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SHARED VALUES AND BOREL EXCEPTIONAL VALUES FOR HIGH ORDER DIFFERENCE OPERATORS

Liangwen Liao and Jie Zhang

Abstract. In this paper, we investigate the high order difference counterpart of Brück's conjecture, and we prove one result that for a transcendental entire function f of finite order, which has a Borel exceptional function a whose order is less than one, if $\Delta^n f$ and f share one small function d other than a CM, then f must be form of $f(z) = a + ce^{\beta z}$, where c and β are two nonzero constants such that $\frac{d-\Delta^n a}{d-a} = (e^\beta - 1)^n$. This result extends Chen's result from the case of $\sigma(d) < 1$ to the general case of $\sigma(d) < \sigma(f)$.

1. Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the following standard notations in value distribution theory (see [8, 14, 15]):

$$T(r, f),\ m(r, f),\ N(r, f),\ \overline{N}(r, f), \ldots.$$

And we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o\{T(r, f)\} \quad \text{as } r \to \infty,$$

possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ if and only if $T(r, a) = S(r, f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f respectively.

We say that two meromorphic functions $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). Classic Nevanlinna four

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values theorem says that if two nonconstant meromorphic functions f and g share four values CM, then $f \equiv g$ or f is a Möbius transformation of g. The condition "4 CM" has been weakened to "2 CM+2 IM" by Gundersen [10], as well as by Mues [13]. But whether the condition can be weakened to "1 CM+3 IM" is still an open question.

We define the difference operators

$$\Delta_\eta f = f(z+\eta) - f(z), \ i.e., \ \Delta f = f(z+1) - f(z)$$

and

$$\Delta_\eta^n f = \Delta_\eta^{n-1}(\Delta_\eta f), \ i.e., \ \Delta^n f = \Delta^{n-1}(\Delta f).$$

Moreover, we see

$$\Delta^n f = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z+j)$$

by mathematical induction.

In 1996, R. Brück [1] studied the uniqueness theory about some entire functions sharing one value with their derivatives and posed the following interesting and famous conjecture.

Conjecture. *Let $f(z)$ be non-constant entire function satisfying*

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is neither infinity nor a positive integer. If $f(z)$ and $f'(z)$ share one finite value a CM, then

$$\frac{f'-a}{f-a} = c$$

holds for some constant $c \neq 0$.

This conjecture has been verified in the special cases when $a = 0$ [1], or when f is of finite order [12], or when $\sigma_2(f) < \frac{1}{2}$ [4]. It is well known that Δf can be regarded as the difference counterpart of f'. Recently, many authors started to consider the complex difference equations and the uniqueness of meromorphic functions sharing values with their difference operators or shifts. For example, the authors in [3] considered the problem that $\Delta_\eta^n f$ and f sharing one function b CM and proved the following theorem.

Theorem A ([3])**.** *Let $f(z)$ be a finite order transcendental entire function such that $\lambda(f(z)-a(z)) < \sigma(f)$, where $a(z)$ is an entire function and satisfies $\sigma(a) < 1$, let n be a positive integer. If $\Delta_\eta^n f(z)$ and $f(z)$ share an entire function $b(z)$ ($b(z) \not\equiv a(z)$ and $\sigma(b) < 1$) CM, where $\eta(\in C)$ satisfies $\Delta_\eta^n f(z) \not\equiv 0$, then*

$$f(z) = a(z) + ce^{c_1 z},$$

where c, c_1 are two nonzero constants.

Remark 1.1. In Theorem A, it is easy to see that if $\Delta_\eta^n f(z) \equiv 0$ holds, then $\Delta_\eta^n f$ and $f(z)$ can not share the function b ($b \not\equiv a$) CM by a simple discussion. So we can remove the assumption $\Delta_\eta^n f(z) \not\equiv 0$. In addition, we also can find out the exact solution of the equation above by a simple calculation, that is

$$f(z) = a(z) + ce^{\beta z},$$

where c, β are some nonzero constants satisfying $\frac{b-\Delta_\eta^n a}{b-a} = (e^\beta - 1)^n$.

Heittokangas et al. [9] considered the uniqueness of meromorphic functions sharing values with their shifts and proved the following theorem.

Theorem B ([9]). *Let f be a meromorphic function with $\sigma(f) < 2$, and let $c \in C$. If $f(z)$ and $f(z+c)$ share the values $a \in C$ and ∞ CM, then*

$$\frac{f(z+c) - a}{f(z) - a} = \tau$$

holds for some constant τ.

