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ON $\pi\mathfrak{F}$-EMBEDDED SUBGROUPS OF FINITE GROUPS

Wenbin Guo, Haifeng Yu, and Li Zhang

Abstract. A chief factor H/K of G is called $\mathfrak{F}$-central in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. A normal subgroup N of G is said to be $\pi\mathfrak{F}$-hypercentral in G if either $N = 1$ or $N \neq 1$ and every chief factor of G below N of order divisible by at least one prime in π is $\mathfrak{F}$-central in G. The symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-hypercentre of G, that is, the product of all the normal $\pi\mathfrak{F}$-hypercentral subgroups of G. We say that a subgroup H of G is $\pi\mathfrak{F}$-embedded in G if there exists a normal subgroup T of G such that HT is s-quasinormal in G and $(H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$, where H_G is the maximal normal subgroup of G contained in H. In this paper, we use the $\pi\mathfrak{F}$-embedded subgroups to determine the structures of finite groups. In particular, we give some new characterizations of p-nilpotency and supersolvability of a group.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group, p denotes a prime and π denotes a non-empty subset of the set $\mathbb{P}$ of all primes. Moreover, $|G|_p$ is the order of Sylow p-subgroups of G, $\pi(G)$ denotes the set of all prime factors of $|G|$ and $\pi(\mathfrak{F}) = \bigcup\{\pi(G) \mid G \in \mathfrak{F}\}$, where $\mathfrak{F}$ is a non-empty class of groups. All unexplained notation and terminology are standard, as in [4], [7] and [15].

Let $\mathfrak{F}$ be a class of groups containing 1 and $G^{\mathfrak{F}} = \bigcap\{N \mid N \trianglelefteq G, G/N \in \mathfrak{F}\}$. $\mathfrak{F}$ is called a *formation* if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$; *$S$-closed* (*$S_n$-closed*) if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$ ($H \trianglelefteq G \in \mathfrak{F}$, respectively).

We use $\mathfrak{N}$, $\mathfrak{U}$, and $\mathfrak{S}$ to denote the saturated formations of all nilpotent groups, supersolvable groups and solvable groups, respectively.

For a class $\mathfrak{F}$ of groups, a chief factor H/K of G is called $\mathfrak{F}$-central in G if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. Following [11], a normal subgroup N of G is said to be $\pi\mathfrak{F}$-hypercentral in G if either $N = 1$ or $N \neq 1$ and every chief factor of

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G below N of order divisible by at least one prime in π is $\mathfrak{F}$-central in G. The symbol $Z_{\pi\mathfrak{F}}(G)$ denotes the $\pi\mathfrak{F}$-hypercentre of G, that is, the product of all normal $\pi\mathfrak{F}$-hypercentral subgroups of G. When $\pi = \mathbb{P}$ is the set of all primes, $Z_{\mathbb{P}\mathfrak{F}}(G)$ is called the $\mathfrak{F}$-hypercentre of G and denoted by $Z_{\mathfrak{F}}(G)$ (see [4] p. 389). Clearly, for any non-empty set π of primes, $Z_{\mathfrak{F}}(G) \leq Z_{\pi\mathfrak{F}}(G)$.

It is well known that the $\mathfrak{F}$-hypercentre essentially influences the structure of a group. For example, if all subgroups of G with prime order and order 4 are contained in $Z_\infty(G)$, then G is nilpotent (N. Itŏ). If all subgroups with prime order and order 4 are in $Z_{\mathfrak{U}}(G)$, then G is supersolvable (B. Huppert, K. Doerk). Recently, by using the $\mathfrak{F}$-hypercentre to study the structure of a group, a large number of new results were obtained (see, for example, [1,3,5,9–11,18–20,23]). In connection with this, we naturally ask: what effect does the $\pi\mathfrak{F}$-hypercentre have on the structure of a group?

Recall that a subgroup H of G is said to be s-quasinormal in G [17] if H permutes with every Sylow subgroup of G. Following [17], we use $Syl(G)^{\perp}$ to denote the set of all s-quasinormal subgroups of G.

In this paper, we will use the $\pi\mathfrak{F}$-hypercentre to study the structure of a group. Our tool is following.

Definition 1.1. Let $\mathfrak{F}$ be a non-empty class of groups. A subgroup H of G is called $\pi\mathfrak{F}$-embedded in G if there exists a normal subgroup T of G such that HT is s-quasinormal in G (that is, $HT \in Syl(G)^{\perp}$) and $(H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$, where H_G is the maximal normal subgroup of G contained in H.

In Section 2, we give some properties of the $\pi\mathfrak{F}$-embedded subgroups and some related results. In Section 3, we give new characterizations of p-nilpotence and supersolvability of a group. In Section 4, we list some applications of our results.

