Bull. Korean Math. Soc. **53** (2016), No. 1, pp. 39–48
http://dx.doi.org/10.4134/BKMS.2016.53.1.039

SEMI-ASYMPTOTIC NON-EXPANSIVE ACTIONS OF SEMI-TOPOLOGICAL SEMIGROUPS

Massoud Amini, Alireza Medghalchi, and Fouad Naderi

Abstract. In this paper we extend Takahashi's fixed point theorem on discrete semigroups to general semi-topological semigroups. Next we define the semi-asymptotic non-expansive action of semi-topological semigroups to give a partial affirmative answer to an open problem raised by A.T-M. Lau.

1. Introduction

A (not necessarily linear) self-mapping $T : E \to E$ on a Banach space E is called *non-expansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in E$. In 1963 R. DeMarr proved the following common fixed point theorem for commuting families of non-expansive self-mappings [6].

Theorem 1.1. *For any non-empty compact convex subset K of a Banach space E each commuting family of non-expansive self mappings on K has a common fixed point in K.*

This generalizes the celebrated Markov-Kakutani fixed point theorem (for the case of Banach spaces) on commuting families of continuous linear transformations on Hausdorff topological vector spaces leaving certain nonempty compact convex subset invariant. DeMarr's theorem has been generalized in several directions by Belluce and Kirk [2,3], Takahashi [16], Mitchell [14], Lau and Holmes [7,8].

DeMarr theorem suggests that the action of certain commutative semigroup has a fixed point. It is then natural to seek the same type of fixed point property for the actions of more general semigroups.

Let S be a semi-topological semigroup, that is a semigroup with a Hausdorff topology with separately continuous multiplication. We say that S is *right reversible* if it has finite intersection property for closed left ideals. An action of S on a topological space E is a mapping $(s, x) \mapsto s(x)$ from $S \times E$ into

Received October 31, 2014; Revised May 22, 2015.
2010 *Mathematics Subject Classification.* Primary 22A22; Secondary 46L05.
Key words and phrases. non-expansive mappings, normal structure, semi-topological semigroups, amenable, left reversible.

E such that $(st)(x) = s(t(x))$ for $s, t \in S$, $x \in E$. Every action of S on E induces a representation of S as a semigroup $\mathcal{S}$ of mappings on E, and the two semigroups are usually identified. The action is separately continuous if it is continuous in each variable when the other is fixed. In this case, each member of $\mathcal{S}$ is continuous on E. When E is a normed space the action of S on E is called *non-expansive* if $\|s(x) - s(y)\| \leq \|x - y\|$ for $s \in S$ and $x, y \in E$ and *right asymptotically non-expansive* if for each $x, y \in E$ there is a left ideal $J \subseteq S$ such that $\|s(x) - s(y)\| \leq \|x - y\|$ for $s \in J$ [9]. The left and two sided versions of the above notions are defined similarly.

Let $l^\infty(S)$ be the C^*-algebra of all bounded complex-valued functions on S with supremum norm and point-wise multiplication. For each $s \in S$ and $f \in l^\infty(S)$, denote by $l_s(f)$ and $r_s(f)$ the left and right translates of f by s respectively, that is $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for $t \in S$. Let X be a closed subspace of $l^\infty(S)$ containing the constant functions and being invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$, and a left invariant mean (LIM) if moreover $m(l_s(f)) = m(f)$ for $s \in S$, $f \in X$. Let $C_b(S)$ be the space of all bounded continuous complex-valued functions on S with supremum norm and $LUC(S)$ be the space of left uniformly continuous functions on S, i.e., all functions $f \in C_b(S)$ for which the mapping $s \to l_s f : S \to C_b(S)$ is continuous when $C_b(S)$ has the sup-norm topology. Then $LUC(S)$ is a C^*-subalgebra of $C_b(S)$ invariant under translations and containing constant functions. Then S is called *left amenable* if $LUC(S)$ has a LIM. The space of all right uniformly continuous functions, $RUC(S)$, and right amenability are defined similarly. The semi-topological semigroup S is called amenable if there is a mean on $C_b(S)$ which is both left and right invariant. For a discrete semigroup S this amounts to saying that S is amenable if it is both left and right amenable [5]. Left amenable semi-topological semigroups include commutative semigroups, as well as compact and solvable groups. The free (semi)group on two or more generators is not left amenable. When S is discrete, $LUC(S) = l^\infty(S)$ and (left) amenability of S yields the (left) reversibility of S. For more details on amenability, examples and relations see [4], [5], [11], [12], [13], [15].