In the same paper, they also given the example $f(z) = e^{z^2} + 1$, which showed that the condition $\sigma(f) < 2$ can not be relaxed to $\sigma(f) \leq 2$. Without loss of generality, we just need to consider the case Δf ($\eta, c = 1$). Using the same restrictive condition $\sigma(f) < 2$ in Theorem B, we once proved one result as follows.

Theorem C ([16]). *Let $f(z)$ be a transcendental entire function such that $\sigma(f) < 2$, and $\alpha(z) \not\equiv 0$ be an entire function such that $\sigma(\alpha) < \sigma(f)$ and $\lambda(f - \alpha) < \sigma(f)$. If $\Delta^n f - \alpha(z)$ and $f(z) - \alpha(z)$ share 0 CM, then $\alpha(z)$ is a polynomial with degree at most $n - 1$ and f must be form of*

$$f(z) = \alpha(z) + H(z)e^{dz},$$

where $H(z)$ is a polynomial such that $cH(z) = -\alpha(z)$, and c, d are two nonzero constants such that $e^d = 1$.

In Theorem C, we are still not sure whether the condition $\sigma(f) < 2$ is necessary or not, and also fail to deal with the general case of $\Delta^n f$ and f sharing some functions other than α. The main purpose of this article is utilizing complex difference equation to prove the high order difference counterpart of Brück's conjecture. We prove the following main theorem, which extends Theorem A from the case of $\sigma(b) < 1$ to the general case of small function such that $\sigma(b) < \sigma(f)$.

Theorem 1.2. *Let $f(z)$ be a transcendental entire function of finite order, which has a Borel exceptional small function $a(z)$ whose order is less than 1. If $\Delta^n f$ and $f(z)$ share one function $d(z)$ ($d(z) \not\equiv a(z)$) such that $\sigma(d) < \sigma(f)$ CM, then*

$$\frac{\Delta^n f - d}{f - d} = \frac{d - \Delta^n a}{d - a}.$$

Furthermore f is form of

$$f(z) = a(z) + ce^{\beta z},$$

where c and β are two nonzero constants such that $\frac{d(z)-\Delta^n a}{d(z)-a(z)} = (e^\beta - 1)^n$.

2. Some lemmas

To prove our results, we need some lemmas as follows.

Lemma 2.1 (see [14]). *Let $f(z)$ be a nonconstant meromorphic function in the complex plane and*

$$R(f) = \frac{p(f)}{q(f)},$$

where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If the coefficients $a_k(k = 0, 1, \ldots, p)$, $b_j(j = 0, 1, \ldots, q)$ are small functions of $f(z)$ and $a_p(z) \not\equiv 0, b_q(z) \not\equiv 0$, then

$$T\big(r, R(f)\big) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 2.2 (see [14]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(iii) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$ $(r \to \infty, r \notin E)$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.3 (see [5]). *Let $f(z)$ be a transcendental meromorphic function with finite order σ. Then for each $\varepsilon > 0$, we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.4 (see [5]). *Let $f(z)$ be a transcendental meromorphic function with finite order σ and η be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r),$$
$$i.e.,\ T(r, f(z+\eta)) = T(r, f) + S(r, f).$$

Lemma 2.5 (see [11]). *Let $w(z)$ be a transcendental meromorphic function with $\sigma(f) < \infty$. Let $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \ldots, m$. Also let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in \Gamma$, one has*

$$\left|\frac{w^{(k)}}{w^{(j)}}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.6 (see [6]). *Let $f(z)$ be a nonconstant meromorphic function of order $\sigma(f) < \infty$, and let λ' and λ'' be, respectively, the exponent of convergence of the zeros and poles of $f(z)$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of $|z| = r$ of finite logarithmic measure, so that*

$$2\pi i n_{z,\eta} + \log \frac{f(z+\eta)}{f(z)} = \eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon}), \tag{1}$$

or equivalently,

$$\frac{f(z+\eta)}{f(z)} = e^{\eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})}, \tag{2}$$

holds for $r \notin E \cup [0, 1]$, where $n_{z,\eta}$ in (1) is an integer depending on both z and η, $\beta = \max\{\sigma - 2, 2\lambda - 2\}$ if $\lambda < 1$ and $\beta = \max\{\sigma - 2, \lambda - 1\}$ if $\lambda \geq 1$, and $\lambda = \max\{\lambda', \lambda''\}$.

