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DYNAMIC ANALYSIS OF A MODIFIED STOCHASTIC PREDATOR-PREY SYSTEM WITH GENERAL RATIO-DEPENDENT FUNCTIONAL RESPONSE

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Abstract. In this paper, we study a modified stochastic predator-prey system with general ratio-dependent functional response. We prove that the system has a unique positive solution for given positive initial value. Then we investigate the persistence and extinction of this stochastic system. At the end, we give some numerical simulations, which support our theoretical conclusions well.

1. Introduction

The dynamical relationship between predators and their preys is one of the dominant themes in ecology [5]. Let $x(t)$ and $y(t)$ represent population densities of prey and predator at time t, respectively. Aziz-Alaoui and Okiye [4] proposed the following modified predator-prey model with Holling type II functional response:

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)\left(\alpha_1 - \beta_1 x(t) - \dfrac{v_1 y(t)}{m_1 + x(t)}\right), \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)\left(\alpha_2 - \dfrac{v_2 y(t)}{m_2 + x(t)}\right), \end{cases} \tag{1}$$

where α_1 is the growth rate of prey x, β_1 measures the strength of competition among individuals of species x, v_1 is the maximum value which per capita reduction rate of x can attain, m_1 (respectively, m_2) measures the extent to which environment provides protection to prey x (respectively, to predator y), α_2 describes the growth rate of y, and v_2 has a similar meaning to v_1.

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Nindjin et al. [16] and Yafia et al. [22] incorporated time delay into system (1) and studied the dynamic behaviors of the model. Song and Li [19] incorporated impulsive effects into system (1).

However, in reality, uncertainties are always exist. Too often these uncertainties are ignored, which limits our prediction. Recently, Ji et al. [10] incorporated white noise in each equations of system (1), which is as follows:

$$\begin{cases} \mathrm{d}x(t) = x(t)\left(\alpha_1 - \beta_1 x(t) - \dfrac{v_1 y(t)}{m_1 + x(t)}\right)\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)\left(\alpha_2 - \dfrac{v_2 y(t)}{m_2 + x(t)}\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t). \end{cases} \tag{2}$$

Here $B_i(t)$ $(i = 1, 2)$ are independent standard Brownian motions. $\sigma_i > 0$ $(i = 1, 2)$ represent the intensities of $B_i(t)$ $(i = 1, 2)$, respectively.

A general representation of functional response (see Kazarinoff and van de Driessche [11], Real [17, 18]) is

$$g(x) = \frac{x^l}{c + x^l}, \quad c > 0,\ l \geq 1.$$

When $l = 1$ and $l = 2$, the functional response is called Holling type II and III functional response, respectively.

Some biological and physiological evidences (Arditi et al. [2], Arditi and Saiah [3], Gutierrez [7], etc) that in many situations especially when the predators have to search for food (and therefore have to share or compete for food). A more suitable to consider predator-prey models is the so called ratio-dependent theory. Based on the Holling type II function, Arditi and Ginzburg [1] proposed a ratio-dependent functional response of the form

$$g\left(\frac{x}{y}\right) = \frac{\frac{x}{y}}{c + \frac{x}{y}} = \frac{x}{cy + x}, \quad c > 0.$$

For more details about ratio-dependent functional response, we refer to [9, 13, 14, 20, 21] and references therein.

Then, based on system (2), Mandal and Banerjee [15] considered a modified stochastic predator-prey model with ratio-dependent functional response.

Motivated by the works in [11, 15, 17, 18], in this paper, we will study a modified stochastic predator-prey model with general ratio-dependent functional response as follows:

$$\begin{cases} \mathrm{d}x(t) = x(t)\left(\alpha_1 - \beta_1 x(t) - \dfrac{v_1 x^{n-1}(t) y(t)}{m_1 y^n(t) + x^n(t)}\right)\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t) = y(t)\left(\alpha_2 - \dfrac{v_2 y(t)}{m_2 + x(t)}\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t), \end{cases} \tag{3}$$

where $n \geq 1$.

The rest of the paper is organized as follows. In Section 2, we give some preliminaries which will be applied in this paper. In Section 3, we show that the

existence of unique positive global solution of system (3) for any positive initial value. In Section 4, we establish that the stochastic system (3) is persistent in mean and extinct under some conditions. In Section 5, numerical simulations are presented to illustrate our theoretical results. Finally, we conclude our work in Section 6.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets).