Takahashi [16] studied DeMarr's theorem and realized that one can use a commutative (discrete) semigroup of self-mappings instead of a commuting family of self-mappings. On the other hand every commutative (discrete) semigroup is amenable, so he could generalize DeMarr theorem for amenable discrete semigroups:

Theorem 1.2. *Let K be a non-empty, compact, convex subset of a Banach space E and S be an amenable discrete semigroup which acts on K separately continuous and non-expansive, then S has a common fixed point in K.*

Mitchell [14] noticed that every left amenable discrete semigroup is left reversible, so he generalized Takahashi's theorem:

Theorem 1.3. *Let K be a non-empty, compact, convex subset of a Banach space E and S be a left reversible discrete semigroup which acts on K separately continuous and non-expansive, then S has a common fixed point in K.*

In order to generalize DeMarr's, Takahashi's, Mitchell's fixed point theorems in their utmost generality, Lau and Holmes defined the property (B) for the action of a semi-topologicals semigroup S on E as follows, which is automatic when S is commutative and the action is separately continuous.

(B) For each $x \in E$ whenever a net $\{s_\alpha(x) : \alpha \in I\}$, $s_\alpha \in S$, converges to x then for each $a \in S$ the net $\{s_\alpha a(x) : \alpha \in I, s_\alpha \in S\}$ converges to $a(x)$.

In [7], Lau and Holmes generalized DeMarr's fixed point theorem as follows:

Theorem 1.4. *Any right reversible semi-topological semigroup acting separately continuous and asymptotically non-expansive on a non-empty compact convex subset K of E with property* (B) *has a common fixed point in K.*

In his excellent review article [11], Lau asked if right reversibility of S and property (B) in the above theorem might be replaced by amenability of S.

This motivated the authors to seek other situations where such a fixed point property holds. In Section 2 we extend Takahashi's fixed point theorem on discrete semigroups [16] to general semi-topological semigroups. Our theorem in this section is new and is not a result of any previous works. In Section 3 we introduce the class of semi-asymptotic non-expansive mappings (which include non-expansive ones) and finally in Section 4 we prove a fixed point theorem for the action of reversible discrete semigroups by semi-asymptotic non-expansive mappings to give a partial affirmative answer to the Lau's problem.

2. Takahashi's theorem for general semi-topological semigroup

Takahashi proved his theorem for discrete semigroups [16], here we generalize it for general semi-topological semigroups. First we prove an assistant lemma then we state our theorem.

Lemma 2.1. *Let M be a nonempty compact subset of a Banach space E, and S be a semi-topological semigroup acting on M such that the action is separately continuous and non-expansive. Then for each $x \in M$ and each $f \in C(M)$ we have $f_x \in RUC(S)$ where $f_x(s) = f(sx)$ $(s \in S)$.*