Lemma 2.7 (see [2]). *Let g be a transcendental function of order less than 1 and h be a positive constant. Then there exists an ε set E_ε such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1 \ \text{as } z \to \infty \ \text{in } C \backslash E_\varepsilon$$

uniformly in η for $|\eta| \leq h$. Further, the set E_ε may be chosen so that for large $|z| \notin E_\varepsilon$, the function g has no zeroes or poles in $|\zeta - z| \leq h$.

Remark 2.8. According to Hayman [7], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E_ε is an ε set, then the set of $r \geq 1$ for which the circle $S(0, r)$ meets E_ε has finite logarithmic measure and for almost all real θ the intersection of E_ε with the ray $\arg z = \theta$ is bounded.

3. The proof of main theorem

Proof of Theorem 1.2. On the one hand, from our assumption that $a(z)$ is a small Borel exceptional function of f, there exist an entire function $H(z)$, which is from the canonical product of the zeros of $f(z) - a(z)$, and a nonconstant polynomial $h(z)$ such that

$$f(z) - a(z) = H(z)e^{h(z)}, \tag{3}$$

where

$$\sigma(H) = \lambda(H) = \lambda(f - a) < k := \sigma(f) = \deg h(z).$$

And then we see that f is of regular growth.

On the other hand, since $\Delta^n f$ and $f(z)$ share the small function $d(z)$ CM, then there exists a polynomial $\alpha(z)$ with degree l not greater than k such that

$$\frac{\Delta^n f - d(z)}{f - d(z)} = e^{\alpha(z)}. \tag{4}$$

Set

$$H_1 = H(z+1)e^{\Delta h} - H, \ldots, H_n = H_{n-1}(z+1)e^{\Delta h} - H_{n-1}$$

by recurrence relations. It follows Lemma 2.4 and the fact $\deg \Delta h = k-1$ that

$$\sigma(H_j) < k, \ i.e., \ T(r, H_j) = o(T(r, e^{z^k})) \quad \text{for} \quad j = 1, 2, \ldots, n.$$

From the combination of equations (3)-(4) and the definition of H_n above, we can obtain

$$H_n e^h = (d - \Delta^n a) + (a - d)e^{\alpha} + He^{\alpha+h}. \tag{5}$$

If $d \equiv 0$, then from equation (4) and Lemma 2.3, we obtain that

$$m(r, e^{\alpha}) = m\big(r, \frac{\Delta f}{f}\big) = m\big(r, \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)}\big) = O(r^{k-1+\varepsilon})$$

holds for any $\varepsilon > 0$. That is to say $l \leq k-1$ because ε can be set small enough. But from equation (5), we see

$$(H_n - He^{\alpha})e^h = ae^{\alpha} - \Delta^n a,$$

which leads to

$$H_n - He^{\alpha} = 0$$

and

$$ae^{\alpha} - \Delta^n a = 0.$$

Recall $\sigma(a) < 1$, if $a \not\equiv 0$, then by Lemma 2.7, we see

$$e^{\alpha} = \frac{\Delta^n a}{a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{a(z+j)}{a(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = 0 \tag{6}$$

as $r \to \infty, \ r \notin E_{\varepsilon}$, which is impossible. That is to say $a \equiv 0$, which contradicts our assumption $a \not\equiv d$. So we can assume $d \not\equiv 0$.
Set

$$h(z) = a_k z^k + \cdots + a_0 \quad \text{and} \quad \alpha(z) = b_l z^l + \cdots + b_0$$

respectively, where $a_k(\neq 0), \ldots, a_0$ and $b_l(\neq 0), \ldots, b_0$ are some constants. In the next section, in order to our discussion, we shall consider the following two cases: 1. $l = k$ and 2. $l < k$ separately.

Case 1. $l = k$. In this case, we shall divide our proof into three subcases: 1.1 $a_k = b_k$, 1.2 $a_k = -b_k$ and 1.3 $a_k \neq \pm b_k$ respectively.

