Bull. Korean Math. Soc. **53** (2016), No. 1, pp. 139–151
http://dx.doi.org/10.4134/BKMS.2016.53.1.139

TOTAL COLORINGS OF PLANAR GRAPHS WITH MAXIMUM DEGREE AT LEAST 7 AND WITHOUT ADJACENT 5-CYCLES

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Abstract. A k-total-coloring of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The total chromatic number $\chi''(G)$ of G is the smallest integer k such that G has a k-total-coloring. Let G be a planar graph with maximum degree Δ. In this paper, it's proved that if $\Delta \geq 7$ and G does not contain adjacent 5-cycles, then the total chromatic number $\chi''(G)$ is $\Delta + 1$.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. And we follow [2] for the terminologies and notations not defined here. Let G be a planar graph which has been embedded in the plane. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, F, Δ and δ) to denote the vertex set, the edge set, the face set, the maximum degree and the minimum degree of G, respectively. A k-cycle is a cycle of length k, two cycles are said to be intersecting if they are incident with a common vertex, and adjacent if they share at least one edge.

A *k-total-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph is *totally k-colorable* if it admits a k-total-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G is totally k-colorable. It's clear that $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [15] independently posed the famous conjecture, which is known as the Total Coloring Conjecture (TCC).

Conjecture A. *For any graph G, $\Delta + 1 \leq \chi''(G) \leq \Delta + 2$.*

Received January 5, 2015; Revised June 2, 2015.
2010 *Mathematics Subject Classification.* 05C15.
Key words and phrases. planar graph, total coloring, adjacent 5-cycle.
This work was supported by National Natural Science Foundation of China(11401386), Natural Science Foundation of Shandong Province (ZR2013AM006) and A Project of Shandong Province Higher Educational Science and Technology Program(J14LI55).

This conjecture was confirmed for $\Delta \leq 5$ (see [24]). For planar graphs, the only open case is $\Delta = 6$ (see [10, 12]). Moreover, if G is a planar graph with maximum degree $\Delta \geq 9$, then $\chi''(G) = \Delta + 1$ (see [3, 11, 22]). However, for $4 \leq \Delta \leq 8$, it's unknown if every planar graph with maximum degree Δ is totally $(\Delta + 1)$-colorable. The study of this has been attracted considerable attention. Some related results can be found in [4-9, 13, 14, 16-23]. Wang, Sun, et al. [23] proved that planar graphs with $\Delta \geq 7$ and without 5-cycles with chords are totally $(\Delta + 1)$-colorable. Wang and Wu [16] proved that if G is a planar graph with $\Delta \geq 7$ and without intersecting 5-cycles, then $\chi''(G) = \Delta + 1$. In this paper, we get the following theorem.

Theorem 1. *If G is a planar graph with $\Delta \geq 7$ and without adjacent 5-cycles, then $\chi''(G) = \Delta + 1$.*

For convenience, we introduce some more notations and definitions. Let $G = (V, E, F)$ be a planar graph. A k-, k^+- or k^--vertex is a vertex of degree k, at least k or at most k, respectively. The degree of f, denoted by $d(f)$, is the number of edges incident with it, where each cut edge counts twice. The notations of k-, k^+- or k^--face are defined analogously as for the vertices. A k-face with consecutive vertices $v_1, v_2, \ldots, v_k$ along its boundary is often said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face. For $v \in V(G)$, we use $N(v)$ to denote the set of vertices which are adjacent to v, $n_i(v)$ to denote the number of i-vertices adjacent to v, $f_i(v)$ to denote the number of i-faces incident with v.

2. Reducible configurations

In [19], Theorem 1 was proved for $\Delta \geq 8$. So it suffices to consider the case that $\Delta = 7$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1 in terms of vertices and edges. Then every proper subgraph of G is totally 8-colorable. Let $L = \{1, 2, \ldots, 8\}$ be the color set for simplicity. We first give some lemmas for G.

Lemma 2 ([3, 6, 13])**.** *The graph G has the following properties*:

(a) *G is 2-connected. Hence $\delta(G) \geq 2$ and the boundary of each face is exactly a cycle.*
(b) *Let $uv \in E(G)$. If $d(u) \leq 3$, then $d_G(u) + d_G(v) \geq \Delta + 2 = 9$. Hence the two neighbors of a 2-vertex are 7-vertices*; *and the three neighbors of a 3-vertex are 6^+-vertices.*
(c) *G contains no even cycle $(v_1, v_2, \ldots, v_{2t})$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$.*
(d) *G has no $(4, 4, 7^-)$-face.*
(e) *If v is a 7-vertex of G with $n_2(v) \geq 1$, then $n_{4^+}(v) \geq 1$.*

Note that in all figures of the paper, the vertices marked by • have no other neighbors in G other than those shown.