Proof. We have to show that for each $x \in M$, $f_x \in C_b(S)$ and the mapping $s \to r_s f_x$ from S into $C_b(S)$ is continuous. Let $s_\alpha \to s$ in S then $s_\alpha(x) \to s(x)$ and by continuity of f, $f_x(s_\alpha) = f(s_\alpha(x)) \to f(s(x)) = f_x(s)$ which means that f_x is continuous. Note that the range of f_x is bounded, since $f_x(S) = f(Sx) \subseteq f(M)$ and M is compact, hence $f_x \in C_b(S)$. Fix $x \in M$, we want to show that the mapping $s \to r_s f_x$ from S into $C_b(S)$ is continuous. Let again $s_\alpha \to s$ in S, by separate continuity of the multiplication, $ts_\alpha \to ts$ in S. Also $s_\alpha(x) \to s(x)$, $ts_\alpha(x) \to ts(x)$ and $\|ts_\alpha(x) - ts(x)\| \leq \|s_\alpha(x) - s(x)\|$ in M, because the action is separately continuous and non-expansive. Uniform continuity of f implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that if $\|u-v\| < \delta$ then $\|f(u) - f(v)\| < \varepsilon$. On the other hand for $\delta > 0$ there is an α_0 such that for any $\alpha \geq \alpha_0, t \in S$, we have $\| ts_\alpha(x) - ts(x)\| \leq \|s_\alpha(x) - s(x)\| \leq \delta$. Putting these inequalities together we see that for $\varepsilon > 0$ there is an α_0 such that for all $\alpha \geq \alpha_0, t \in S$ the inequality $\mid f(ts_\alpha x) - f(tsx) \mid \leq \varepsilon$ holds, hence

$$\|r_{s_\alpha}(f_x) - r_s(f_x)\| = \sup\{\mid f(ts_\alpha x) - f(tsx) \mid : t \in S\} \leq \varepsilon.$$

Therefore $f_x \in RUC(S)$. □

Now we use the above lemma to extend Takahashi's theorem on discrete semigroups [16] to general semi-topological semigroups.

Theorem 2.2. *Let K be a non-empty compact convex subset of a Banach space E, and S be an amenable semi-topological semigroup acting on K such that the action is separately continuous and non-expansive. Then S has a common fixed point in K.*

Proof. An application of Zorn's lemma shows that there exists a minimal non-empty compact convex and S-invariant subset $X \subseteq K$. A second application of Zorn's lemma shows that there is a minimal non-empty compact and S-invariant subset $M \subseteq X$. We claim that M is S-preserved, i.e., $M = sM$ for all $s \in S$. Let ν be an invariant mean on $RUC(S)$ and define $\mu(f) = \nu(f_x)$. Then by Riesz representation theorem, μ induces a regular probability measure on M such that $\mu(sB) = \mu(B)$ for all Borel S-invariant sets $B \subseteq M$ as follows: Since μ is invariant we have $\mu(s^{-1}B) = \mu(B)$ for all Borel sets $B \subseteq M$, where $s^{-1}B$ denotes the pre-image of B under s. Also $B \subseteq s^{-1}sB$, hence for every Borel S-invariant set B, $\mu(B) \leq \mu(s^{-1}sB) = \mu(sB) \leq \mu(B)$, and therefore $\mu(sB) = \mu(B)$. Take F to be the support of μ, then $F \subseteq sM$ for each $s \in S$, since s defines a measurable function from M into M and $\mu(sM) = \mu(M) = 1$. Let χ_F be the characteristic function of F. For each $s \in S$,

$$1 = \mu(F) = \int_M \chi_F(y) d\mu = \int_M \chi_F(sy) d\mu = \mu(s^{-1}F),$$

and so by the definition of support, $F \subseteq s^{-1}F$, for each s in S, hence F is S-invariant. Now the minimality of M yields $F = M$, and $\mu(sM) = \mu(M) = 1$ implies that $M \subseteq sM$, but M was S-invariant so $sM = M$ for each s in S.