2. Preliminaries

Lemma 2.1 ([11, Lemma 2.2], [3, Lemma 2.8])**.** *Let $\mathfrak{F}$ be a saturated formation and $\pi \subseteq \pi(\mathfrak{F})$. Let N be a normal subgroup of G and $A \leq G$. Then*:

(1) *Every G-chief factor of $Z_{\pi\mathfrak{F}}(G)$ of order divisible by at least one prime in π is $\mathfrak{F}$-central.*

(2) $Z_{\pi\mathfrak{F}}(G)N/N \leq Z_{\pi\mathfrak{F}}(G/N)$.

(3) $Z_{\pi\mathfrak{F}}(A)N/N \leq Z_{\pi\mathfrak{F}}(AN/N)$.

(4) *If $\mathfrak{F}$ is* (S_n*-closed*) *S-closed and A is a* (*normal*) *subgroup of G, then* $Z_{\pi\mathfrak{F}}(G) \cap A \leq Z_{\pi\mathfrak{F}}(A)$.

(5) *If $\mathfrak{G}_{\pi'}\mathfrak{F} = \mathfrak{F}$ and $G/Z_{\pi\mathfrak{F}}(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.*

(6) *Suppose that $\mathfrak{F}$ is* (S_n*-closed*) *S-closed and A is a* (*normal*) *subgroup of G. If $\mathfrak{G}_{\pi'}\mathfrak{F} = \mathfrak{F}$ and $A \in \mathfrak{F}$, then $Z_{\pi\mathfrak{F}}(G)A \in \mathfrak{F}$.*

Lemma 2.2 (see [17])**.** *Let G be a group, $H \leq K \leq G$ and $A \leq G$.*

(1) *$Syl(G)^{\perp}$ is a proper sublattice of the lattice consisting of all subnormal subgroups of G.*

(2) *If* $A, H \in Syl(G)^{\perp}$, *then* $\langle A, H\rangle \in Syl(G)^{\perp}$, *where* $\langle A, H\rangle$ *is the smallest subgroup of* G *containing* A *and* H.

(3) *If* $H \in Syl(G)^{\perp}$, *then* $H \in Syl(K)^{\perp}$ *and* $H \cap A \in Syl(A)^{\perp}$.

(4) *Suppose that* A *is normal in* G. *If* $H \in Syl(G)^{\perp}$, *then* $HA/A \in Syl(G/A)^{\perp}$. *Moreover, the converse holds in case* $A \leq H$.

(5) *Let* A *be a* p*-subgroup of* G *for some prime* p. *Then* $A \in Syl(G)^{\perp}$ *if and only if* $O^p(G) \leq N_G(A)$.

Lemma 2.3 ([2, Lemma 2.12])**.** *Let* p *be a prime divisor of* $|G|$ *with* $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$. *If* $H \trianglelefteq G$ *with* $p^{n+1} \nmid |H|$ *and* G/H *is* p*-nilpotent, then* G *is* p*-nilpotent. In particular, if* $p^{n+1} \nmid |G|$, *then* G *is* p*-nilpotent.*

Lemma 2.4. *Let* $\mathfrak{F}$ *be a saturated formation,* G *be a group and* $H \leq K \leq G$.

(1) H *is* $\pi\mathfrak{F}$*-embedded in* G *if and only if there exists a normal subgroup* T *of* G *such that* $HT \in Syl(G)^{\perp}$, $H_G \leq T$ *and* $(H \cap T)/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$.

(2) *Suppose that* H *is normal in* G. *Then* K/H *is* $\pi\mathfrak{F}$*-embedded in* G/H *if and only if* K *is* $\pi\mathfrak{F}$*-embedded in* G.

(3) *Suppose that* H *is normal in* G. *Then for every* $\pi\mathfrak{F}$*-embedded subgroup* E *of* G *satisfying* $(|H|, |E|) = 1$, HE/H *is* $\pi\mathfrak{F}$*-embedded in* G/H.

(4) *Suppose that* H *is* $\pi\mathfrak{F}$*-embedded in* G. *If* $\mathfrak{F}$ *is* (S_n*-closed*) S*-closed and* K *is a* (*normal*) *subgroup of* G, *then* H *is* $\pi\mathfrak{F}$*-embedded in* K.

(5) *If* $G \in \mathfrak{F}$, *then every subgroup of* G *is* $\pi\mathfrak{F}$*-embedded in* G.

(6) *Every subgroup of a* π'*-group* G *is* $\pi\mathfrak{F}$*-embedded in* G.

Proof. (1) The sufficiency is clear. Now assume that H is $\pi\mathfrak{F}$-embedded in G and let T be a normal subgroup of G such that $HT \in Syl(G)^{\perp}$ and $(H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$. Let $T_0 = TH_G$. Then $HT_0 = HT \in Syl(G)^{\perp}$ and $(H \cap T_0)/H_G = (H \cap T)H_G/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$.

