, then

$$0 \neq \mathfrak{T}_h = [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{g^{-1}}] \subset I$$

by Σ^1-multiplicativity, Equation (9) and the fact $\dim \mathfrak{T}_h = 1$. Therefore, we have as a consequence of $\mathfrak{T}_1 = \sum_{g\in\Sigma^1, h\in\Sigma^1\cup\{1\}} [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}]$ and Equation (9) that

$$\mathfrak{T}_1 \subset I. \tag{10}$$

Let us denote by

$$\Sigma^1_I := \{g \in \Sigma^1 : \mathfrak{T}_g \subset I\}$$

and by

$$J := \bigoplus_{h\in\Sigma^1\setminus\Sigma^1_I} \mathfrak{T}_h. \tag{11}$$

By the above, we can assert that given any $g \in \Sigma^1$ either g or g^{-1} belongs to Σ^1_I. We also have that in case $g, g^{-1} \in \Sigma^1_I$ for some $g \in \Sigma^1$, then $\Sigma^1_I = \Sigma^1$. Indeed, given any $h \in \Sigma^1$, $h \notin \{g, g^{-1}\}$, there exists a connection

$$\{g_1, \dots, g_{2n+1}\} \subset \Sigma^1 \cup \{1\}$$

such that $g_1 = h$,

$$g_1g_2g_3, \dots, g_1g_2\cdots g_{2n+1} \in \Sigma^1,$$
$$g_1g_2, \dots, g_1g_2\cdots g_{2n} \in \Sigma^0$$

and

$$g_1g_2\cdots g_{2n+1} \in \{g, g^{-1}\}.$$

From here, we also have the connection

$$\{g_1g_2\cdots g_{2n+1}, g^{-1}_{2n+1}, g^{-1}_{2n}, \dots, g^{-1}_2\} \subset \Sigma^1 \cup \{1\}$$

which satisfies

$$g_1g_2\cdots g_{2n+1}, g_1g_2\cdots g_{2n-1}, \dots, g_1 \in \Sigma^1,$$
$$g_1g_2\cdots g_{2n}, g_1g_2\cdots g_{2n-2}, \dots, g_2 \in \Sigma^0,$$
$$g_1g_2\cdots g_{2n+1} \in \{g, g^{-1}\}$$

and $g_1 = h$. By Σ^1-multiplicativity,

$$[[\dots[\mathfrak{T}_{g_1g_2\cdots g_{2n+1}}, \mathfrak{T}_{g^{-1}_{2n+1}}, \mathfrak{T}_{g^{-1}_{2n}}], \dots], \mathfrak{T}_{g_3^{-1}}, \mathfrak{T}_{g_2^{-1}}] = \mathfrak{T}_h.$$

Since $\mathfrak{T}_g + \mathfrak{T}_{g^{-1}} \subset I$ we obtain $\mathfrak{T}_h \subset I$ and so $\Sigma^1_I = \Sigma^1$. From here, and taking into account Equation (10), we can assert:

Lemma 5.2. *If $g, g^{-1} \in \Sigma^1_I$ for some $g \in \Sigma^1$, then $I = \mathfrak{T}$.*

We also have in this framework the next result.

Lemma 5.3. *If $I \neq \mathfrak{T}$, the following assertions hold.*

- (i) *For any $\bar{g} \in \Sigma^1 \setminus \Sigma^1_I$, we have $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}] = 0$.*
- (ii) *For any $\bar{g} \in \Sigma^1 \setminus \Sigma^1_I$ and $l \in \Sigma^1 \cup \{1\}$, we have $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_l, \mathfrak{T}_1] = 0$.*
- (iii) *For any $\bar{g} \in \Sigma^1 \setminus \Sigma^1_I$ and $g, h \in \Sigma^1_I$ with $\bar{g} \neq g^{-1}$, we have $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_h] = 0$.*
- (iv) *For any $\bar{g} \in \Sigma^1 \setminus \Sigma^1_I$ and $g, h \in \Sigma^1_I$ we have $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{\bar{g}}] = 0$.*

(v) *For any* $\bar{g}, \bar{h} \in \Sigma^1 \setminus \Sigma^1_I$ *and* $g \in \Sigma^1_I$, *we have* $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_{\bar{h}}] = 0$.

(vi) *For any* $\bar{g}, \bar{h}, \bar{k} \in \Sigma^1 \setminus \Sigma^1_I$, *we have* $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] = 0$ *if* $\bar{g}\bar{h}\bar{k} = 1$ *and* $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] \subset J$ *if* $\bar{g}\bar{h}\bar{k} \neq 1$.