Now if M is singleton we are done, otherwise if $\delta(M) = diam(M) > 0$, we get a contradiction by DeMarr's Lemma [6], which implies that

$$\exists u \in \overline{co}(M), r_0 = \sup\{\|m - u\| : m \in M\} < \delta(M).$$

Define $X_0 = \bigcap_{m \in M} B[m, r_0]$, and note that X_0 is non-empty (as $u \in X_0$) compact and convex such that $sX_0 \subseteq X_0$, for each s in S. But this contradicts the minimality of X. Therefor M contains only one point which is a common fixed point for the action of S. □

This implies a result of Takahashi [16]:

Corollary 2.3. *Let K be a non-empty, compact, convex subset of a Banach space E, and S be an amenable discrete semigroup acting on K such that the action is separately continuous and non-expansive. Then S has a common fixed point in K.*

In Section 4 we will see that Takahashi's theorem can be deduced from another theorem (see Corollary 4.3).

3. Semi-asymptotic non-expansive action of semi-topological semigroups

Definition 3.1. The action of a semigroup S on a normed space E is left semi-asymptotic non-expansive if for each $x \in E$ there is a left ideal $J \subseteq S$ such that $\|s(x) - s(y)\| \leq \|x - y\|$ for $s \in J$, $y \in E$. Right and two sided semi-asymptotic non-expansive actions are defined similarly.

Remark 3.2. (i) The following implications hold for the action of a (right reversible) semigroup S on E:

non-expansive $\rightarrow$ left semi-asymptotic non-expansive,
left semi-asymptotic non-expansive $\rightarrow$ right asymptotic non-expansive.

Care is needed in the expressions "left semi-asymptotic non-expansive" and "left asymptotic non-expansive" because we have used different ideals in their definitions.

(ii) Recall that S is right reversible if S has finite intersection property for closed left ideals. This is clearly equivalent to the assumption that any two principal closed left ideals meet: Because any left ideal contains a left principal ideal and given principal left ideals Sa, Sb, Sc, choose $d \in \overline{Sa} \bigcap \overline{Sb}$, then $\overline{Sd} \subseteq \overline{Sa} \bigcap \overline{Sb}$ and since $\overline{Sd} \bigcap \overline{Sc} \neq \emptyset$, we have $\overline{Sa} \bigcap \overline{Sb} \bigcap \overline{Sc} \neq \emptyset$. The same holds for any finite number of principal closed left ideals by an easy induction.

The converses of implications in (i) do not hold even for discrete semigroups.

Example 3.3. (i) *property* (B) $+$*asymptotic non-expansive* $\not\Rightarrow$ *semi-asymptotic non-expansive*: Let $K = \{(r, \theta) : 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi\}$ be the closed unit disc in $\mathbb{R}^2$ in polar coordinates. Define continuous self mappings f, g on K by $f(r, \theta) = (\frac{r}{2}, \theta)$ and $g(r, \theta) = (r, 2\theta \pmod{2\pi})$. The discrete semigroup S of continuous mappings from K to K generated by f and g under composition is commutative, and so has property (B). Lau has shown that this action is asymptotically non-expansive, but it is not non-expansive (g ruins everything!) [7]. For the sake of completeness we give the details of asymptotic non-expansivity and then we show that the action is not even semi-asymptotic non-expansive. Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers $\{0, 1, 2, \ldots\}$ and T be the semigroup $\mathbb{N}_0 \times \mathbb{N}_0 \setminus \{(0, 0)\}$ under pointwise addition. Then T acts on K by $(m, n)(r, \theta) = f^m g^n(r, \theta)$, and we can identify T with S via $(m, n) \mapsto f^m g^n$. The ideals of S are of the form $I_{(m_0,n_0)} = \{f^m g^n : m \geq m_0,$