Subcase 1.1. $a_k = b_k$. We rewrite equation (5) as the following form

$$He^{A_1} e^{2a_k z^k} = H_n e^{A_2} e^{a_k z^k} - (d - \Delta^n a) - (a - d)e^{A_3} e^{a_k z^k}, \tag{7}$$

where A_1, A_2, A_3 are some polynomials with degree at most $k-1$. Then

$$\begin{cases} T(r, He^{A_1}) = o(T(r, e^{a_k z^k})), \\ T(r, H_n e^{A_2}) = o(T(r, e^{a_k z^k})), \\ T(r, e^{A_3}) = o(T(r, e^{a_k z^k})), \\ T(r, d - \Delta^n a) = o(T(r, e^{a_k z^k})). \end{cases}$$

Applying Lemma 2.1 to (7), we see

$$2T(r, e^{a_k z^k}) = T(r, e^{a_k z^k}) + S(r, e^{a_k z^k}),$$

which is impossible.

Subcase 1.2. $a_k = -b_k$. We rewrite equation (5) as the following form

$$H_n e^{B_1} e^{2a_k z^k} = (d - \Delta^n a + He^{B_2})e^{a_k z^k} + (a - d)e^{B_3}, \tag{8}$$

where B_1, B_2, B_3 are some polynomials with degree at most $k-1$. Then

$$\begin{cases} T(r, H_n e^{B_1}) = o(T(r, e^{a_k z^k})), \\ T(r, d - \Delta^n a + He^{B_2}) = o(T(r, e^{a_k z^k})), \\ T(r, (a-d)e^{B_3}) = o(T(r, e^{a_k z^k})). \end{cases}$$

If $H_n \not\equiv 0$, then applying Lemma 2.1 to equation (8), we also see

$$2T(r, e^{a_k z^k}) \leq T(r, e^{a_k z^k}) + S(r, e^{a_k z^k}),$$

which is impossible.

If $H_n \equiv 0$, then

$$(d - \Delta^n a + He^{B_2})e^{a_k z^k} + (a - d)e^{B_3} = 0.$$

It means

$$a - d = 0$$

and

$$d - \Delta^n a + He^{B_2} = 0,$$

which is a contradiction to our assumption $a \not\equiv d$.

Subcase 1.3. $a_k \neq \pm b_k$. Applying Lemma 2.2 to (5), we see

$$H_n = d - \Delta^n a = a - d = H = 0,$$

which is impossible.

Case 2. $l < k$. We rewrite equation (5) as the following form

$$(H_n - He^{\alpha})e^h = (d - \Delta^n a) + (a - d)e^{\alpha}, \tag{9}$$

which leads to

$$H_n - He^{\alpha} = 0$$

and

$$d - \Delta^n a + (a - d)e^{\alpha} = 0$$

in a similar way. So

$$K := e^{\alpha} = \frac{H_n}{H} = \frac{d - \Delta^n a}{d - a}.$$

Substitution the equation above into equation (4) yields

$$(d - a)(\Delta^n f - \Delta^n a) = (f - a)(d - \Delta^n a). \tag{10}$$

Set

$$g(z) := f(z) - a(z) = He^h,$$

and then equation (10) becomes

$$\frac{\Delta^n g}{g} = K. \tag{11}$$

We also shall consider two subcases: 2.1 $k \geq 2$ and 2.2 $k = 1$ in this case respectively as follows.

Subcase 2.1. $k \geq 2$. Employing the definition of g, it turns out that

$$\sigma(g) = k \geq 2 \text{ and } \lambda(g) = \sigma(H) < k.$$

By applying Lemma 2.6 to g, for any given $\varepsilon > 0$, there exists a set E with finite logarithmic measure such that

$$\frac{g(z+j)}{g(z)} = e^{j\frac{g'(z)}{g(z)} + O(r^{\beta+\varepsilon})}, \tag{12}$$

as $r \to \infty$, and $r \notin E \cup [0, 1]$, where

$$\beta = \begin{cases} k-2, & \text{if } \sigma(H) < 1; \\ \max\{k-2, \sigma(H)-1\}, & \text{if } \sigma(H) \geq 1. \end{cases}$$

Combining the fact $\sigma(H) < k$ and the equation above, we get $\beta < k - 1$. But the definition of g gives

$$\frac{g'(z)}{g(z)} = h'(z) + \frac{H'(z)}{H(z)}.$$

From Lemma 2.5, we see

$$|\frac{H'}{H}| \leq |z|^{\sigma(H)-1+\varepsilon},\ r \notin E \cup [0, 1]$$

holds for any given $\varepsilon > 0$. That is to say:

$$|\frac{H'}{H}| = o(r^{k-1}),\ r \notin E \cup [0, 1] \tag{13}$$

by taking ε small enough. Thus the combination of (12) and (13) gives:

$$\frac{g(z+j)}{g(z)} = e^{ja_k z^{k-1}(1+o(1))} \quad \text{as } r \to \infty, \quad r \notin E \cup [0, 1]. \tag{14}$$