(vii) *For any* $g \in \Sigma^1_I$ *we have* $[\mathfrak{T}_g, \mathfrak{T}_1, \mathfrak{T}_{g^{-1}}] + [\mathfrak{T}_{g^{-1}}, \mathfrak{T}_g, \mathfrak{T}_g] = 0$.

(viii) *For any* $g \in \Sigma^1_I$ *and* $h \in \Sigma^1_I \cup \{1\}$, *we have* $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] = 0$.

(ix) *For any* $g, h \in \Sigma^1_I$ *with* $h \neq g$ *we have* $[\mathfrak{T}_{g^{-1}}, \mathfrak{T}_h, \mathfrak{T}_g] + [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{g^{-1}}] = 0$.

Proof. (i) Suppose $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}] \neq 0$, then the Σ^1-multiplicativity and maximal length of $\mathfrak{T}$ give us $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}_1] = \mathfrak{T}_{\bar{g}}$. Equation (10) implies now $0 \neq \mathfrak{T}_{\bar{g}} \subset I$ and so $\bar{g} \in \Sigma^1_I$, a contradiction. Hence $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}] = 0$.

(ii) By item (i) we know $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}_1] = 0$. Hence, we just have to verify that $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_m, \mathfrak{T}_1] = 0$ for any $m \in \Sigma^1$. Suppose $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_m, \mathfrak{T}_1] \neq 0$, if $m^{-1} = \bar{g}$ we have

$$0 \neq [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{(\bar{g})^{-1}}, \mathfrak{T}_1] \subseteq [\mathfrak{T}_{(\bar{g})^{-1}}, \mathfrak{T}_1, \mathfrak{T}_{\bar{g}}] + [\mathfrak{T}_1, \mathfrak{T}_{\bar{g}}, \mathfrak{T}_{(\bar{g})^{-1}}].$$

Since $[\mathfrak{T}_1, \mathfrak{T}_{\bar{g}}, \mathfrak{T}_{(\bar{g})^{-1}}] = 0$ by item (i) then $[\mathfrak{T}_{(\bar{g})^{-1}}, \mathfrak{T}_1, \mathfrak{T}_{\bar{g}}] \neq 0$, so $(\bar{g})^{-1} \in \Sigma^0$ and consequently $\bar{g} \in \Sigma^0$. Hence, by the Σ^1-multiplicativity and maximal length of $\mathfrak{T}$ we get $\mathfrak{T}_{\bar{g}} = [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_1, \mathfrak{T}_1] \subset I$ and $\bar{g} \in \Sigma^1_I$, a contradiction. If $m^{-1} \neq \bar{g}$, then the Σ^1-multiplicativity and maximal length of $\mathfrak{T}$ give us now $\mathfrak{T}_{\bar{g}} = [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_m, \mathfrak{T}_{m^{-1}}] \subset I$ by Equation (9) and so $\bar{g} \in \Sigma^1_I$, a contradiction. Hence $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_m, \mathfrak{T}_1] = 0$ for any $m \in \Sigma^1$.

(iii) Suppose $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_h] \neq 0$. We have by the Σ^1-multiplicativity and maximal length of $\mathfrak{T}$ that $\mathfrak{T}_{\bar{g}} = [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_{g^{-1}}]$. From here, Equation (9) gives us $\bar{g} \in \Sigma^1_I$, a contradiction.

(iv) By Jacobi identity, if $\bar{g} \neq g^{-1}$ and $\bar{g} \neq h^{-1}$, then item (iii) completes the assertion. If $\bar{g} = g^{-1}$ and $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{\bar{g}}] \neq 0$, then $gh \in \Sigma^0$ and so $(gh)^{-1} \in \Sigma^0$. Therefore $\mathfrak{T}_{\bar{g}} = [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_h] \subset I$ by Equation (9). From here, $\bar{g} \in \Sigma^1_I$ a contradiction. If $\bar{g} = h^{-1}$ the proof is similar.

(v) If $\bar{g} = g^{-1}$ and $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_{\bar{h}}] \neq 0$, then $\bar{h} \in \Sigma^1_I$ a contradiction. Suppose now $\bar{g} \neq g^{-1}$ and $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_{\bar{h}}] \neq 0$, then $\mathfrak{T}_{\bar{g}} = [\mathfrak{T}_{\bar{g}}, \mathfrak{T}_g, \mathfrak{T}_{g^{-1}}] \subset I$ by Equation (9). From here, $\bar{g} \in \Sigma^1_I$ a contradiction.