In this section, we introduce the following definition and lemmas, which will be used in the following sections.

Definition (see [6]). The system is said to be *persistent in mean*, if

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t x(s)ds > 0, \quad \liminf_{t\to+\infty} \frac{1}{t}\int_0^t y(s)ds > 0.$$

Next, we introduce a lemma, which will be applied to show the solution of system (3) is global.

Lemma 2.1 (see [10]). *Consider one-dimensional stochastic differential equation*

$$\mathrm{d}X(t) = X(t)[(a - bX(t))\mathrm{d}t + \sigma \mathrm{d}B(t)], \tag{4}$$

where parameters a, b and σ are positive, $B(t)$ is a standard Brownian motion. Suppose $a > \frac{\sigma^2}{2}$, and $X(t)$ is the solution of equation (4) *with any initial value $X_0 > 0$, then we have*

$$\lim_{t\to\infty} \frac{\ln X(t)}{t} = 0,$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds = \frac{a - \frac{\sigma^2}{2}}{b},$$

almost surely (a.s.).

Consider the following stochastic differential equation.

$$\mathrm{d}X(t) = \mu(X(t), t)\mathrm{d}t + \sigma(X(t), t)\mathrm{d}B(t). \tag{5}$$

Then we have:

Lemma 2.2 (see [12]). *Suppose $X(t)$ is the solution of* (5). *If $S(-\infty) > -\infty$ and $S(+\infty) = +\infty$, then*

$$\lim_{t\to\infty} X(t) = -\infty,$$

where the scale function

$$S(u) = \int_0^u e^{-\int_0^v \frac{2\mu(s)}{\sigma^2(s)}ds} dv.$$

3. Existence and uniqueness of the positive solution

Because $x(t)$ and $y(t)$ in system (3) are population densities of the prey and the predator at time t, respectively. we are only interested in positive solutions. We first prove that there exists a unique positive local solution of system (3) and then show that this solution is actually global by using comparison theorem for stochastic equations. We have the following result.

Theorem 3.1. *Given positive initial value (x_0, y_0), system (3) has a unique positive global solution $(x(t), y(t))$ on $t \in [0, \infty)$.*

Proof. We consider the following system

$$
(6) \quad \begin{cases} \mathrm{d}u(t) = \left(\alpha_1 - \dfrac{\sigma_1^2}{2} - \beta_1 e^{u(t)} - \dfrac{v_1 e^{(n-1)u(t)} e^{v(t)}}{m_1 e^{nv(t)} + e^{nu(t)}}\right) \mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}v(t) = \left(\alpha_2 - \dfrac{\sigma_2^2}{2} - \dfrac{v_2 e^{v(t)}}{m_2 + e^{u(t)}}\right) \mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \end{cases}
$$

on $t \geq 0$ with initial value $u(0) = \ln x_0$ and $v(0) = \ln y_0$. It is obvious that the coefficients of system (6) satisfy the local Lipschitz condition, then there is a unique local solution $(u(t), v(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Therefore, by Itô formula, it is easy to see $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$ is the unique positive local solution to system (6) with initial value $x_0 > 0$, $y_0 > 0$.

Now, we show that this solution is global, i.e., $\tau_e = \infty$. Since the solution is positive, we get

$$
\mathrm{d}x(t) \leq x(t) \left(\alpha_1 - \beta_1 x(t)\right) \mathrm{d}t + \sigma_1 x(t) \mathrm{d}B_1(t).
$$

If $x(t) \leq y(t)$, then

$$
\frac{v_1 x^{n-1}(t) y(t)}{m_1 y^n(t) + x^n(t)} = \frac{v_1}{\left(\frac{y(t)}{x(t)}\right)^{-1} + m_1 \left(\frac{y(t)}{x(t)}\right)^{n-1}} \leq \frac{v_1}{m_1}.
$$

If $x(t) > y(t)$, then

$$
\frac{v_1 x^{n-1}(t) y(t)}{m_1 y^n(t) + x^n(t)} = \frac{y(t)}{x(t)} \frac{v_1 x^n(t)}{m_1 y^n(t) + x^n(t)} \leq v_1.
$$