$n \geq n_0,\ (m,n) \neq (0,0)\}$ for $(m_0, n_0) \in T$. Let us observe that the action of S is asymptotic non-expansive. Let $(r,\theta), (r',\theta')$ be distinct points in K, for $n \in \mathbb{N}_0$ put $u_n = r^2 + r'^2 - 2rr' \cos 2^n(\theta - \theta'); u = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$ then by Archimedean property of reals, there is $m_0 \in \mathbb{N}_0$ such that $\frac{1}{2^{2m}} u_n \leq u$ for all $m \geq m_0$, namely

$$\|f^m g^n(r,\theta) - f^m g^n(r',\theta')\|^2 \leq \|(r,\theta) - (r',\theta')\|^2,$$

and using the left ideal $J = I_{(m_0,0)}$, the action is asymptotically non-expansive. To show that the action is not semi-asymptotic non-expansive, let $(1,0) \in K$, if I is the ideal such that $\|s(r,\theta) - s(1,0)\| \leq \|(r,\theta) - (1,0)\|$ for $s \in I, (r,\theta) \in K$, then I has the form $I = I_{(m_0,n_0)}$ for some (m_0,n_0), let $s = f^m g^n \in I$ act on $(1, \frac{\pi}{2^n})$ then $\|(\frac{1}{2^m}, \pi) - (\frac{1}{2^m}, 0)\| \leq \|(1, \frac{\pi}{2^n}) - (1,0)\|$ which in turn gives $\frac{1}{2^{2m-2}} \leq 2(1 - \cos(\frac{\pi}{2^n}))$ for $m \geq m_0,\ n \geq n_0,\ (m,n) \neq (0,0)$, but this leads to a contradiction by fixing m and letting n vary indefinitely.

(ii) *semi-asymptotic non-expansive $\not\Rightarrow$ non-expansive*: We proceed by defining a map T on a set M, for which the powers $T^n,\ n \geq 2$, are non-expansive, but T itself isn't, and consider the corresponding action of $(\mathbb{N}, +)$ on M. Let M be the closed unit ball of $\mathbb{R}^2$ and $f : \mathbb{R} \to [-1,1]$ be continuous with $f(0) = 0$ which isn't non-expansive. Consider the self-mapping T on M, defined by $T(x_1, x_2) = (f(x_2), 0)$, then T isn't non-expansive but $T^n = 0$ for $n \geq 2$. One can generalize this example to $\mathbb{R}^p$ with $S = (p-k) + \mathbb{N} = \{p-k+1, p-k+2, \ldots\}$, $k - 1 \leq \lfloor \frac{p}{2} \rfloor = \max\{m \in \mathbb{Z} : m \leq \frac{p}{2}\}$, by taking f as before and $T(x_1, x_2, \ldots, x_p) = (f(x_2), \ldots, f(x_p), 0)$.

(iii) *semi-asymptotic non-expansive $\not\Rightarrow$ property* (B): Let K be as in (i), define the self mappings f and g on K by $f(r,\theta) = (r, \frac{\pi}{2} + \theta)$ and $g(r,\theta) = (r, \frac{\pi}{2} - \theta)$. Let S be the discrete semigroup generated by f and g under composition. In this case, S is non-commutative. The actions of f and g are non-expansive, as f is a rotation and g is the composition of a reflection and a rotation. To be more precise, let $(r,\theta), (r',\theta')$ be two points in K, then $\|f(r,\theta) - f(r',\theta')\|^2 = \|(r, \frac{\pi}{2} + \theta) - (r', \frac{\pi}{2} + \theta')\|^2 = r^2 + r'^2 - 2rr' \cos(\frac{\pi}{2} + \theta - (\frac{\pi}{2} + \theta')) = r^2 + r'^2 - 2rr' \cos(\theta - \theta') \leq \|(r,\theta) - (r',\theta')\|^2$ which shows that f is non-expansive, similar calculations show that g is also non-expansive. Therefore the action of S is non-expansive and semi-asymptotic non-expansive, but it fails to have property (B): Consider $x = (1, \frac{\pi}{4}) \in K, s_n = g, n \in \mathbb{N}; a = f \in S$, then $s_n(x) = (1, \frac{\pi}{4}) \to (1, \frac{\pi}{4}) = x$, but $s_n a(x) \to (1, -\frac{\pi}{4}) \neq a(x)$.