(2) First assume that K/H is $\pi\mathfrak{F}$-embedded in G/H. Then by (1), G/H has a normal subgroup T/H such that

$$(K/H)(T/H) = KT/H \in Syl(G/H)^{\perp}, \quad (K/H)_{G/H} = K_G/H \leq T/H$$

and

$$((K/H) \cap (T/H))/(K/H)_{G/H} \leq Z_{\pi\mathfrak{F}}((G/H)/(K/H)_{G/H}).$$

Note that

$$((K/H) \cap (T/H))/(K/H)_{G/H} \cong (T \cap K)/K_G$$

and

$$Z_{\pi\mathfrak{F}}((G/H)/(K/H)_{G/H}) \cong Z_{\pi\mathfrak{F}}(G/K_G).$$

Also, $KT \in Syl(G)^{\perp}$ by Lemma 2.2(4). Hence K is $\pi\mathfrak{F}$-embedded in G. Analogously, one can show that if K is $\pi\mathfrak{F}$-embedded in G, then K/H is $\pi\mathfrak{F}$-embedded in G/H.

(3) Assume that H is normal in G and E is $\pi\mathfrak{F}$-embedded in G with $(|H|, |E|) = 1$. Then by (1), G has a normal subgroup T such that $ET \in Syl(G)^{\perp}$, $E_G \leq T$ and $(E \cap T)/E_G \leq Z_{\pi\mathfrak{F}}(G/E_G)$. We now prove that HE/H is $\pi\mathfrak{F}$-embedded in G/H. By (2), we only need to prove that HE is $\pi\mathfrak{F}$-embedded in G. It is

clear that $(HE)T = H(ET) \in Syl(G)^{\perp}$ by Lemma 2.2(2). Since $(|H|, |E|) = 1$, $(|HE \cap T : H \cap T|, |HE \cap T : E \cap T|) = 1$. So $HE \cap T = (H \cap T)(E \cap T)$ (see [4, A, 1.6]). Let $D = (HE)_G$. Then $(HE \cap T)D/E_G = (E \cap T)D/E_G \leq Z_{\pi\mathfrak{F}}(G/E_G)(D/E_G)$. Thus $(HE \cap T)D/D \leq Z_{\pi\mathfrak{F}}(G/D)$ by Lemma 2.1(2). This shows that HE is $\pi\mathfrak{F}$-embedded in G.

(4) Let T be a normal subgroup of G such that $HT \in Syl(G)^{\perp}$, $H_G \leq T$ and $(H \cap T)/H_G \leq Z_{\pi\mathfrak{F}}(G/H_G)$. Assume that $T_1 = K \cap T$. Then $HT_1 = K \cap HT \in Syl(K)^{\perp}$ by Lemma 2.2(3) and $(H \cap T_1)/H_G = (H \cap T)/H_G \cap K/H_G \leq Z_{\pi\mathfrak{F}}(K/H_G)$ by Lemma 2.1(4). Since $H_G \leq H_K$, $(T_1 \cap H)H_K/H_K \leq Z_{\pi\mathfrak{F}}(K/H_K)$ by Lemma 2.1(2). Hence H is $\pi\mathfrak{F}$-embedded in K.

(5) and (6) are obvious. □

Lemma 2.5 (see [17]). (1) *Let H be a p-subgroup of G for some prime p. Then H is subnormal in G if and only if $H \leq O_p(G)$.*

(2) *Let H be a subgroup of G with p-power index for some prime p. Then H is subnormal in G if and only if $O^p(G) \leq H$.*

Lemma 2.6 ([12, Lemma 2.12]). *Let p be a prime divisor of G with $(|G|, p-1) = 1$. Suppose that P is a Sylow p-subgroup of G such that every maximal subgroup of P has a p-nilpotent supplement in G. Then G is p-nilpotent.*

Lemma 2.7 ([8, Lemma 2.3]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Main results

Theorem 3.1. *Let p be a prime divisor of $|G|$ such that $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. If there exists a Sylow p-subgroup P of G such that every n-maximal subgroup (if exists) of P is $p\mathfrak{U}$-embedded in G, then G is p-nilpotent.*

Proof. Suppose that the assertion is false and let (G, P) be a counterexample such that $|G| + |P|$ is minimal. Then $p^{n+1} \mid |G|$ by Lemma 2.3.

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) > 1$. Let $M/O_{p'}(G)$ be an n-maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = O_{p'}(G)(M \cap P)$, where $M \cap P$ is an n-maximal subgroup of P since $|P : M \cap P| = |PO_{p'}(G) : M| = p^n$. Thus $M/O_{p'}(G)$ is $p\mathfrak{U}$-embedded in $G/O_{p'}(G)$ by Lemma 2.4(3). This shows that

$$(G/O_{p'}(G), PO_{p'}(G)/O_{p'}(G))$$

satisfies the hypothesis for (G, P). Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. It follows that G is p-nilpotent, a contradiction.

(2) $Z_{p\mathfrak{U}}(G) = 1$.