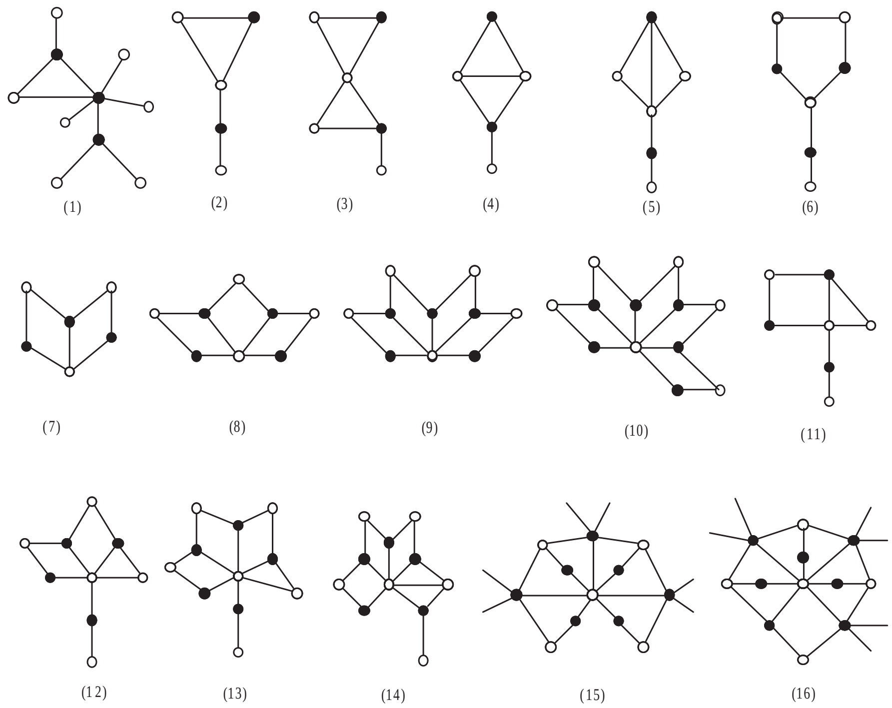


FIGURE 1. Reducible configurations

Lemma 3. *G contains no subgraph isomorphic to the configurations depicted in Fig. 1.*

The proof that G contains no configurations depicted in Fig. 1(1)-(16) can be found in [3, 5, 8, 11, 16, 23].

Let φ be a (partial) 8-total-coloring of G. For each element $x \in V \cup E$, we denote by $C(x)$ the set of colors of vertices and edges incident or adjacent to x. If $v \in V$, we set $S(v) := \{\varphi(uv), u \in N(v)\}$ and $\overline{S}(v) := S(v) \cup \varphi(v)$. Call φ is nice if only some 3^{-}-vertices are not colored. Note that every nice coloring can be greedily extended to a 8-total-coloring of G, since each 3^{-}-vertex has at most 6 forbidden colors. Therefore, in the rest of this paper, we shall always suppose that such vertices are colored at the very end.

Lemma 4. *G has no subgraph isomorphic to the configurations depicted in Fig. 2.*

Proof. (1) On the contrary, suppose G contains a configuration as depicted in Fig. 2(1). By the minimality of G, $G' = G - vv_1$ has a proper 8-total-coloring φ. Without loss of generality, suppose that $\varphi(vv_i) = i$ $(i = 2, 3, \ldots, 7)$ and $\varphi(v) = 8$. If $\varphi(v_1x) \neq 1$, we can color vv_1 with 1 to obtain a nice coloring of G,

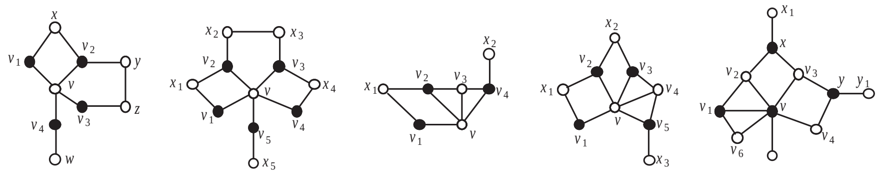


FIGURE 2. Reducible configurations

a contradiction. Thus $\varphi(v_1x) = 1$. Moreover, we infer that $\varphi(v_3z) = 1$. Since otherwise, we can recolor vv_3 with 1, and color vv_1 with 3, a contradiction. Similarly, $\varphi(v_4w) = \varphi(v_2y) = 1$. First we recolor the edges v_2y and v_3z with $\varphi(yz)$ and yz with 1. In the following, we split the proof into three cases.

Case 1. $\varphi(yz) = 2$. Then we can recolor vv_2 with 1, and color vv_1 with 2.

Case 2. $\varphi(yz) = 3$. If $\varphi(v_2x) \neq 3$, then we can recolor vv_3 with 1, and color vv_1 with 3. Otherwise, we can interchange the colors of v_1x and v_2x, and recolor vv_3 with 1, vv_4 with 3, and color vv_1 with 4.

Case 3. $\varphi(yz) \notin \{2, 3\}$. Then we interchange the colors of v_1x and v_2x, and color vv_1 with 1, a contradiction.

(2) Suppose G contains a configuration as depicted in Fig. 2(2). Then $G' = G - vv_1$ has a proper 8-total-coloring φ. Without loss of generality, suppose that $\varphi(vv_i) = i$ $(i = 2, 3, \ldots, 7)$ and $\varphi(v) = 8$. Obviously, $\varphi(v_1x_1) = 1$. If $1 \notin S(v_2)$, we can recolor vv_2 with 1, and color vv_1 with 2 to obtain a nice coloring of G, a contradiction. Thus $\varphi(v_2x_2) = 1$. Similarly, $\varphi(v_3x_3) = \varphi(v_4x_4) = \varphi(v_5x_5) = 1$. First we recolor v_2x_2 and v_3x_3 with $\varphi(x_2x_3)$ and x_2x_3 with 1. In the following, we split the proof into three cases.

Case 1. $\varphi(x_2x_3) = 2$. If $\varphi(x_3x_4) \neq 2$, we can recolor vv_2 with 1, and color vv_1 with 2. Otherwise, then interchange the colors of v_3x_4 and v_4x_4, and recolor vv_2 with 1, color vv_1 with 2.

Case 2. $\varphi(x_2x_3) = 3$. If $\varphi(v_2x_1) \neq 3$, then recolor vv_3 with 1, and color vv_1 with 3. Otherwise, then interchange the colors of v_2x_1 and v_1x_1, and recolor vv_3 with 1, vv_4 with 3, color vv_1 with 4.

Case 3. $\varphi(x_2x_3) \notin \{2, 3\}$. Then interchange the colors of v_2x_1 and v_1x_1, and also of v_3x_4 and v_4x_4. If $\varphi(v_3x_4) \neq 4$, then color vv_1 with 1. If $\varphi(v_3x_4) = 4$, $\varphi(v_2x_1) \neq 4$, then recolor vv_4 with 1, and color vv_1 with 4. If $\varphi(v_3x_4) = \varphi(v_2x_1) = 4$, then recolor vv_4 with 1, vv_5 with 4, and color vv_1 with 5.