(vi) In case $\bar{g}\bar{h}\bar{k} = 1$ then $\bar{k} = (\bar{g}\bar{h})^{-1}$. If $\bar{h} = (\bar{g})^{-1}$, item (ii) shows $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] = 0$; and if $\bar{h} \neq (\bar{g})^{-1}$, Lemma 5.2 gives us $(\bar{h})^{-1} \in \Sigma^1_I$ and then the Σ^1-multiplicativity of $\mathfrak{T}$ allows us to get that in case $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] \neq 0$ then $0 \neq [\mathfrak{T}_{(\bar{g})^{-1}}, \mathfrak{T}_{(\bar{h})^{-1}}, \mathfrak{T}_1] = \mathfrak{T}_{(\bar{g}\bar{h})^{-1}} \subset I$, that is $\bar{k} \in \Sigma^1_I$ what is a contradiction. Hence $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] = 0$.

Finally, suppose $\bar{g}\bar{h}\bar{k} \neq 1$ with $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] \neq 0$ and $\bar{g}\bar{h}\bar{k} \in \Sigma^1_I$. We have by Σ^1-multiplicativity, (taking into account that

$$\bar{g}\bar{h} \neq 1$$

by Equation (9)), that $[\mathfrak{T}_{\bar{g}\bar{h}\bar{k}}, \mathfrak{T}_{(\bar{k})^{-1}}, \mathfrak{T}_{(\bar{h})^{-1}}] = \mathfrak{T}_{\bar{g}} \subset I$. From here, $\bar{g} \in \Sigma^1_I$, a contradiction. We conclude that in case $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] \neq 0$ with $\bar{g}\bar{h}\bar{k} \neq 1$ then $\bar{g}\bar{h}\bar{k} \in \Sigma^1 \setminus \Sigma^1_I$, that is, $[\mathfrak{T}_{\bar{g}}, \mathfrak{T}_{\bar{h}}, \mathfrak{T}_{\bar{k}}] \subset J$.

(vii) Suppose $[\mathfrak{T}_g, \mathfrak{T}_1, \mathfrak{T}_{g^{-1}}] \neq 0$. Since $I \neq \mathfrak{T}$, Lemma 5.2 tells us that $g^{-1} \in \Sigma^1 \setminus \Sigma^1_I$. Jacobi identity and items (i) and (ii) give us now a contradiction. From here $[\mathfrak{T}_g, \mathfrak{T}_1, \mathfrak{T}_{g^{-1}}] = 0$.

Suppose now $[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_g] \neq 0$. Taking into account $\dim \mathfrak{T}_g = 1$ for any $g \in \Sigma^1$ we get

$$[[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}], [\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}]] = 0.$$

From here, and taking into account that Lemma 5.2 gives us $g^{-1} \in \Sigma^1 \setminus \Sigma^1_I$, item (v) allows us to obtain

$$[[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_g], \mathfrak{T}_{g^{-1}}] = 0.$$

However, $0 = [[\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}, \mathfrak{T}_g], \mathfrak{T}_{g^{-1}}] = [\mathfrak{T}_g, \mathfrak{T}_{g^{-1}}]$ a contradiction.

(viii) If $h = 1$ the assertion if consequence of the fact that Lemma 5.2 gives us $g^{-1} \in \Sigma^1 \setminus \Sigma^1_I$ and items (i) and (ii). If $h \neq 1$ suppose $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}] \neq 0$. Since $h \neq g^{-1}$ by Lemma 5.2, then $0 \neq \mathfrak{T}_{gh} = [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_1] \subset I$ and so $gh \in \Sigma^1_I$. By Lemma 5.2 we have $(gh)^{-1} \in \Sigma^1 \setminus \Sigma^1_I$ what contradicts item (iv).

(ix) As $I \neq \mathfrak{T}$, Lemma 5.2 gives us that $g^{-1} \in \Sigma^1 \setminus \Sigma^1_I$ then by item (iii) we have $[\mathfrak{T}_{g^{-1}}, \mathfrak{T}_h, \mathfrak{T}_g] = 0$. If $[\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{g^{-1}}] \neq 0$, then $gh \in \Sigma^0$ and so $(gh)^{-1} \in \Sigma^0$. Therefore $0 \neq \mathfrak{T}_{g^{-1}} = [\mathfrak{T}_{g^{-1}}, \mathfrak{T}_{h^{-1}}, \mathfrak{T}_h] \subset I$ by Equation (9). Hence $g^{-1} \in \Sigma^1_I$ a contradiction. $\square$