Suppose that $Z_{p\mathfrak{U}}(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $Z_{p\mathfrak{U}}(G)$. Then by (1) $N \leq Z_{\mathfrak{U}}(G)$ is a subgroup of order p.

Consequently, $N \leq Z(G)$ since $(|G|, p-1) = 1$. By Lemma 2.4(2), $(G/N, P/N)$ satisfies the hypothesis for (G, P). Hence G/N is p-nilpotent and so G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

If $O_p(G) = 1$, then $(P_n)_G = 1$ for any n-maximal subgroup P_n of P. Hence by the hypothesis and (2), G has a normal subgroup T such that $P_nT \in Syl(G)^{\perp}$ and $P_n \cap T = 1$. Clearly $|T|_p \leqslant p^n$, so T is p-nilpotent by Lemma 2.3. Thus $T = 1$ by the assumption $O_p(G) = 1$ and (1). This shows that $P_n \in Syl(G)^{\perp}$, so $P_n \leq O_p(G) = 1$ by Lemma 2.2(1) and Lemma 2.5(1), which contradicts $p^{n+1} \mid |G|$. Thus $O_p(G) \neq 1$.

(4) $O_p(G)$ *is a minimal normal subgroup of* G *and* $G = O_p(G) \rtimes M$*, where* M *is* p*-nilpotent.*

Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then G/N is p-nilpotent similar as the proof in (2). Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $N \nsubseteq \Phi(G)$. It follows that $G = N \rtimes M$ for some maximal subgroup M of G. By [4, A, 8.4], $O_p(G) \cap M \trianglelefteq G$, so $O_p(G) \cap M = 1$ by the uniqueness of N. It follows that $O_p(G) = N(O_p(G) \cap M) = N$. Thus $O_p(G)$ is a minimal normal subgroup of G.

(5) *Final contradiction.*

Let P_n be an arbitrary n-maximal subgroup of P. Then $(P_n)_G = 1$ or $O_p(G)$ by (4). If $(P_n)_G = O_p(G)$ for any n-maximal subgroup P_n of P, then $G = O_p(G)M = P_nM$. This shows that every n-maximal subgroup of P has a p-nilpotent supplement in G. Consequently, every maximal subgroup of P has a p-nilpotent supplement in G. So G is p-nilpotent by Lemma 2.6. This contradiction shows that there exists at least one non-trivial n-maximal subgroup P_n of P with $(P_n)_G = 1$. Then by the hypothesis, G has a normal subgroup T such that $P_nT \in Syl(G)^{\perp}$ and $P_n \cap T = 1$ by (2). Now by Lemma 2.3, T is p-nilpotent. Hence $T = 1$ or $O_p(G)$ by (1) and (4). Assume that $T = O_p(G)$. Since $P_nT \in Syl(G)^{\perp}$, we have that $P_n \leq O_p(G) = T$ by Lemma 2.2(1) and Lemma 2.5(1). Thus $P_n = P_n \cap T = 1$, a contradiction. Therefore $T = 1$. Then $P_n \in Syl(G)^{\perp}$, so $P_n \leq O_p(G)$ and $O^p(G) \leq N_G(P_n)$ by Lemma 2.2(1)(5) and Lemma 2.5(1). Clearly, the number of subgroups in the conjugate class of P_n in P is equal to $|P : P \cap N_G(P_n)| = |G : N_G(P_n)| > 1$, which is a p-power. Let $|O_p(G)| = p^d$ and $|P_n| = p^k$. As $O_p(G)$ is elementary abelian by (4), the number of subgroups of order $|P_n|$ is

$$f(d,k) = \frac{(p^d-1)(p^{d-1}-1)\cdots(p^{d-k+1}-1)}{(p^k-1)(p^{k-1}-1)\cdots(p-1)}$$

(see [15, III, 8.5]). But $p \nmid f(d,k)$, a contradiction. This completes the proof. □

Corollary 3.2. *Let* p *be a prime divisor of* $|G|$ *with* $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ *for some integer* $n \geqslant 1$*. Suppose that* G *has a normal subgroup* H *such* *that G/H is p-nilpotent. If H has a Sylow p-subgroup P such that every n-maximal subgroup (if exists) of P is $p\mathfrak{U}$-embedded in G, then G is p-nilpotent.*

Proof. First suppose that $H = P$. Let K/P be the normal Hall p'-subgroup of G/P. By the Schur-Zassenhaus Theorem $K = P \rtimes K_{p'}$, for some Hall p'-subgroup $K_{p'}$ of K. Obviously, $K_{p'}$ is also a Hall p'-subgroup of G. By Lemma 2.4(4) every n-maximal subgroup of P is $p\mathfrak{U}$-embedded in K. Hence K is p-nilpotent by Theorem 3.1 and so $K = P \times K_{p'}$. Then $K_{p'}$ is normal in G. Consequently G is p-nilpotent.