(3) suppose G contains a configuration as depicted in Fig. 2(3). Consider a nice coloring φ of $G' = G - vv_1$. Without loss of generality, suppose that $\varphi(vv_i) = i$ $(i = 2, 3, \ldots, 7)$ and $\varphi(v) = 8$. Obviously, $\varphi(v_1x_1) = 1$. If $1 \notin S(v_2)$, then recolor vv_2 with 1, and color vv_1 with 2. Thus we can get a nice coloring of G, a contradiction. Hence $\varphi(v_2v_3) = 1$. Similarly, $\varphi(v_4x_2) = 1$. Now we

interchange the colors of vv_3 and v_2v_3. If $\varphi(v_2x_1) \neq 3$, then color vv_1 with 3. Otherwise, then interchange the colors of v_2x_1 and v_1x_1, and recolor vv_4 with 3, color vv_1 with 4.

(4) The proof is similar to the previous case, we omit here.

(5) suppose G contains a configuration as depicted in Fig. 2(5). Consider a nice coloring φ of $G' = G - vv_1$. Without loss of generality, suppose that $\varphi(vv_i) = i + 1$ $(i = 2, 3, \ldots, 6)$, $\varphi(v) = 8$, and $\varphi(v_6v_1) = 1$, $\varphi(v_1v_2) = 2$. We split the proof into two cases.

Case 1. $\varphi(v_2x) \neq 4$.

Suppose $2 \notin S(x)$. First we interchange the colors of v_2x and v_2v_1. If $\varphi(v_2x) \neq 1$, then color vv_1 with 2. Otherwise, then interchange the colors of v_1v_6 and vv_6, and color vv_1 with 2.

Suppose $2 \in S(x)$, $3 \notin S(x)$. First recolor v_2x with 3, v_1v_2 with $\varphi(v_2x)$, and recolor vv_2 with 2. If $\varphi(v_2x) \neq 1$, then color vv_1 with 3. Otherwise, then interchange the colors of v_1v_6 and vv_6, and color vv_1 with 3.

Suppose $2 \in S(x)$, $3 \in S(x)$. Without loss of generality, let $\varphi(v_3x) = 2, \varphi(xx_1) = 3$. We can interchange the colors of vv_3 and v_3x, and color vv_1 with 4.

Case 2. $\varphi(v_2x) = 4$.

If $2 \notin S(x)$, then we interchange the colors of v_2x and v_2v_1, and color vv_1 with 2. If $2 \in S(x)$, $3 \notin S(x)$, then we recolor v_2x with 3, v_1v_2 with 4, vv_2 with 2, and color vv_1 with 3.

If $2 \in S(x)$, $3 \in S(x)$, then without loss of generality, let $\varphi(v_3x) = 2$, $\varphi(xx_1) = 3$.

Subcase 2.1. $2 \notin S(y)$.

First interchange the colors of v_3x and v_3y. If $\varphi(v_3y) = 3$, then recolor vv_3 with 3, v_3x with 4, v_2x with 2, vv_2 with 4, v_1v_2 with 3, and color vv_1 with 2. Otherwise, then interchange the colors of v_1v_2 and v_2x, and color vv_1 with 2.

Subcase 2.2. $2 \in S(y)$. Then $\varphi(v_4y) = 2$ or $\varphi(yy_1) = 2$.

Subcase 2.2.1. $\varphi(v_4y) = 2$.

Suppose $\varphi(v_3y) \neq 3$. First interchange the colors of v_3y and v_3x. If $\varphi(yy_1) = 4$, then interchange the colors of v_4y and vv_4, and color vv_1 with 5. Otherwise, then recolor v_3y with 4, vv_3 with 2, and color vv_1 with 4.

Suppose $\varphi(v_3y) = 3$, $\varphi(yy_1) = 4$. Then interchange the colors of vv_4 and v_4y, and color vv_1 with 5.

Suppose $\varphi(v_3y) = 3$, $\varphi(yy_1) \neq 4$. Then interchange the colors of vv_3 and v_3y, and also of vv_2 and v_1v_2, and color vv_1 with 4.

Subcase 2.2.2. $\varphi(yy_1) = 2$.

Suppose $\varphi(v_3y) \neq 3$. First we recolor v_3x with $\varphi(v_3y)$, v_3y with 4, and vv_3 with 2. If $\varphi(v_4y) = 4$, then interchange the colors of v_4y and vv_4, and color vv_1 with 5. Otherwise, we color vv_1 with 4.

Suppose $\varphi(v_3y) = 3$, $\varphi(v_4y) = 4$. Then we interchange the colors of vv_4 and v_4y, and also of vv_3 and v_3y, and vv_2 and v_1v_2. Finally, we color vv_1 with 5.

Suppose $\varphi(v_3y) = 3$, $\varphi(v_4y) \neq 4$. Then we interchange the colors of vv_3 and v_3y, and also of vv_2 and v_1v_2. Finally, we color vv_1 with 4. Thus we can obtain a nice coloring of G, a contradiction. □

3. Discharging

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v) - 6) + \sum_{f\in F}(d(f) - 6) = -12 < 0. \tag{1}$$

We define ch to be the initial charge. Let $ch(v) = 2d(v)-6$ for each $v \in V(G)$ and $ch(f) = d(f) - 6$ for each $f \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12. \tag{2}$$

In the following, we will show that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to (2), which completes the proof.

For a k-face $f = v_1v_2 \dots v_k$, we use $(d(v_1), d(v_2), \dots, d(v_k)) \to (c_1, c_2, \dots, c_k)$ to denote the vertex v_i sends f the amount of charge c_i for $i = 1, 2, \dots, k$.