It follows equations (11) and (14) that

$$e^{\alpha} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{g(z+j)}{g(z)} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} e^{ja_k z^{k-1}(1+o(1))} \tag{15}$$

as $r \to \infty$, $r \notin E \cup [0, 1]$, and then in this situation, we see $e^{na_k z^{k-1}(1+o(1))}$ is the only maximal magnitude of module term in the right of equation (15) by taking such $z = re^{i\theta}$ that $\delta(\theta) = \cos((k-1)\theta + \arg a_k) > 0$, which means $l = k - 1$. Recall

$$e^{\alpha} = \frac{d - \Delta^n a}{d - a}. \tag{16}$$

So $\sigma(a) < 1 \leq k-1 \leq \sigma(d)$.

If $a - \Delta^n a \equiv 0$, then $a \equiv 0$ by the same discussion in equation (6) at the beginning of proof, which leads to $e^\alpha = 1$ by equation (16), a contradiction to $l = k-1$. So we can assume $a - \Delta^n a \not\equiv 0$. From equation (16), we see

$$N(r, \frac{1}{d-a}) \leq N(r, \frac{1}{a-\Delta^n a}),\ \textit{i.e.},\ \lambda(d-a) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{d-a})}{\log r} < 1.$$

So we can set $d - a = ue^v$, where $\lambda(u) = \sigma(u) = \lambda(d-a) < 1 \leq \deg v = k-1$, and substitute it into equation (16), we see

$$e^\alpha - 1 = (\frac{a - \Delta^n a}{u})e^{-v}.$$

Thus e^α has three Borel values $0, 1, \infty$, which is impossible.

Subcase 2.2. $k = 1$, i.e., $\sigma(d) < 1$. In this subcase, we can get our conclusion from Theorem A immediately, and here, we give a simple proof in some sense. Now equation (3) can be set as the following form

$$f(z) - a = H(z)e^{\beta z}, \tag{17}$$

where β is a nonzero constant and H is an entire function with order less than 1. Now we shall show H is a nonzero constant. Recall

$$H_j(z) = e^{\Delta h}H_{j-1}(z+1) - H_{j-1}(z) = e^\beta H_{j-1}(z+1) - H_{j-1}(z)$$

for $j = 1, 2, \ldots, n$, and then

$$\Delta^j g = H_j(z)e^{\beta z}, \quad j = 1, 2, \ldots, n. \tag{18}$$

From equation (11), we obtain

$$K = \frac{\Delta^n g}{g} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{g(z+j)}{g(z)} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{H(z+j)}{H(z)} e^{j\beta}. \tag{19}$$

By applying Lemma 2.7 to equation (19), we see

$$K = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{H(z+j)}{H(z)} e^{j\beta} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} e^{j\beta} \tag{20}$$

as $z \to \infty$ in $C \backslash E_\varepsilon$, where E_ε is an ε set. Then from equation (20), we can obtain $K = e^\alpha$ is a constant, *i.e.*, α is a constant and

$$K = \sum_{j=0}^{n} C_n^j (-1)^{n-j} e^{j\beta} = (e^\beta - 1)^n. \tag{21}$$

From equations (11) and (18), we see $H_n = KH$. By substituting equation (21) and $H_n = KH$ into (19), we obtain

$$\sum_{j=0}^{n} e^{j\beta} C_n^j (-1)^{n-j} (H(z+j) - H(z)) = 0. \tag{22}$$

Set

$$B(z) = \Delta H(z),$$

then from Lemma 2.4, it is easy for us to see $\sigma(B) \leq \sigma(H) < 1$. From the definition of $B(z)$, we can obtain

$$\begin{aligned}
&H(z+1) - H(z) = B,\\
&H(z+2) - H(z) = \Delta B + 2B,\\
&H(z+3) - H(z) = \Delta^2 B + 3\Delta B + 3B,\\
&\ldots\\
&H(z+j) - H(z) = \Delta^{j-1} B + \cdots + jB,\\
&\ldots.
\end{aligned}$$