As $\mathfrak{T}_1 = \sum_{g \in \Sigma^1, h \in \Sigma^1 \cup \{1\}} [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}]$, the fact that for any $g \in \Sigma^1$ either g or g^{-1} belongs to Σ^1_I together with Lemma 5.3-(i), (ii), (iii), (v) and (vi) allow us to write

$$\mathfrak{T}_1 = \sum_{g \in \Sigma^1_I, h \in \Sigma^1_I \cup \{1\}} [\mathfrak{T}_g, \mathfrak{T}_h, \mathfrak{T}_{(gh)^{-1}}].$$

So Lemma 5.3-(viii) gives us that in case $I \neq \mathfrak{T}$ we have $\mathfrak{T}_1 = 0$. That is:

Lemma 5.4. *If $\mathfrak{T}_1 \neq 0$, then $I = \mathfrak{T}$.*

Lemma 5.5. *If $\mathfrak{T}_1 = 0$, then either $I = \mathfrak{T}$ or $\mathfrak{T} = I \oplus J$ with I and J simple ideals of $\mathfrak{T}$ and satisfying $[I, \mathfrak{T}, J] + [J, \mathfrak{T}, I] = 0$ in the second case.*

Proof. If $g, g^{-1} \in \Sigma^1_I$ for some $g \in \Sigma^1$, then Lemma 5.2 gives us $I = \mathfrak{T}$. Suppose then that for any $g \in \Sigma^1_I$ we have $g^{-1} \in \Sigma^1 \setminus \Sigma^1_I$ and consider the graded linear space J given by Equation (11). Let us verify that J is actually an ideal of $\mathfrak{T}$. Indeed, observe that we can write

$$[J, \mathfrak{T}, \mathfrak{T}]$$

$$= [\bigoplus_{\bar{g} \in \Sigma^1 \setminus \Sigma^1_I} \mathfrak{T}_{\bar{g}}, (\bigoplus_{g \in \Sigma^1_I} \mathfrak{T}_g) \oplus (\bigoplus_{\bar{h} \in \Sigma^1 \setminus \Sigma^1_I} \mathfrak{T}_{\bar{h}}), (\bigoplus_{h \in \Sigma^1_I} \mathfrak{T}_h) \oplus (\bigoplus_{\bar{k} \in \Sigma^1 \setminus \Sigma^1_I} \mathfrak{T}_{\bar{k}})]. \tag{12}$$

But Lemma 5.3-(iii), (vii) and (ix) give us that

$$[\bigoplus_{\bar{g} \in \Sigma^1 \setminus \Sigma^1_I} \mathfrak{T}_{\bar{g}}, \bigoplus_{g \in \Sigma^1_I} \mathfrak{T}_g, \bigoplus_{h \in \Sigma^1_I} \mathfrak{T}_h] = 0, \tag{13}$$

Lemma 5.3-(v) that

$$(14)\quad [\bigoplus_{\bar{g}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{g}}, \bigoplus_{g\in\Sigma_I^1}\mathfrak{T}_g, \bigoplus_{\bar{k}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{k}}] + [\bigoplus_{\bar{g}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{g}}, \bigoplus_{\bar{h}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{h}}, \bigoplus_{h\in\Sigma_I^1}\mathfrak{T}_h] = 0$$

and Lemma 5.3-(vi) that

$$(15)\qquad [\bigoplus_{\bar{g}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{g}}, \bigoplus_{\bar{h}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{h}}, \bigoplus_{\bar{k}\in\Sigma^1\setminus\Sigma_I^1}\mathfrak{T}_{\bar{k}}] \subset J.$$

Hence, Equations (12), (13), (14), (15) allow us to assert that J is an ideal of $\mathfrak{T}$, being also $[I,\mathfrak{T},J]=0$ as consequence of Lemma 5.3-(iv) and (v) and $[J,\mathfrak{T},I]=0$ by Lemma 5.3-(iii), (vii), (ix) and (v).

Finally, the simplicity of $I=\bigoplus_{g\in\Sigma_I^1}\mathfrak{T}_g$ is obtained by observing that the Σ^1-multiplicativity of $\mathfrak{T}$ gives us that Σ_I^1 has all of its elements Σ_I^1-connected, that is, connected through connections contained in $\Sigma_I^1\cup\{1\}$, and that I is Σ_I^1-multiplicative. We also clearly have $\dim\mathfrak{T}_g=1$ for any $g\in\Sigma_I^1$, and $\mathrm{Ann}_I(I)=0$, $(\mathrm{Ann}_I(I):=\{x\in I:[x,I,I]=0\})$, as consequence of $\mathfrak{T}_1=0$, the fact $[I,\mathfrak{T},J]+[J,\mathfrak{T},I]=0$ and $\mathrm{Ann}(\mathfrak{T})=0$. So, we can argue as in the beginning of this section with the Σ_I^1-multiplicativity of I and the condition $\dim\mathfrak{T}_g=1$ for any $g\in\Sigma_I^1$, taking into account Lemma 5.2, to conclude that any nonzero graded ideal $\widetilde{I}$ of I is necessarily $\widetilde{I}=I$. Hence I is a simple ideal. The same argument applies to show J is also a simple ideal. □

Now we can state the following main results.