Our discharging rules are defined as follows.

R1. Each 2-vertex receives 1 from each of its neighbors.

R2. Suppose $f = v_1v_2v_3$ is a 3-face, let
$(3^-, 6^+, 6^+) \to (0, \frac{3}{2}, \frac{3}{2})$,
$(4, 5^+, 5^+) \to (\frac{2}{3}, \frac{7}{6}, \frac{7}{6})$,
$(5^+, 5^+, 5^+) \to (1, 1, 1)$.

R3. Suppose $f = v_1v_2v_3v_4$ is a 4-face, let
$(3^-, 6^+, 3^-, 6^+) \to (0, 1, 0, 1)$,
$(3^-, 6^+, 4, 6^+) \to (0, \frac{3}{4}, \frac{1}{2}, \frac{3}{4})$,
$(3^-, 6^+, 5, 6^+) \to (0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3})$,
$(3^-, 6^+, 6^+, 6^+) \to (0, \frac{1}{2}, 1, \frac{1}{2})$,
$(4^+, 4^+, 4^+, 4^+) \to (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

R4. Suppose $f = v_1v_2v_3v_4v_5$ is a 5-face, let
$(3^-, 6^+, 6^+, 3^-, 6^+) \to (0, \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3})$,
$(3^-, 6^+, 4^+, 4^+, 6^+) \to (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$,
$(4^+, 4^+, 4^+, 4^+, 4^+) \to (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$.

The rest of this paper is to check that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$.

It's obvious that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices $v \in V$ by Lemma 2 and our discharging rules. So we only need to check that $ch'(v) \geq 0$ for all 3^+-vertices in G.

If $d(v) = 3$, then $ch'(v) = ch(v) = 0$.

If $d(v) = 4$, then $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = f_5(v) = 0$. Thus $ch'(v) \geq ch(v) - \frac{2}{3} \times 3 = 0$. If $f_3(v) = 2$ and the two 3-faces are adjacent, then $f_4(v) + f_5(v) \leq 1$. Otherwise, if the two 3-faces are not adjacent, then $f_4(v) = 0$. Thus $ch'(v) \geq ch(v) - \max\{\frac{2}{3} \times 2 + \frac{1}{2}, \frac{2}{3} \times 2 + \frac{1}{4} \times 2\} = \frac{1}{6} > 0$. If $f_3(v) = 1$, then $f_4(v) \leq 2$. Thus $ch'(v) \geq ch(v) - \frac{2}{3} - \frac{1}{2} \times 2 - \frac{1}{4} = \frac{1}{12} > 0$. If $f_3(v) = 0$, then $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$.

If $d(v) = 5$, then $f_3(v) \leq 3$. If $f_3(v) = 3$, then $f_4(v) = 0$. Thus $ch'(v) \geq ch(v) - \frac{7}{6} \times 3 - \frac{1}{4} \times 2 = 0$. If $f_3(v) = 2$, then $f_4(v) \leq 2$. Thus $ch'(v) \geq ch(v) - \frac{7}{6} \times 2 - \frac{2}{3} \times 2 - \frac{1}{4} = \frac{1}{12} > 0$. If $f_3(v) \leq 1$, then $ch'(v) \geq ch(v) - \frac{7}{6} - \frac{2}{3} \times 4 = \frac{1}{6} > 0$.

If $d(v) = 6$, then $f_3(v) \leq 4$. If $f_3(v) = 4$, then $f_4(v) = 0$. And by Lemma 3(1), we can get $n_3(v) \leq 1$. So $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{7}{6} \times 2 - \frac{1}{3} \times 2 = 0$. If $f_3(v) = 3$, then $f_4(v) \leq 1$. Thus $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{7}{6} - 1 - \frac{2}{3} = \frac{1}{6} > 0$ by Lemma 3. If $f_3(v) = 2$, then $f_4(v) \leq 3$. And if the two 3-faces are adjacent, then $ch'(v) \geq ch(v) - \max\{\frac{3}{2} + \frac{7}{6} + 3 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{1}{2} + \frac{1}{3}, \frac{3}{2} \times 2 + 1 \times 2 + \frac{1}{3} \times 2\} = 0$ by Lemmas 3(1), 4(5) and our discharging rules. Otherwise, $ch'(v) \geq ch(v) - \frac{3}{2} - \frac{7}{6} - 3 - \frac{1}{3} = 0$ by Lemma 3(1). If $f_3(v) \leq 1$, then $ch'(v) \geq ch(v) - \max\{\frac{3}{2} + 4 + \frac{1}{3}, 6\} = 0$.

If $d(v) = 7$, this situation is very complicated. For convenience, we introduce some more notations and definitions. An *l-fan* (or simply F_l) is a configuration consisting of l consecutive faces around v and their edges incident with v, starting and ending with $(2, 7)$-edges, and containing no other $(2, 7)$-edge incident with v. An l-fan with consecutive faces $f_1, f_2, \ldots, f_l$ is often said to be a $(k_1^+, k_2^+, \ldots, k_l^+)$-fan if $d(f_1) \geq k_1, d(f_2) \geq k_2, \ldots, d(f_l) \geq k_l$. And we use $\tau(v \to x)$ to denote the sum of the charges that v sends to x.