Let $\gamma = \max\left\{\frac{v_1}{m_1}, v_1\right\}$. Then we have

$$
\mathrm{d}x(t) \geq x(t) \left(\alpha_1 - \gamma - \beta_1 x(t)\right) \mathrm{d}t + \sigma_1 x(t) \mathrm{d}B_1(t).
$$

Let $X_1(t)$ and $X_2(t)$ be the solutions of following stochastic equations, respectively.

$$
(7) \qquad \mathrm{d}X_1(t) = X_1(t) \left(\alpha_1 - \gamma - \beta_1 X_1(t)\right) \mathrm{d}t + \sigma_1 X_1(t) \mathrm{d}B_1(t)
$$

and

$$
(8) \qquad \mathrm{d}X_2(t) = X_2(t) \left(\alpha_1 - \beta_1 X_2(t)\right) \mathrm{d}t + \sigma_1 X_2(t) \mathrm{d}B_1(t),
$$

with initial value $x_0 > 0$. Consequently, by the comparison theorem for stochastic equations, we obtain

$$X_1(t) \le x(t) \le X_2(t), \quad \text{a.s.} \quad t \in [0, \tau_e), \tag{9}$$

where

$$X_1(t) = \frac{e^{\left[\left(\alpha_1 - \gamma - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{\frac{1}{x_0} + \beta_1 \int_0^t e^{\left[\left(\alpha_1 - \gamma - \frac{\sigma_1^2}{2}\right)s + \sigma_1 B_1(s)\right]} ds}$$

and

$$X_2(t) = \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{\frac{1}{x_0} + \beta_1 \int_0^t e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)s + \sigma_1 B_1(s)\right]} ds}.$$

On the other hand, from the second equation of system (3), we get

$$\mathrm{d}y(t) \le y(t)\left(\alpha_2 - \frac{v_2}{m_2 + X_2(t)} y(t)\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t)$$

and

$$\mathrm{d}y(t) \ge y(t)\left(\alpha_2 - \frac{v_2}{m_2} y(t)\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t).$$

Let $Y_1(t)$ and $Y_2(t)$ be the solutions of following stochastic equations, respectively.

$$\mathrm{d}Y_1(t) = Y_1(t)\left(\alpha_2 - \frac{v_2}{m_2} Y_1(t)\right)\mathrm{d}t + \sigma_2 Y_1(t)\mathrm{d}B_2(t) \tag{10}$$

and

$$\mathrm{d}Y_2(t) = Y_2(t)\left(\alpha_2 - \frac{v_2}{m_2 + X_2(t)} Y_2(t)\right)\mathrm{d}t + \sigma_2 Y_2(t)\mathrm{d}B_2(t),$$

with initial value $y_0 > 0$. Using the comparison theorem for stochastic equations, we have

$$Y_1(t) \le y(t) \le Y_2(t), \quad \text{a.s.} \quad t \in [0, \tau_e),$$

where

$$Y_1(t) = \frac{e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right]}}{\frac{1}{y_0} + \frac{v_2}{m_2}\int_0^t e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)s + \sigma_2 B_2(s)\right]} ds}$$

and

$$Y_2(t) = \frac{e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right]}}{\frac{1}{y_0} + \int_0^t \frac{v_2}{m_2 + X_2(s)} e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)s + \sigma_2 B_2(s)\right]} ds}.$$

From the expression of the solutions $X_1(t)$, $X_2(t)$, $Y_1(t)$ and $Y_2(t)$, it is clear that they are all existence on $t \in [0, +\infty)$, which implies that $\tau_e = +\infty$. This completes the proof. □

4. Persistence and extinction

In this section, we aim to establish the persistent and extinct conditions for system (3). From Lemma 2.1, equations (7) and (8), we have the following theorem.