Finally, assume that $H > P$. Then by Lemma 2.4(4) and Theorem 3.1, H is p-nilpotent. Let $H_{p'}$ be the normal Hall p'-subgroup of H. Now by Lemma 2.4(3), $(G/H_{p'}, H/H_{p'})$ satisfies the assumptions. Hence $G/H_{p'}$ is p-nilpotent by induction. It follows that G is p-nilpotent. □

We use $\mathfrak{N}^p$ to denote the saturated formation of all p-nilpotent groups.

Theorem 3.3. *Let p be a prime divisor of $|G|$ with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geqslant 1$. Let H be a normal subgroup of G such that G/H is p-nilpotent and P be an arbitrary Sylow p-subgroup of H. Suppose that every subgroup L of $P \cap G^{\mathfrak{N}^p}$ of order p^n or 4 (when $p = 2$, $n = 1$, P is non-abelian and L is cyclic) not contained in $Z_\infty(G)$ is $p\mathfrak{U}$-embedded in G. Then G is p-nilpotent.*

Proof. Suppose that the result is false and let (G, H) be a counterexample for which $|G| + |H|$ is minimal. Clearly, $G^{\mathfrak{N}^p} \leq H$. We proceed via the following steps.

(1) $|P| \geqslant p^{n+1}$ (it follows directly from Lemma 2.3).

(2) $O_{p'}(G) = 1$.

Assume that $N = O_{p'}(G) > 1$. If $|(G/N)^{\mathfrak{N}^p}|_p = |G^{\mathfrak{N}^p}N/N|_p < p^{n+1}$, then G/N is p-nilpotent by Lemma 2.3. We may, therefore, assume that $|G^{\mathfrak{N}^p}N/N|_p \geqslant p^{n+1}$. Let L/N be a subgroup of $PN/N \cap G^{\mathfrak{N}^p}N/N$ of order p^n or 4 (when $p = 2$ and $n = 1$, PN/N is non-abelian and L/N is cyclic) not contained in $Z_\infty(G/N)$, where P is an arbitrary Sylow p-subgroup of H. Since $L = (L \cap P)N$ and $(|N|, p) = 1$, $|L/N| = |L \cap P| = p^n$ or 4. Also, since $L \cap P \leq G^{\mathfrak{N}^p}N$ and $(|L \cap P|, |G^{\mathfrak{N}^p}N : G^{\mathfrak{N}^p}|) = 1$, we have $L \cap P \leq G^{\mathfrak{N}^p}$. By Lemma 2.1(2) $L \cap P \nsubseteq Z_\infty(G)$. Suppose that $|L \cap P| = 4$. Then P is non-abelian and $L \cap P$ is cyclic owing to the G-isomorphism $L/N \cong L \cap P$. Hence by hypothesis and Lemma 2.4(3), L/N is $p\mathfrak{U}$-embedded in G/N. This shows that $(G/N, HN/N)$ satisfies the hypothesis. The choice of (G, H) implies that G/N is p-nilpotent. Thus G is p-nilpotent, a contradiction.

(3) *Every proper subgroup M of G is p-nilpotent.*

Considering $(M, M \cap H)$. Clearly, every Sylow p-subgroup of $M \cap H$ has the form $M \cap P$ for some Sylow p-subgroup P of H. By Lemma 2.3, we may assume that $|(H \cap M) \cap M^{\mathfrak{N}^p}|_p = |M^{\mathfrak{N}^p}|_p \geqslant p^{n+1}$. Let L be a subgroup of $(P \cap M) \cap M^{\mathfrak{N}^p}$ of order p^n or 4 (when $p = 2$ and $n = 1$, $P \cap M$ is non-abelian and L is cyclic) not contained in $Z_\infty(M)$. Obviously $P\cap M^{\mathfrak{N}^p}\leq P\cap(M\cap G^{\mathfrak{N}^p})\leq P\cap G^{\mathfrak{N}^p}$. Also, $L\nsubseteq Z_\infty(G)$ by Lemma 2.1(4). Hence by the hypothesis and Lemma 2.4(4), L is $p\mathfrak{U}$-embedded in M. The choice of (G,H) implies that M is p-nilpotent.

(4) *G is a minimal non-nilpotent group.*

By (3), G is a minimal non-p-nilpotent group. Then G is a minimal non-nilpotent group by Itŏ's Theorem (see [15, IV, 5.4]). Hence by [7, (3.4.7) and (3.4.11)], $G=P\rtimes Q$, where P is the normal Sylow p-subgroup of G and Q a cyclic Sylow q-subgroup of G with $q\neq p$, and the following hold: (i) $P/\Phi(P)$ is a chief factor of G; (ii) $P=G^{\mathfrak{N}}=G^{\mathfrak{N}_p}$; (iii) the exponent of P is p or 4 (when P is a non-abelian 2-subgroup); (iv) $\Phi(G)=Z_\infty(G)$; (v) $F(G)=F_p(G)=P\Phi(G)$.