Lemma 5. (1) $\tau(v \to F_2) \leq \frac{3}{2}$. *Especially, if F_2 is a $(4, 4)$-fan and incident with a 6^+-neighbor of v, or if F_2 is a $(4, 5)$-fan and v is adjacent to at least three 2-vertices, or if F_2 is a $(4^+, 6^+)$-fan or a $(5^+, 5^+)$-fan, then $\tau(v \to F_2) \leq 1$.*

(2) $\tau(v \to F_3) \leq \frac{5}{2}$. *Especially, if F_3 is a $(4, 5, 4)$-fan and v is adjacent to at least three 2-vertices, or if F_3 is a $(4^+, 6^+, 4^+)$-fan, or a $(4^+, 4^+, 6^+)$-fan, or a $(4^+, 5^+, 5^+)$-fan, or a $(5^+, 4^+, 5^+)$-fan, or if F_3 is a $(4, 4, 4)$-fan and incident with two 4^+-neighbors of v or at least a 6^+-neighbor of v, then $\tau(v \to F_3) \leq 2$.*

(3) $\tau(v \to F_4) \leq \frac{15}{4}$. *Especially, if F_4 contains at most one 3-face, then $\tau(v \to F_4) \leq \frac{7}{2}$.*

(4) $\tau(v \to F_5) \leq \frac{29}{6}$. *Especially, if F_5 contains exactly three 3-faces or at most one 3-face, then $\tau(v \to F_5) \leq \frac{9}{2}$.*

(5) $\tau(v \to F_6) \leq 6$.

Proof. (1) Suppose F_2 is a 2-fan incident with v. Then it must be a $(4^+, 4^+)$-fan by Lemma 3(2). Let u be the 3^+-neighbor of v that incident with F_2. If

F_2 is a $(4,4)$-fan, then $\tau(v \to F_2) \leq \frac{3}{4} \times 2 = \frac{3}{2}$ by Lemma 3(7). Otherwise, $\tau(v \to F_2) \leq 1 + \frac{1}{3} = \frac{4}{3} < \frac{3}{2}$. And especially, if F_2 is a (4, 4)-fan and $d(u) \geq 6$, then $\tau(v \to F_2) \leq \frac{1}{2} \times 2 = 1$ by our discharging rules. If F_2 is a $(4,5)$-fan and v is adjacent to at least three 2-vertices, then $\tau(v \to F_2) \leq \max\{\frac{3}{4} + \frac{1}{4}, \frac{1}{2} + \frac{1}{3}\} = 1$ by Lemma 4(1). The rest of the proof is obvious by our discharging rules.

(2) Suppose F_3 is a 3-fan incident with v, and f_1, f_2, f_3 are the three consecutive faces. Let vu be the common edge between f_1 and f_2, and vw be the common edge between f_2 and f_3. Without loss of generality, we can assume that $d(f_1) \leq d(f_3)$. If $d(f_2) = 3$, $d(f_1) = 4$, then $d(f_3) \geq 6$. Thus $\tau(v \to F_3) \leq \max\{\frac{3}{4} + \frac{7}{6}, \frac{3}{2} + \frac{1}{2}\} \leq 2 < \frac{5}{2}$ by Lemma 3(11). If $d(f_2) = 3$, $d(f_1) = 5$, then $\tau(v \to F_3) \leq \frac{3}{2} + \frac{2}{3} < \frac{5}{2}$. Otherwise, $\tau(v \to F_3) \leq \max\{\frac{3}{2}, \frac{3}{4} \times 2 + 1, 2 + \frac{1}{3}\} = \frac{5}{2}$ by Lemma 3(8). Especially, if F_3 is a $(4,5,4)$-fan and v is adjacent to at least three 2-vertices, then $\tau(v \to F_3) \leq \max\{1 + \frac{3}{4} + \frac{1}{4}, 1 + \frac{2}{3} + \frac{1}{3}, \frac{3}{4} \times 2 + \frac{1}{3}\} \leq 2$ by Lemma 4(2). If F_3 is a $(4,4,4)$-fan and incident with two 4^+-neighbors of v or at least a 6^+-neighbor of v, then $\tau(v \to F_3) \leq \max\{\frac{3}{4} \times 2 + \frac{1}{2}, \frac{1}{2} \times 2 + 1\} = 2$ by our discharging rules. The rest of the proof is obvious, we omit here.

(3) Suppose F_4 is a 4-fan incident with v. Then F_4 contains at most two 3-faces. If F_4 contains exactly two 3-faces, then it contains at most one 4-face. Thus $\tau(v \to F_4) \leq \max\{\frac{3}{4} + \frac{7}{6} + \frac{3}{2} + \frac{1}{3}, \frac{3}{2} \times 2 + \frac{2}{3}\} = \frac{15}{4}$ by Lemmas 3(5), 3(11) and our discharging rules. Otherwise, $\tau(v \to F_4) \leq \max\{2 + \frac{3}{2}, \frac{3}{2} + 1 + \frac{2}{3}, \frac{3}{4} \times 2 + 2, 3 + \frac{1}{3}\} = \frac{7}{2}$ by Lemma 3(9).

(4) Suppose F_5 is a 5-fan incident with v. Then F_5 contains at most three 3-faces. If F_5 contains exactly three 3-faces, then it contains no 4-face and 5-face. Thus $\tau(v \to F_5) \leq \max\{1 + \frac{3}{2} \times 2, \frac{3}{2} + \frac{7}{6} \times 2\} = 4$ by Lemma 3(5). If F_5 contains exactly two 3-faces, then it contains at most two 4-faces. Thus $\tau(v \to F_5) \leq \max\{\frac{7}{6} + \frac{3}{2} + \frac{3}{4} + 1 + \frac{1}{3}, \frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{3}{2} \times 2 + 1 + \frac{1}{3} \times 2\} = \frac{29}{6}$ by Lemmas 3(5), 3(11) and 3(12). Otherwise, $\tau(v \to F_5) \leq \max\{\frac{3}{2} + 3, \frac{3}{2} + 2 + \frac{2}{3}, 4 + \frac{1}{3}, \frac{3}{4} \times 2 + 3\} = \frac{9}{2}$ by Lemma 3(10).