Theorem 4.1. *If $\alpha_1 - \gamma - \frac{\sigma_1^2}{2} > 0$, then for any initial value $x_0 > 0$, the solution $x(t)$ of system (3) satisfies*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0, \quad a.s.$$

Furthermore, we can prove:

Theorem 4.2. *If $\alpha_1 - \gamma - \frac{\sigma_1^2}{2} > 0$, then for any initial value $x_0 > 0$, the solution $x(t)$ of system (3) has the property*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\alpha_1 - \gamma - \frac{\sigma_1^2}{2}}{\beta_1} > 0, \quad a.s.$$

Proof. Let $V(x) = \ln x(t)$. Applying the Itô formula, we have

$$\ln x(t) = \ln x_0 + \left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t - \beta_1\int_0^t x(s)ds - v_1\int_0^t \frac{x^{n-1}(s)y(s)}{m_1 y^n(s) + x^n(s)}ds + \sigma_1 B_1(t).$$

Hence,

$$\frac{\ln x(t)}{t} = \frac{\ln x_0}{t} + \left(\alpha_1 - \frac{\sigma_1^2}{2}\right) - \beta_1\frac{\int_0^t x(s)ds}{t} - v_1\frac{\int_0^t \frac{x^{n-1}(s)y(s)}{m_1 y^n(s)+x^n(s)}ds}{t} + \sigma_1\frac{B_1(t)}{t}.$$

Letting $t \to \infty$ and by the strong law of numbers of local martingales and Theorem 4.1, we get

$$\lim_{t\to\infty}\left[\beta_1\frac{\int_0^t x(s)ds}{t} + v_1\frac{\int_0^t \frac{x^{n-1}(s)y(s)}{m_1 y^n(s)+x^n(s)}ds}{t}\right] = \alpha_1 - \frac{\sigma_1^2}{2}.$$

Thus,

$$\alpha_1 - \frac{\sigma_1^2}{2} \leq \beta_1 \liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds + \gamma,$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\alpha_1 - \gamma - \frac{\sigma_1^2}{2}}{\beta_1} > 0, \quad \text{a.s.}$$

□

Theorem 4.3. *If $\alpha_1 - \gamma - \frac{\sigma_1^2}{2} > 0$ and $\alpha_2 - \frac{\sigma_2^2}{2} > 0$, then for any initial value $y_0 > 0$, the solution $y(t)$ of system (3) has the property*

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = 0 \quad a.s.$$

Proof. From Lemma 2.1, equation (9) and Theorem 4.1, we know that for any $\varepsilon > 0$, there exists $T > 0$ such that

$$e^{-\varepsilon t} \le X_i(t) \le e^{\varepsilon t} \quad \text{for} \quad t \ge T, \quad i = 1, 2.$$

Then by the strong law of large numbers of local martingales, for $\varepsilon > 0$ and $T > 0$ above, we get

$$-\varepsilon t \le \sigma_i B_i(t) \le \varepsilon t \quad \text{for} \quad t \ge T, \quad i = 1, 2.$$

When $\alpha_2 - \frac{\sigma_2^2}{2} > 0$, from Lemma 2.1 and equation (10), we can also obtain

$$\lim_{t\to\infty} \frac{\ln Y_1(t)}{t} = 0, \quad \text{a.s.}$$

In the following, we choose T, such that $\frac{1}{2}e^{\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t} \ge 1$ for $t \ge T$, then when $s \ge T$, we have

$$\begin{aligned}
X_2(t) &= \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{\frac{1}{x_0} + \beta_1 \int_0^t e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)s + \sigma_1 B_1(s)\right]} ds} \le \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{\beta_1 \int_0^t e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)s + \sigma_1 B_1(s)\right]} ds} \\
&\le \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{\beta_1 e^{-\varepsilon t} \int_0^t e^{\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)s} ds} = \frac{\alpha_1 - \frac{\sigma_1^2}{2}}{\beta_1} \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{e^{-\varepsilon t}\left[e^{\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t} - 1\right]} \\
&\le \frac{2\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)}{\beta_1} \frac{e^{\left[\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1 B_1(t)\right]}}{e^{-\varepsilon t} e^{\left(\alpha_1 - \frac{\sigma_1^2}{2}\right)t}} \le \frac{2\alpha_1 - \sigma_1^2}{\beta_1} e^{2\varepsilon t}.
\end{aligned}$$