Theorem 5.1. *If $\mathfrak{T}_1\neq 0$, then $\mathfrak{T}$ is simple if and only if the support has all of its elements connected.*

Proof. The if condition is Theorem 4.1-2 and the converse is a consequence of Lemma 5.4. □

Theorem 5.2. *If $\mathfrak{T}_1=0$, then $\mathfrak{T}$ is simple if and only if the support has all of its elements connected and $\mathfrak{T}$ is prime.*

Proof. The if condition is Theorem 4.1-2 and the converse is a consequence of Lemma 5.5. □

Theorem 5.3. *A graded Lie triple system $\mathfrak{T}$ is the direct sum of the family of its minimal ideals, each one being a simple graded Lie triple system I_j having its support, $\Sigma_{I_j}^1$, with all of its elements connected.*

Proof. By Corollary 4.1, $\mathfrak{T}=\bigoplus_{[g]\in\Sigma^1/\sim}I_{[g]}$ is the direct sum of the ideals

$$I_{[g]}=\mathfrak{T}_{1,[g]}\oplus V_{[g]}=(\sum_{h\in[g],k\in[g]\cup\{1\}}[\mathfrak{T}_h,\mathfrak{T}_k,\mathfrak{T}_{(hk)^{-1}}])\oplus(\bigoplus_{h\in[g]}\mathfrak{T}_h),$$

having any $I_{[g]}$ its support, $\Sigma_{I_{[g]}}^1=[g]$, with all of its elements connected. Taking into account that $\Sigma_{I_{[g]}}^1=[g]$ and the Σ^1-multiplicativity of $\mathfrak{T}$, we have that

$\Sigma^1_{I_{[g]}}$ has all of its elements $\Sigma^1_{I_{[g]}}$-connected. We also have that any of the $I_{[g]}$ is $\Sigma^1_{I_{[g]}}$-multiplicative as consequence of the Σ^1-multiplicativity of $\mathfrak{T}$. Clearly $\dim \mathfrak{T}_g = 1$ for any $g \in \Sigma^1_{I_{[g]}}$, and finally $\mathrm{Ann}_{I_{[g]}}(I_{[g]}) = 0$ as a consequence of $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ if $[g] \neq [h]$ (Corollary 4.1), and $\mathrm{Ann}(\mathfrak{T}) = 0$. If $\mathfrak{T}_{1,[g]} \neq 0$, we can apply Theorem 5.1 to obtain $I_{[g]}$ is simple. If $\mathfrak{T}_{1,[g]} = 0$, Lemma 5.5 implies either $I_{[g]}$ is simple (we will denote $\overline{I}_{[g]} := I_{[g]}$ in this case), or $I_{[g]} = \widetilde{I}_{[g]} \oplus \widetilde{J}_{[g]}$, where $\widetilde{I}_{[g]}, \widetilde{J}_{[g]}$ are simple ideals of $I_{[g]}$ satisfying $[\widetilde{I}_{[g]}, \mathfrak{T}, \widetilde{J}_{[g]}] + [\widetilde{J}_{[g]}, \mathfrak{T}, \widetilde{I}_{[g]}] = 0$. The fact $[I_{[g]}, \mathfrak{T}, I_{[h]}] = 0$ if $[g] \neq [h]$ ensures $\widetilde{I}_{[g]}, \widetilde{J}_{[g]}$ are also simple ideals of $\mathfrak{T}$. We conclude

$$\mathfrak{T} = \Big(\bigoplus_{\substack{[g]\in\Sigma^1/\sim;\\ \mathfrak{T}_{1,[g]}\neq 0}} I_{[g]}\Big) \oplus \Big(\bigoplus_{\substack{[h]\in\Sigma^1/\sim;\\ \mathfrak{T}_{1,[h]}=0}} \overline{I}_{[h]}\Big) \oplus \Big(\bigoplus_{\substack{[k]\in\Sigma^1/\sim;\\ \mathfrak{T}_{1,[k]}=0}} \widetilde{I}_{[k]}\Big) \oplus \Big(\bigoplus_{\substack{[k]\in\Sigma^1/\sim;\\ \mathfrak{T}_{1,[k]}=0}} \widetilde{J}_{[k]}\Big).$$

From the above, it is easy to verify that this decomposition satisfies the assertions of the theorem. □

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