(5) Suppose F_6 is a 6-fan incident with v. Then F_6 contains at most three 3-faces. If F_6 is a $(4^+,3,3,3,4^+,4^+)$-fan, then $\tau(v \to F_6) \leq \max\{\frac{7}{6} \times 3 + \frac{3}{4} \times 2 + 1, 1 + \frac{7}{6} + \frac{3}{2} + \frac{3}{4} \times 3, \frac{7}{6} \times 2 + \frac{3}{2} + \frac{3}{4} \times 2 + \frac{2}{3}, \frac{7}{6} \times 2 + \frac{3}{2} + \frac{3}{4} + 1 + \frac{1}{3}, \frac{7}{6} + \frac{3}{2} \times 2 + \frac{3}{4} \times 2 + \frac{1}{3}, \frac{7}{6} + \frac{3}{2} \times 2 + 1 + \frac{2}{3}\} = 6$ by Lemmas 3(5), (11), (12). If F_6 is a $(4^+,3,4^+,3,3,4^+)$-fan, then $\tau(v \to F_6) \leq \max\{\frac{3}{2} \times 3 + \frac{1}{3} \times 3, \frac{3}{2} \times 2 + \frac{7}{6} + \frac{3}{4} + \frac{2}{3}, \frac{3}{2} \times 3 + \frac{1}{3} + 1, \frac{7}{6} + \frac{3}{2} \times 2 + \frac{3}{4} \times 2\} = \frac{35}{6} \leq 6$. If F_6 contains exactly two 3-faces, then F_6 contains at most three 4-faces. Thus $\tau(v \to F_6) \leq \max\{\frac{3}{2} \times 2 + 2 \times 1 + \frac{1}{3} \times 2, \frac{3}{2} \times 2 + 3 \times 1, \frac{7}{6} + \frac{3}{2} + \frac{3}{4} + 2 + \frac{1}{3}, \frac{3}{2} \times 2 + \frac{3}{4} \times 2 + 1 + \frac{1}{3}\} = 6$ by Lemmas 3(5), (11), (12), (13). Otherwise, $\tau(v \to F_6) \leq \max\{\frac{3}{2} + 4 \times 1 + \frac{1}{3}, 6 \times 1\} = 6$. $\square$

Now we come back to check the new charge of 7-vertex v and consider eight cases in the following.

Case 1. $n_2(v) = 7$. Then $f_{6^+}(v) = 7$ by Lemmas 2 and 3(6). So $ch'(v) \geq ch(v) - 7 = 1$.

Case 2. $n_2(v) = 6$. Then $f_3(v) = 0$ and $f_{6^+}(v) \geq 5$ by Lemmas 3(2) and 3(6). Thus $ch'(v) \geq ch(v) - 6 - 2 = 0$.

Case 3. $n_2(v) = 5$. Then there are three possibilities in which 2-vertices are located. Various situations can see Fig. 3.

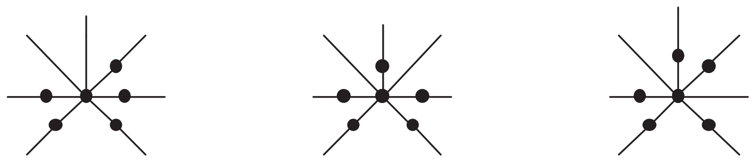


FIGURE 3. $n_2(v) = 5$

For Fig. 3(1), $f_{6^+}(v) \geq 4$. So $ch'(v) \geq ch(v) - 5 - \frac{5}{2} = \frac{1}{2} > 0$ by Lemma 5(2). For Fig. 3(2) and 3(3), $f_{6^+}(v) \geq 3$. So $ch'(v) \geq ch(v) - 5 - \frac{3}{2} \times 2 = 0$ by Lemma 5(1).

Case 4. $n_2(v) = 4$. Then there are four possibilities in which 2-vertices are located. Various situations can see Fig. 4.

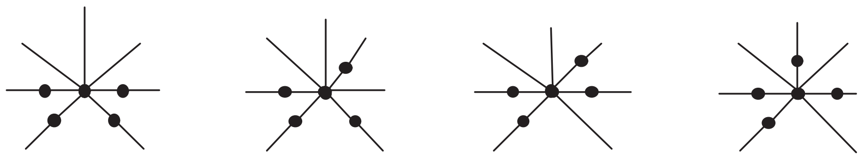


FIGURE 4. $n_2(v) = 4$

For Fig. 4(1), $f_{6^+}(v) \geq 3$. So $ch'(v) \geq ch(v) - 4 - \frac{15}{4} = \frac{1}{4} > 0$ by Lemma 5(3). For Fig. 4(2) and 4(3), $f_{6^+}(v) \geq 2$ and v is incident with a 2-fan and a 3-fan. So $ch'(v) \geq ch(v) - 4 - \frac{3}{2} - \frac{5}{2} = 0$ by Lemmas 5(1) and 5(2). For Fig. 4(4), $f_{6^+}(v) \geq 1$ and v is incident with three 2-fans. If they are all (4, 4)-fan, then v is adjacent to a 6^+-vertex by Lemma 3(15). So $ch'(v) \geq ch(v) - 4 - \frac{3}{2} \times 2 - 1 = 0$ by Lemma 5(1).

Case 5. $n_2(v) = 3$. Then there are four possibilities in which 2-vertices are located. Various situations can see Fig. 5.