On the other hand, from the expression of $Y_2(t)$ that for any $t > s \ge T$, we have

$$\begin{aligned}
\frac{1}{Y_2(t)} &= \frac{1}{Y_2(T)} e^{\left[-\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)(t-T) - \sigma_2(B_2(t) - B_2(T))\right]} \\
&\quad + e^{-\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right]} \int_T^t \frac{v_2}{m_2 + X_2(s)} e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)s + \sigma_2 B_2(s)\right]} ds \\
&\ge e^{-\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right]} \int_T^t \frac{v_2}{m_2 + X_2(s)} e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)s + \sigma_2 B_2(s)\right]} ds \\
&\ge e^{-\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2 B_2(t)\right]} \frac{\beta_1 v_2}{\beta_1 m_2 + 2\alpha_1 - \sigma_1^2} \int_T^t e^{-2\varepsilon t} e^{\left[\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)s + \sigma_2 B_2(s)\right]} ds \\
&\ge e^{-4\varepsilon t} K(t),
\end{aligned}$$

where

$$K(t) = \frac{2\beta_1 v_2}{(\beta_1 m_2 + 2\alpha_1 - \sigma_1^2)(2\alpha_2 - \sigma_2^2)} \left(1 - e^{-\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)(t-T)}\right).$$

Therefore, we obtain

$$-\ln Y_2(t) \geq \ln K(t) - 4\varepsilon t.$$

Then

$$\frac{\ln Y_2(t)}{t} \leq -\frac{\ln K(t)}{t} + 4\varepsilon.$$

Moreover

$$\frac{\ln K(t)}{t} \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, for arbitrary $\varepsilon > 0$, we get

$$\limsup_{t\to\infty} \frac{\ln Y_2(t)}{t} \leq 0, \quad \text{a.s.}$$

Consequently

$$\begin{aligned} 0 \leq \liminf_{t\to\infty} \frac{\ln Y_1(t)}{t} \leq \liminf_{t\to\infty} \frac{\ln y(t)}{t} \leq \limsup_{t\to\infty} \frac{\ln y(t)}{t} \\ \leq \limsup_{t\to\infty} \frac{\ln Y_2(t)}{t} \leq 0. \end{aligned}$$

This implies that

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = 0, \quad \text{a.s.} \qquad \square$$

The following theorem shows that system (3) is persistent in mean.

Theorem 4.4. *Assume that* $\alpha_1 - \gamma - \frac{\sigma_1^2}{2} > 0$ *and* $\alpha_2 - \frac{\sigma_2^2}{2} > 0$. *Then for any initial value* $x_0,\ y_0 > 0$, *system* (3) *is persistent in mean.*

Proof. Let $V(y) = \ln y(t)$. Using the Itô formula, we obtain

$$\ln y(t) = \ln y_0 + \left(\alpha_2 - \frac{\sigma_2^2}{2}\right) t - v_2 \int_0^t \frac{y(s)}{m_2 + x(s)} ds + \sigma_2 B_2(t).$$

Thus,

$$\frac{v_2}{t} \int_0^t \frac{y(s)}{m_2 + x(s)} ds = -\frac{\ln y(t)}{t} + \frac{\ln y_0}{t} + \left(\alpha_2 - \frac{\sigma_2^2}{2}\right) + \sigma_2 \frac{B_2(t)}{t}.$$

Letting $t \to \infty$ and using the strong law of numbers of local martingales again and Theorem 4.3, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{y(s)}{m_2 + x(s)} ds = \frac{\alpha_2 - \frac{\sigma_2^2}{2}}{v_2}, \quad \text{a.s.}$$

Then, we get

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds \geq m_2 \liminf_{t\to\infty}\frac{1}{t}\int_0^t \frac{y(s)}{m_2+x(s)}ds = \frac{m_2\left(\alpha_2-\frac{\sigma_2^2}{2}\right)}{v_2} > 0, \quad \text{a.s.}$$

From this, Definition 2 and Theorem 4.2, we obtain that system (3) is persistent in mean. □

Next, we establish extinct results for prey or predator species.

Theorem 4.5. *Let $(x(t), y(t))$ be a solution of system (3) with any initial value x_0, $y_0 > 0$. Then we have*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty}\frac{1}{t}\int_0^t y(s)ds = \frac{m_2\left(\alpha_2-\frac{\sigma_2^2}{2}\right)}{v_2}, \quad \textit{a.s.}$$

when $\alpha_1 - \frac{\sigma_1^2}{2} < 0$ and $\alpha_2 - \frac{\sigma_2^2}{2} > 0$.