(5) *$F(G)=P$ and $\Phi(G)=\Phi(P)$.*

By (2) and (4), $F_p(G)=P=F(G)=P\Phi(G)$. Then $\Phi(P)\leq\Phi(G)\leq P$, so $\Phi(G)=\Phi(P)$ or P since $P/\Phi(P)$ is a chief factor of G. If $\Phi(G)=P$, then $G=Q$, a contradiction. Thus $\Phi(G)=\Phi(P)$.

(6) *P has a proper subgroup L of order p^n or 4 such that $L\nsubseteq\Phi(P)$.*

Take $x\in P\setminus\Phi(P)$ and let $E=\langle x\rangle$. Then $|E|=p$ or 4 (when P is a non-abelian 2-subgroup). It follows that P has a subgroup L of order p^n or cyclic of order 4 such that $E\leq L$ and $L\nsubseteq\Phi(P)$.

(7) *Final contradiction.*

Clearly $L_G\leq\Phi(P)$. By the hypothesis L is $p\mathfrak{U}$-embedded in G. Let T be a normal subgroup of G such that $LT\in Syl(G)^\perp$, $L_G\leq T$ and $(L\cap T)/L_G\leq Z_{p\mathfrak{U}}(G/L_G)$. If $T=G$, then $(L\cap T)/L_G=L/L_G\leq Z_{p\mathfrak{U}}(G/L_G)$. It follows from Lemma 2.1(2) that $L\Phi(P)/\Phi(P)\leq Z_{p\mathfrak{U}}(G/\Phi(P))$. Hence $P/\Phi(P)\cap Z_{p\mathfrak{U}}(G/\Phi(P))\neq 1$, so $P/\Phi(P)\leq Z_{p\mathfrak{U}}(G/\Phi(P))$ by (i) in (4). Then $|P/\Phi(P)|=p$ and $P/\Phi(P)\leq Z(G/\Phi(P))$. Therefore $G/\Phi(P)$ is p-nilpotent since $P=G^{\mathfrak{N}_p}$, and so G is p-nilpotent, a contradiction. Now assume $T<G$. Then $T\leq F_p(G)=P$ by (3). As $T\Phi(P)$ is normal in G, $T\Phi(P)=\Phi(P)$ or P by (i). If $T\Phi(P)=P$, then $P=T$ and so $L\cap T=L$. Similar as the above, we obtain a contradiction. Hence $T\leq\Phi(P)$. As $LT\in Syl(G)^\perp$, $Q\leq O^p(G)\leq N_G(LT)$ by Lemma 2.2(5). Then $Q\leq N_G(LT\Phi(P))=N_G(L\Phi(P))$. Also, since $P/\Phi(P)$ is elementary abelian by (i), it follows that $L\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Thus $L\Phi(P)=P=L$. The final contradiction completes the proof. □

Theorem 3.4. *G is supersolvable if and only if G has a normal subgroup H such that G/H is supersolvable and every maximal subgroup of any non-cyclic Sylow p-subgroup of H is $p\mathfrak{U}$-embedded in G, for any prime $p\in\pi(H)$.*

Proof. Since the necessity is obvious, we only need to prove the sufficiency. Suppose that the sufficiency is false and let (G,H) be a counterexample such that $|G|+|H|$ is minimal. Then:

(1) *Let N is a normal subgroup of G contained in H. If N is either a p-subgroup for some prime $p\in\pi(H)$ or a Hall subgroup of H, then the hypothesis holds for $(G/N,H/N)$ and so G/N is supersolvable.*

Firstly, $(G/N)/(H/N) \cong G/H$ is supersolvable. Assume that N is a p-group for some prime $p \in \pi(H)$. Let Q/N be a non-cyclic Sylow q-subgroup of H/N and M/N be an arbitrary maximal subgroup of Q/N. If $q = p$, then Q is a non-cyclic Sylow p-subgroup of H and M/N is $p\mathfrak{U}$-embedded in G/N by Lemma 2.4(2). Now assume $q \neq p$. Let M_1 be a Sylow q-subgroup of M and Q_1 be a Sylow q-subgroup of Q containing M_1. Then $Q = N \rtimes Q_1$ and $M = N \rtimes M_1$ by the Schur-Zassenhaus Theorem. Clearly $|Q_1 : M_1| = |Q/N : M/N| = q$ and Q_1 is a non-cyclic Sylow q-subgroup of H. Hence M/N is $q\mathfrak{U}$-embedded in G/N by Lemma 2.4(3). This shows that the hypothesis holds for $(G/N, H/N)$. Analogously, one can show that the hypothesis still holds for $(G/N, H/N)$ when N is a Hall subgroup of H.

(2) *H is a Sylow tower group of supersolvable type.*

Let p be the smallest prime divisor of $|H|$ and H_p be a Sylow p-subgroup of H. If H_p is cyclic, then H is p-nilpotent by [16, (10.1.9)]. Otherwise, H is still p-nilpotent by Lemma 2.4(4) and Theorem 3.1. Let U be the normal Hall p'-subgroup of H. Then by Lemma 2.4(4), U satisfies the hypothesis. Therefore H is a Sylow tower group of supersolvable type by induction.