For Fig. 5(1), $f_{6^+}(v) \geq 2$. Then $ch'(v) \geq ch(v) - 3 - \frac{29}{6} = \frac{1}{6} > 0$ by Lemma 5(4). For Fig. 5(2), v is incident with a 2-fan and a 4-fan. If F_4 contains at most one 3-face, then $ch'(v) \geq ch(v) - 3 - \frac{3}{2} - \frac{7}{2} = 0$ by Lemmas 5(1) and 5(3). If F_4 contains two 3-faces and F_2 is a $(4^+, 5^+)$-fan, then $ch'(v) \geq ch(v) - 3 - 1 - \frac{15}{4} = \frac{1}{4} > 0$ by Lemma 5. Otherwise, if F_2 is a $(4, 4)$-fan, then F_4 must be a

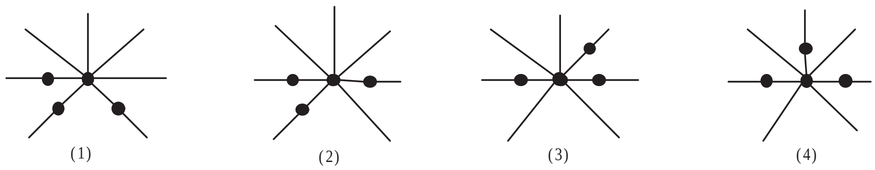


FIGURE 5. $n_2(v) = 3$

$(4^+, 3, 3, 6^+)$-fan. So $ch'(v) \geq ch(v) - 3 - \frac{3}{2} - \max\{\frac{3}{4} + \frac{7}{6} + \frac{3}{2}, \frac{3}{2} \times 2 + \frac{1}{3}\} = \frac{1}{12} > 0$ by Lemmas 3(5) and 3(11). For Fig. 5(3), v is incident with two 3-fans. Thus $ch'(v) \geq ch(v) - 3 - \frac{5}{2} \times 2 = 0$ by Lemma 5(2). For Fig. 5(4), v is incident with a 3-fan and two 2-fans. Suppose F_3 contains one 3-face. If v is incident with two $(4, 4)$-fan, then F_3 must be a $(6^+, 3, 6^+)$-fan. So $ch'(v) \geq ch(v) - 3 - \frac{3}{2} \times 2 - \frac{3}{2} = \frac{1}{2} > 0$ by Lemma 5(1). Otherwise, if v is incident with at least a $(4^+, 5^+)$-fan, then $ch'(v) \geq ch(v) - 3 - \frac{3}{2} - 1 - \frac{5}{2} = 0$. Suppose F_3 contains no 3-face. If v is incident with two $(4, 4)$-fan, then F_3 must be a $(4, 4^+, 4)$-fan, or a $(4, 4^+, 6^+)$-fan, or a $(6^+, 4^+, 6^+)$-fan. Thus $ch'(v) \geq ch(v) - 3 - \max\{\frac{3}{2} \times 2 + 2, \frac{3}{2} + 1 + \frac{5}{2}\} = 0$ by Lemmas 3(8), 3(16) and Lemma 5. Otherwise, v is incident with at least a $(4^+, 5^+)$-fan, then $ch'(v) \geq ch(v) - 3 - \frac{3}{2} - 1 - \frac{5}{2} = 0$.

Case 6. $n_2(v) = 2$. Then there are three possibilities in which 2-vertices are located. Various situations can see Fig. 6.

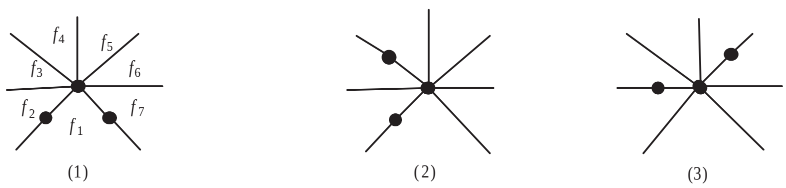


FIGURE 6. $n_2(v) = 2$

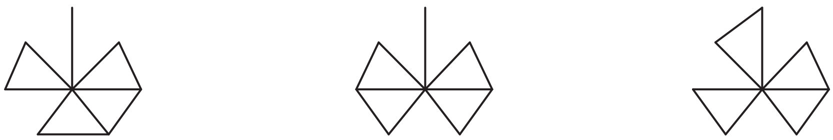
For Fig. 6(1), if $d(f_1) = 5$, then $d(f_2) \geq 6$ and $d(f_7) \geq 6$. Thus $ch'(v) \geq ch(v) - 2 - \frac{3}{2} \times 3 - 1 - \frac{1}{3} = \frac{1}{6} > 0$. Otherwise, $ch'(v) \geq ch(v) - 2 - 6 = 0$ by Lemma 5(5). For Fig. 6(2), v is incident with a 2-fan and a 5-fan. If F_5 contains three 3-faces or at most one 3-face, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} - \frac{9}{2} = 0$ by Lemma 5. Suppose F_5 contains exactly two 3-faces. If F_2 is a $(4^+, 6^+)$-fan or a $(5^+, 5^+)$-fan, then $ch'(v) \geq ch(v) - 2 - 1 - \frac{29}{6} = \frac{1}{6} > 0$ by Lemma 5. Otherwise, if F_2 is a $(4, 4)$-fan or a $(4, 5)$-fan, then $ch'(v) \geq ch(v) - 2 - \frac{3}{2} - \max\{\frac{3}{2} \times 2 + 1 + \frac{1}{3}, \frac{3}{4} + 1 + \frac{7}{6} + \frac{3}{2}, \frac{3}{4} \times 2 + \frac{3}{2} \times 2\} = 0$ by Lemmas 3(5) and 3(12). For Fig. 6(3), v is incident with a 3-fan and a 4-fan. If F_4 contains at most one 3-face, then $ch'(v) \geq ch(v) - 2 - \frac{7}{2} - \frac{5}{2} = 0$ by Lemma 5. Otherwise,

$ch'(v) \geq ch(v) - 2 - \max\{\frac{3}{2} \times 2 + \frac{1}{3} \times 2 + \frac{3}{2}, \frac{3}{2} \times 2 + \frac{1}{3} + \frac{5}{2}, \frac{3}{2} + \frac{7}{6} + \frac{3}{4} + \frac{1}{3} + \frac{3}{2}, \frac{3}{2} + \frac{7}{6} + \frac{3}{4} + \frac{5}{2}\} = \frac{1}{12} > 0$ by Lemmas 3(5) and 3(11) and our discharging rules.