Proof. From the first equation of system (6), we have

$$\begin{aligned}\mathrm{d}u(t) &= \left(\alpha_1 - \frac{\sigma_1^2}{2} - \beta_1 e^{u(t)} - \frac{v_1 e^{(n-1)u(t)}e^{v(t)}}{m_1 e^{nv(t)} + e^{nu(t)}}\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t)\\ &\leq \left(\alpha_1 - \frac{\sigma_1^2}{2}\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t).\end{aligned}$$

Applying the comparison theorem for stochastic equations and the theory of diffusion processes (see Lemma 2.2), for $\mu(t) = \alpha_1 - \frac{\sigma_1^2}{2}$ and $\sigma(t) = \sigma_1$. By basic calculation, when $\alpha_1 - \frac{\sigma_1^2}{2} < 0$, we obtain that $S(-\infty) > -\infty$ and $S(+\infty) = +\infty$. Then

$$\lim_{t\to\infty} u(t) = -\infty, \quad \text{a.s.},$$

i.e.,

$$\lim_{t\to\infty} x(t) = 0, \quad \text{a.s.}$$

Therefore, for arbitrary small $\varepsilon > 0$, there exist t_0 and a set Ω_ε such that $P(\Omega_\varepsilon) \geq 1-\varepsilon$ and $\frac{x(t)}{m_2+x(t)} \leq \varepsilon$ for $t \geq t_0$ and $\omega \in \Omega_\varepsilon$. Thus, from the second equation of system (3), we have

$$\begin{aligned}\mathrm{d}y(t) &= y(t)\left(\alpha_2 - \frac{v_2 y(t)}{m_2 + x(t)}\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t)\\ &= y(t)\left(\alpha_2 - \frac{v_2}{m_2}y(t) + \frac{v_2 x(t) y(t)}{m_2(m_2+x(t))}\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t).\end{aligned}$$

It is obvious that

$$\mathrm{d}y(t) \geq y(t)\left(\alpha_2 - \frac{v_2}{m_2}y(t)\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t)$$

and

$$\mathrm{d}y(t) \leq y(t)\left(\alpha_2 - \frac{v_2}{m_2}(1-\varepsilon)y(t)\right)\mathrm{d}t + \sigma_2 y(t)\mathrm{d}B_2(t).$$

When $\alpha_2 - \frac{\sigma_2^2}{2} > 0$, from Lemma 2.1 and comparison theorem for stochastic equations, we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \geq \frac{m_2\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)}{v_2}, \quad \limsup_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds \leq \frac{m_2\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)}{v_2(1-\varepsilon)}.$$

For the arbitrary of ε, we get

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t y(s)ds = \frac{m_2\left(\alpha_2 - \frac{\sigma_2^2}{2}\right)}{v_2}, \quad \text{a.s.}$$

□

Theorem 4.6. *Let $(x(t), y(t))$ be a solution of system (3) with any initial value $x_0,\ y_0 > 0$. Then when $\alpha_1 - \frac{\sigma_1^2}{2} > 0$ and $\alpha_2 - \frac{\sigma_2^2}{2} < 0$, we have*

$$\lim_{t\to\infty} x(t) = \frac{\alpha_1 - \frac{\sigma_1^2}{2}}{\beta_1}, \quad \int_0^t y(s)ds = 0, \quad a.s.$$

Proof. From the second equation of system (6), we get

$$\begin{aligned} \mathrm{d}v(t) &= \left(\alpha_2 - \frac{\sigma_2^2}{2} - \frac{v_2 e^{v(t)}}{m_2 + e^{u(t)}}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t) \\ &\leq \left(\alpha_2 - \frac{\sigma_2^2}{2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t). \end{aligned}$$

When $\alpha_2 - \frac{\sigma_2^2}{2} < 0$, similar as the proof of Theorem 4.5, we have

$$\lim_{t\to\infty} v(t) = -\infty, \quad \text{a.s.},$$

i.e.,

$$\lim_{t\to\infty} y(t) = 0, \quad \text{a.s.}$$

Hence, for arbitrarily small $\varepsilon > 0$, there exist T_0 and a set Ω_ε such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and $v_1 \frac{y(t)}{x(t)} \leq \varepsilon$ for $t \geq T_0$ and $\omega \in \Omega_\varepsilon$. Then, from the first equation of system (3), we have

$$\mathrm{d}x(t) \geq x(t)\left(\alpha_1 - \varepsilon - \beta_1 x(t)\right)\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_1(t)$$

and

$$\mathrm{d}x(t) \leq x(t)\left(\alpha_1 - \beta_1 x(t)\right)\mathrm{d}t + \sigma_1 x(t)\mathrm{d}B_{21}(t).$$