(3) *If N is a normal Hall subgroup of H, then $N = H$.*

Clearly, (G, N) satisfies the hypothesis. Hence it follows from (1).

(4) *H is a non-cyclic q-subgroup for some prime q (it follows directly from* (2)*,* (3) *and Lemma* 2.7*).*

(5) *H is a minimal normal subgroup of G.*

By (1) and (4), G has an unique minimal normal subgroup R of G contained in H and $R \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G such that $G = R \rtimes M$. By (4), $H \leq F(G) \leq C_G(R)$, so $H \cap M \trianglelefteq G$. This implies that $H \cap M = 1$. Thus $H = R(H \cap M) = R$.

(6) *Final contradiction.*

By (5), $G = H \rtimes M$ where M is a maximal subgroup of G and $M \cong G/H$ is supersolvable. Let M_q be a Sylow q-subgroup of M. Then $G_q = HM_q$ is a Sylow q-subgroup of G. Let $H^* = G_q^* \cap H$ where G_q^* is a maximal subgroup of G_q containing M_q. Then H^* is a non-trivial maximal subgroup of H. Clearly, $(H^*)_G = 1$ by (5). By (4), H^* is $q\mathfrak{U}$-embedded in G. Let T be a normal subgroup of G such that $H^*T \in Syl(G)^{\perp}$ and $H^* \cap T \leq Z_{q\mathfrak{U}}(G)$. If $H \cap T = H$, then $H^* \cap T = H^* \leq Z_{q\mathfrak{U}}(G)$. This implies that $H \leq Z_{q\mathfrak{U}}(G)$ is a cyclic subgroup of order q, which contradicts (4). We may, therefore, assume that $H \cap T = 1$. Then $H^* = H^*(H \cap T) = H \cap H^*T \in Syl(G)^{\perp}$ by Lemma 2.2(1). Therefore $G = O^p(G)G_q \leq N_G(H^*)$ by Lemma 2.2(5). This shows that H^* is normal in G. The contradiction completes the proof. □

The following theorem is a dual of Theorem 3.4.

Theorem 3.5. *G is supersolvable if and only if G has a normal subgroup H such that G/H is supersolvable and every cyclic subgroup of H of order p or* *order* 4 (*when* $p = 2$ *and* H *has a non-abelian Sylow* 2*-subgroup*) *is* $p\mathfrak{U}$*-embedded in* G*, for any prime* $p \in \pi(H)$.

Proof. Since the necessity is obvious, we only need to prove the sufficiency. Suppose that the result is false and let (G, H) be a counterexample such that $|G| + |H|$ is minimal. Then:

(1) *G is a minimal non-supersolvable group.*

Let M be a proper subgroup of G. Consider $(M, M \cap H)$. Firstly, $M/(M \cap H) \cong HM/H \leq G/H$ is superslovable. Also, every cyclic subgroup of $M \cap H$ of order p or order 4 (if $M \cap H$ has a non-abelian Sylow 2-subgroup) is $p\mathfrak{U}$-emdedded in M by Lemma 2.4(4). This shows that $(M, M \cap H)$ satisfies the hypothesis for (G, H). Hence M is supersolvable by the choice of (G, H). This shows that G is a minimal non-supersolvable group, and so G is solvable (see [16, (10.3.4)]).

(2) *H is a q-group for some prime q.*

Let p be the smallest prime divisor of $|H|$. By Lemma 2.4(4), every subgroup of H with order p or 4 (if $p = 2$ and H has a non-abelian Sylow 2-subgroup) is $p\mathfrak{U}$-embedded in H. Hence H is p-nilpotent by Theorem 3.3. Let U be the normal Hall p'-subgroup of H. U satisfies the hypothesis by Lemma 2.4(4). Therefore H is a Sylow tower group of supersolvable type by induction.

Let H_q be the normal Sylow q-subgroup of H, where q is the largest prime divisor of $|H|$. By Lemma 2.4(3), $(G/H_q, H/H_q)$ satisfies the hypothesis. Hence G/H_q is supersolvable. If $H_q < H$, then G is supersolvable by the choice of (G, H). Thus (2) holds.

(3) *q is the largest prime divisor of $|G|$. Consequently $q > 2$.*

Assume that p ($\neq q$) is the largest prime divisor of $|G|$. Let G_p be a Sylow p-subgroup of G. Then HG_p/H is normal in G/H by the supersolvability of G/H. By Lemma 2.4(4) and Theorem 3.3, HG_p is q-nilpotent. It follows that G_p is normal in G. Considering $(G/G_p, HG_p/G_p)$, then G/G_p is supersolvable by Lemma 2.4(3) and the choice of (G, H). This implies that $G \cong G/(H \cap G_p)$ is supersolvable, a contradiction.