Case 7. $n_2(v) = 1$. Note that $f_3(v) \leq 5$. Let u be the unique 2-vertex adjacent to v. We split the proof into six cases.

Subcase 7.1. $f_3(v) = 5$. Then $f_4(v) = f_5(v) = 0$. So $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - \frac{7}{6} \times 4 = \frac{5}{6} > 0$ by Lemmas 3(3) and 3(4).

Subcase 7.2. $f_3(v) = 4$. Various situations are illustrated in Fig. 7.



(1) (2) (3)

FIGURE 7

If uv is incident with a 3-face, then the other three 3-faces are not incident with 3-vertices by Lemmas 3(3) and 3(4). For Fig. 7(1), $f_{6^+}(v) \geq 1$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - \frac{7}{6} \times 3 - 2 = 0$. For Fig. 7(2) and 7(3), $f_4(v) \leq 1$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - \frac{7}{6} \times 3 - 1 - \frac{2}{3} = \frac{1}{3} > 0$. Now suppose uv is not incident with a 3-face. For Fig. 7(1), $f_{6^+}(v) \geq 2$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 4 - 1 = 0$. For Fig. 7(2), $f_4(v) \leq 1$, thus $ch'(v) \geq ch(v) - 1 - \max\{\frac{3}{2} \times 4 + \frac{1}{3} \times 3, \frac{3}{2} \times 3 + \frac{7}{6} + 1 + \frac{1}{3}\} = 0$ by Lemma 4(3).

Subcase 7.3. $f_3(v) = 3$. Various situations are illustrated in Fig. 8.

(1) (2) (3) (4)

FIGURE 8

Suppose uv is incident with a 3-face. For Fig. 8(1), $f_{6^+}(v) \geq 2$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - \frac{7}{6} \times 2 - 2 = \frac{7}{6} > 0$. For Fig. 8(2)-8(4), $f_4(v) \leq 3$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} - \frac{7}{6} \times 2 - \max\{3, 2 + \frac{2}{3}\} = \frac{1}{6} > 0$. Suppose uv is not incident with a 3-face. For Fig. 8(1), $f_{6^+}(v) \geq 2$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 3 - 2 = \frac{1}{2} > 0$. For Fig. 8(2) and 8(3), $f_4(v) \leq 2$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 3 - \max\{2 + \frac{1}{3}, 1 + \frac{1}{3} \times 3\} = \frac{1}{6} > 0$. For Fig. 8(4), $f_4(v) \leq 1$, thus $ch'(v) \geq ch(v) - 1 - \frac{3}{2} \times 3 - 1 - \frac{1}{3} \times 3 = \frac{1}{2} > 0$.

Subcase 7.4. $f_3(v) = 2$. Various situations are illustrated in Fig. 9.

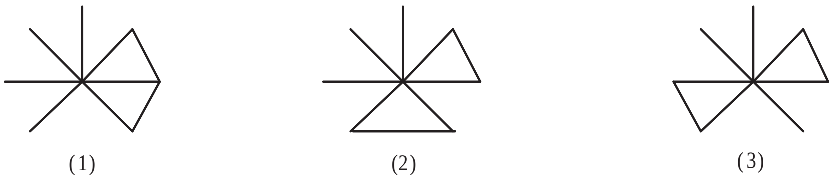


FIGURE 9

It's obvious that $f_4(v) \le 4$. Suppose uv is incident with a 3-face. Then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} - \frac{7}{6} - 4 - \frac{1}{3} = 0$ by Lemmas 3(3) and 3(4). Suppose uv is not incident with a 3-face. If $f_4(v) \le 3$, then $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - 3 - \frac{1}{3} \times 2 = \frac{1}{3} > 0$. Now we consider the case that $f_4(v) = 4$. For Fig. 9(1), $ch'(v) \ge ch(v) - 1 - \max\{\frac{3}{2} \times 2 + \frac{3}{4} \times 2 + 2 + \frac{1}{3}, \frac{3}{2} + \frac{7}{6} + 4 + \frac{1}{3}\} = 0$ by Lemmas 3(5), 3(14) and Lemma 4(4). For Fig. 9(2) and 9(3), $f_{6^+}(v) \ge 1$. So $ch'(v) \ge ch(v) - 1 - \frac{3}{2} \times 2 - 4 = 0$.

Subcase 7.5. $f_3(v) = 1$. In this case, $f_4(v) \le 5$. So $ch'(v) \ge ch(v) - 1 - \frac{3}{2} - 5 - \frac{1}{3} = \frac{1}{6} > 0$.

Subcase 7.6. $f_3(v) = 0$. It's obvious that $ch'(v) \ge ch(v) - 1 - 7 = 0$.

Case 8. $n_2(v) = 0$. In this case, $f_3(v) \le 5$. If $f_3(v) = 5$, then $f_{6^+}(v) = 2$. Thus $ch'(v) \ge ch(v) - \frac{3}{2} \times 5 = \frac{1}{2} > 0$. If $f_3(v) = 4$, then $f_4(v) \le 1$. Thus $ch'(v) \ge ch(v) - \frac{3}{2} \times 4 - 1 - \frac{2}{3} = \frac{1}{3} > 0$. If $1 \le f_3(v) \le 3$, then $f_{5^+}(v) \ge 1$. Thus $ch'(v) \ge ch(v) - \frac{3}{2} \times f_3(v) - (6 - f_3(v)) \times 1 - \frac{1}{3} = \frac{5}{3} - \frac{1}{2} \times f_3(v) > 0$. If $f_3(v) = 0$, then $ch'(v) \ge ch(v) - 7 = 1 > 0$.

Hence we complete the proof of Theorem 1.

Acknowledgement. I would like to thank the referees for their valuable suggestions.

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