When $\alpha_1 - \frac{\sigma_1^2}{2} > 0$, from Lemma 2.1 and comparison theorem for stochastic equations, we get

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \geq \frac{\alpha_1 - \varepsilon - \frac{\sigma_1^2}{2}}{\beta_1}, \quad \limsup_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds \leq \frac{\alpha_1 - \frac{\sigma_1^2}{2}}{\beta_1}.$$

By the arbitrary of ε, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds = \frac{\alpha_1 - \frac{\sigma_1^2}{2}}{\beta_1}, \quad \text{a.s.}$$

□

Similar to Theorems 4.5 and 4.6, we can obtain the following result.

Theorem 4.7. *Let* $(x(t), y(t))$ *be the solution of system* (3) *with any initial value* $x_0,\ y_0 > 0$. *If* $\alpha_1 - \frac{\sigma_1^2}{2} < 0$ *and* $\alpha_2 - \frac{\sigma_2^2}{2} < 0$, *then*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \text{a.s.}$$

5. Numerical simulations

In this section, we carry out numerical simulations to illustrate our theoretical results obtained in previous sections.

Our numerical method is based on the algorithm from [8]. Consider a case of $n = 3$, the discretization equations are given as follows.

$$\begin{cases} x_{k+1} = x_k + \left(\alpha_1 x_k - \beta_1 x_k^2 - \dfrac{v_1 x_k^3 y_k}{m_1 y_k^3 + x_k^3}\right)\triangle t + \sigma_1 x_k \sqrt{\triangle t}\chi_k \\ \qquad\qquad + \dfrac{\sigma_1^2}{2} x_k(\chi_k^2 - 1)\triangle t, \\ y_{k+1} = y_k + \left(\alpha_2 y_k - \dfrac{v_2 y_k^2}{m_2 + x_k}\right)\triangle t + \sigma_2 y_k \sqrt{\triangle t}\eta_k \\ \qquad\qquad + \dfrac{\sigma_2^2}{2} y_k(\eta_k^2 - 1)\triangle t, \end{cases}$$

where $\chi_k,\ \eta_k,\ k = 1, 2, \ldots, n$ are independent Gaussian random variables $N(0, 1)$.

We choose $(x(0), y(0)) = (1, 10)$ as the initial value in system (3), and set parameters as $\alpha_1 = 1.5$, $\beta_1 = 0.25$, $v_1 = 0.2$, $m_1 = 0.5$, $\alpha_2 = 1.6$, $v_2 = 0.35$ and $m_2 = 1$. Then $\gamma = \max\left\{\frac{v_1}{m_1}, v_1\right\} = 0.4$. In Figure 5.1, we choose noise intensity $\sigma_1 = 0.3$ and $\sigma_2 = 0.15$, then the conditions of Theorem 4.4 are satisfied. As expected, system (3) is persistent in mean. Secondly, we fix $\sigma_1 = 2$ and $\sigma_2 = 0.8$. From Theorem 4.5, we have that prey species will become extinct (see Figure 5.2). Then, we let $\sigma_1 = 0.7$ and $\sigma_2 = 3$, which satisfy conditions of Theorem 4.6, it is clear that predator species will become extinct (see Figure 5.3). Finally, we choose $\sigma_1 = 1.8$ and $\sigma_2 = 2.3$. Then all conditions of Theorem 4.7 are satisfied. Thus, we obtain that both prey and predator species will become extinct (see Figure 5.4).