(4) *The Sylow q-subgroup G_q of G is normal in G.*

It follows from (3) and the supersolvability of G/H.

(5) *$H = G_q = G^{\mathfrak{U}}$ with exponent q and $H/\Phi(H)$ is a non-cyclic chief factor of G.*

By (1), (3), (4) and [7, (3.4.2) and (3.4.7)], we have: (i) $G_q = G^{\mathfrak{U}}$; (ii) $G^{\mathfrak{U}}/\Phi(G^{\mathfrak{U}})$ is a chief factor of G; (iii) the exponent of $G^{\mathfrak{U}}$ is q. Since $G_q = G^{\mathfrak{U}} \leq H \leq G_q$, we have that $H = G_q = G^{\mathfrak{U}}$. By Lemma 2.7, we see that $H/\Phi(H)$ is non-cyclic.

(6) *$H/\Phi(H)$ has a minimal subgroup which does not belong to $Syl(G/\Phi(H))^{\perp}$.*

Assume that every minimal subgroup of $H/\Phi(H)$ belongs to $Syl(G/\Phi(H))^{\perp}$. By (5) and Lemma 2.2(2), $H/\Phi(H)$ has a maximal subgroup which belongs to $Syl(G/\Phi(H))^{\perp}$. Then Lemma 2.2(5) implies that $H/\Phi(H)$ has a maximal subgroup which is normal in $G/\Phi(H)$. This implies that $H/\Phi(H)$ is of order q, which contradicts (5).

(7) *Final contradiction.*

Let $X/\Phi(H)$ be a minimal subgroup of $H/\Phi(H)$ which does not belong to $Syl(G/\Phi(H))^{\perp}$. Take $x \in X \setminus \Phi(H)$. Then $L = \langle x\rangle$ is of order q and $L\Phi(H) = X$. Clearly $L_G = 1$. Let T be a normal subgroup of G such that $LT \in Syl(G)^{\perp}$ and $L \cap T \leq Z_{q\mathfrak{U}}(G)$. By (5), $(H \cap T)\Phi(H) = \Phi(H)$ or H. If $(H \cap T)\Phi(H) = H$, then $H \leq T$ and $L \leq Z_{q\mathfrak{U}}(G)$. Now similar as the proof (7) in Theorem 3.3, we have $|H/\Phi(H)| = q$, a contradiction. Hence $(H\cap T)\Phi(H) = \Phi(H)$. Then $X/\Phi(H) = L(H\cap T)\Phi(H)/\Phi(H) = (H/\Phi(H))\cap (LT\Phi(H)/\Phi(H)) \in Syl(G/\Phi(H))^{\perp}$ by Lemma 2.2(1)(4), which contradicts the choice of $X/\Phi(H)$. The contradiction completes the proof. □

4. Some applications

Recall that a subgroup H of G is said to be: c-normal ([22]) in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G$; $\mathfrak{F}_h$-normal in G ([6]) if there exists a normal subgroup K of G such that HK is a normal Hall subgroup of G and $(H\cap K)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$; $\mathfrak{F}_s$-quasinormal in G ([14]) if there exists a normal subgroup T of G such that $HT \in Syl(G)^{\perp}$ and $(H \cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$.

It is easy to see that all above subgroups are $\pi\mathfrak{F}$-embedded for any non-empty set π of primes. However, the following example shows that the converse is not true.

Example. Let $G = S_4$. Then $Z_{\mathfrak{U}}(G) = 1$ and $Z_{3\mathfrak{U}}(G) = G$. Assume that $H = \{1, (12)(34)\}$. Clearly $H_G = 1$. Since $\{1, (123), (132)\}H \neq H\{1, (123), (132)\}$, H is not s-quasinormal in G. For any non-trivial normal subgroup T of G, we have that $HT = T \in Syl(G)^{\perp}$, $H \cap T = H \nsubseteq Z_{\mathfrak{U}}(G)$ and $H \cap T \leq Z_{3\mathfrak{U}}(G)$. This shows that H is $3\mathfrak{U}$-embedded but not $\mathfrak{U}_s$-quasinormal in G.

Many known results are corollaries of our Theorems, for example, Theorem 5.2 in [6], Theorem 3.4 in [13], Theorem 3.3 in [14], Theorem 1 and Theorem 3 in [21], Theorem 4.1 and Theorem 4.2 in [22] and so on.

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Wenbin Guo
Department of Mathematics
University of Science and Technology of China
Hefei 230026, P. R. China
E-mail address: wbguo@ustc.edu.cn

Haifeng Yu
Department of Mathematics and Physics
Hefei University
Hefei, 230601, P. R. China
E-mail address: yuhfslx@hfuu.edu.cn

Li Zhang
Department of Mathematics
University of Science and Technology of China
Hefei 230026, P. R. China
E-mail address: zhang12@mail.ustc.edu.cn