Here we just need to show that the last term in $H(z+j) - H(z)$ is jB, and we prove it by mathematical induction. Firstly, suppose

$$H(z+j) - H(z) = \Delta^{j-1} B + \cdots + jB \tag{23}$$

has holden for $s = j$, then take difference operator of both sides of equation (23) and we see

$$\begin{aligned}
&\Delta^j B + \cdots + j\Delta B\\
&= \Delta(H(z+j) - H(z))\\
&= (H(z+j+1) - H(z+1)) - (H(z+j) - H(z))\\
&= (H(z+j+1) - H(z)) - (H(z+1) - H(z)) - (H(z+j) - H(z))\\
&= (H(z+j+1) - H(z)) - B - (\Delta^{j-1} B + \cdots + jB).
\end{aligned}$$

Thus

$$H(z+j+1) - H(z) = \Delta^j B + \cdots + (j+1)B$$

holds which means equation (23) still holds for $s = j+1$. Therefore, we can obtain the last term in $H(z+j) - H(z)$ is jB by mathematical induction. By substituting equation (23) into equation (22), we see

$$\sum_{j=1}^{n} e^{\beta j} C_n^j (-1)^{n-j} (\Delta^{j-1} B + \cdots + jB) = 0. \tag{24}$$

From equation (24), we can get

$$\sum_{t=1}^{s} a_t \Delta^t B + \big(\sum_{j=1}^{n} e^{\beta j} C_n^j (-1)^{n-j} j\big) B = 0, \tag{25}$$

where $a_t (t = 1, 2, \ldots, s)$ are some constants. If $B(z) \not\equiv 0$, then from equation (25), we can see

$$\sum_{t=1}^{s} a_t \frac{\Delta^t B}{B} + \sum_{j=1}^{n} e^{\beta j} C_n^j (-1)^{n-j} j = 0. \tag{26}$$

Since $\sigma(B) < 1$, then by applying Lemma 2.7 to $\frac{\Delta^t B}{B}$ described in equation (26), we can obtain

$$\frac{\Delta^t B}{B} = \sum_{j=0}^{t} C_t^j (-1)^{t-j} \frac{B(z+j)}{B} \to \sum_{j=0}^{t} C_t^j (-1)^{t-j} = (1-1)^t = 0 \tag{27}$$

as $z \to \infty$ in $C \backslash E_\varepsilon$, where E_ε is an ε set. Thus from equations (26)-(27), we see

$$\sum_{j=1}^{n} e^{\beta j} C_n^j (-1)^{n-j} j = -\sum_{t=1}^{s} a_t \frac{\Delta^t B}{B} \to 0 \tag{28}$$

as $z \to \infty$ in $C \backslash E_\varepsilon$. Thus from equation (28), we see

$$\sum_{j=1}^{n} e^{\beta j} C_n^j (-1)^{n-j} j = 0.$$

That is

$$\sum_{j=1}^{n} e^{\beta j} n C_{n-1}^{j-1} (-1)^{n-j} = 0,$$

or equivalently

$$\sum_{j=1}^{n} e^{\beta j} C_{n-1}^{j-1} (-1)^{n-j} = 0,$$

which implies

$$e^\beta \sum_{s=0}^{n-1} e^{s\beta} C_{n-1}^s (-1)^{n-s-1} = (e^\beta - 1)^{n-1} e^\beta = 0.$$

Therefore, we get $e^\beta = 1$. From equation (21), we see $K = 0$, which is a contradiction.

Therefore, $B(z) \equiv 0$, and then $H(z+1) = H(z)$. If $H(z)$ is not a constant, then from our assumption that $H(z)$ is from the canonical product of the zeros of $f(z) - a$, which means it has a zero z_0 at least, we see $z_0 + 1, z_0 + 2, \ldots$ are some zeros of H. Thus

$$n(r, \frac{1}{H(z)}) \geq r(1 + o(1)),$$

which implies $\sigma(H) \geq 1$. This is a contradiction. So $H(z)$ is a nonzero constant and f is form of

$$f(z) = a(z) + ce^{\beta z},$$

where c and β are two nonzero constants such that $\frac{d - \Delta^n a}{d - a} = (e^\beta - 1)^n$. □

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Liangwen Liao
Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China
E-mail address: maliao@nju.edu.cn

Jie Zhang
College of Science
China University of Mining and Technology
Xuzhou 221116, P. R. China
E-mail address: zhangjie1981@cumt.edu.cn