6. Conclusions

In this paper, we have incorporated general ratio-dependent function to describe the functional response. We have established the persistent and extinct results for system (3). For a special case $n = 1$, system (3) is similar to that considered by Mandal and Banerjee [15]. The persistence of system (3) obtained in Theorem 4.4 are the same as Theorem 3.2 in [15]. However, in this paper, we have also established the extinct results for prey and predator species, which are not discussed by the authors in [15]. By our main results Theorems 4.4-4.7,

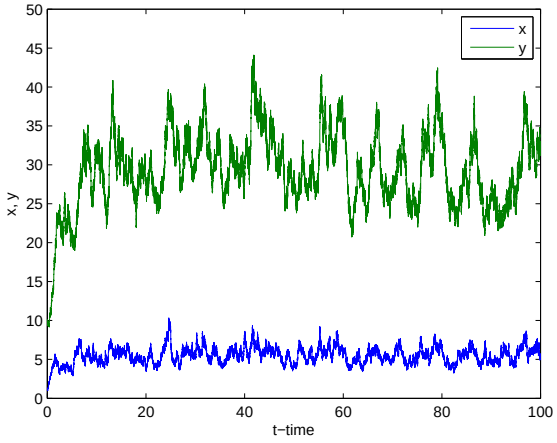

FIGURE 5.1. Numerical simulation of system (3) for $\sigma_1 = 0.3$, $\sigma_2 = 0.15$ and $\Delta t = 0.001$ shows that system (3) is persistent in mean.

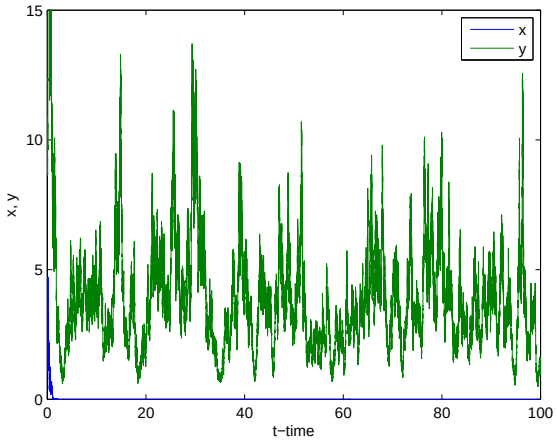

FIGURE 5.2. Numerical simulation of system (3) for $\sigma_1 = 2$, $\sigma_2 = 0.8$ and $\Delta t = 0.001$ shows that prey species go to extinction.

we conclude that the power n has no effect on the persistent and extinct results for system (3).

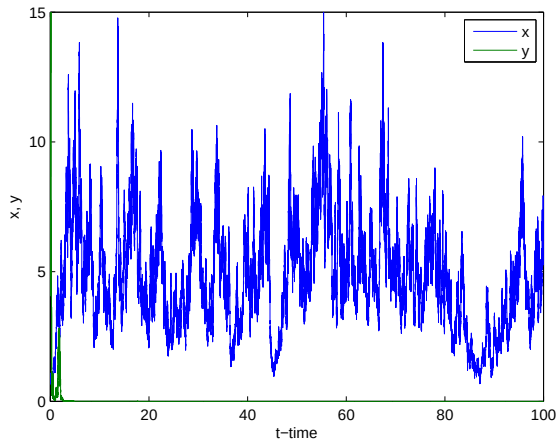

FIGURE 5.3. Numerical simulation of system (3) for $\sigma_1 = 0.7$, $\sigma_2 = 3$ and $\Delta t = 0.001$ shows that predator species go to extinction.

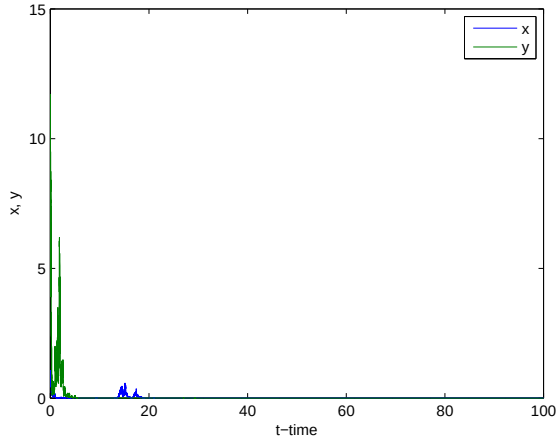

FIGURE 5.4. Numerical simulation of system (3) for $\sigma_1 = 1.8$, $\sigma_2 = 2.3$ and $\Delta t = 0.001$ shows that both prey and predator species go to extinction.

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