Definition 3.4. A semi-topological semigroup S is called totally left reversible if every family of closed right ideals of S has non-empty intersection. The right and two sided versions are defined similarly.

Example 3.5. The commutative semigroup $\mathbb{N}$ of natural numbers is not totally reversible under addition, but it is totally reversible under the min operation $m \wedge n = \min\{m, n\}$, as in the latter case 1 belongs to every ideal.

Proposition 3.6. *If a totally right reversible semi-topological semigroup act right asymptotically non-expansive on a normed space, the action is left semi-asymptotic non-expansive.*

Proof. Let x be an arbitrary element in E. By asymptotic assumption for each $y \in E$ there is a left ideal $J_{x,y} \subseteq S$ such that $\|s(x) - s(y)\| \leq \|x - y\|$ for all $s \in J_{x,y}$. Now the left ideal $J_x = \bigcap_{y \in E} J_{x,y}$ satisfies the conditions in the definition of left semi-asymptotic non-expansive action. □

4. Lau's problem for discrete semigroups with semi-asymptotic action

In this section we prove the analogue of a result of Lau and Holmes for semi-asymptotic non-expansive actions of discrete semigroups without the property (B), adapting the technique of the proof of [7, Theorem 3.1] and [14] to our setting. Recall that when the semigroup S is discrete no mention of topology is needed, in this case S is left(right) reversible if it has finite intersection property for right(left) ideals and S is reversible if it is both left and right reversible. Also S is left amenable if $l^\infty(S)$ has a left invariant mean, right and two sided amenability are defined similarly.

If H and K are non-empty subsets of a Banach space E and H is bounded, for $k \in K$, define $r(H,k) = \sup\{\|h-k\| : h \in H\}$. Put $r(H,K) = \inf\{r(H,k) : k \in K\}$ and let $C(H,K) = \{k \in K : r(H,k) = r(H,K)\}$. When K is convex, we say that K has *normal structure* if for each bounded closed convex subset W of K with more than one point, there exists $x \in W$ such that $r(W,x) < \delta(W) = diam(W)$, or equivalently, $C(W,W)$ is a proper subset of W. DeMarr [6] showed that every compact convex subset of a Banach space has normal structure, and Alspach [1] observed that this is not true for weakly compact convex subsets.

Theorem 4.1. *Let K be a non-empty compact convex subset of a Banach space E and S be a reversible discrete semigroup which acts on K separately continuous and left semi-asymptotic non-expansive, then S has a common fixed point in K.*

Proof. Using Zorn's lemma we get a minimal non-empty compact convex subset $A_0 \subseteq K$ satisfying:

($*$) There exists a collection $\mathcal{C}$ of closed subsets of K such that $A_0 = \cap\mathcal{C}$ and for each $x \in A_0, B \in \mathcal{C}$ there is a left ideal $J \subseteq S$ such that $J(x) \subseteq B$.

To see this, consider the family $\mathcal{M}$ of all non-empty compact convex subsets D of K such that for each $x \in D$ there is a left ideal $J \subseteq S$ such that $J(x) \subseteq D$. Order this family by reverse inclusion. Let $\{D_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{M}$, by compactness of K, $D = \bigcap\{D_\lambda : \lambda \in \Lambda\} \neq \varnothing$. Let $x \in D$, for each $\lambda \in \Lambda$ there is a left ideal J_λ such that $J_\lambda(x) \subseteq D_\lambda$. For any finite subset Γ of Λ, using the reversibility of S and Remark 3.2(ii) we see that $\varnothing \neq (\bigcap\{J_\lambda : \lambda \in \Gamma\})(x) \subseteq \bigcap\{J_\lambda(x) : \lambda \in \Gamma\} \subseteq K$, hence the family $\{J_\lambda(x) : \lambda \in \Lambda\}$ has finite intersection property, and by compactness of K we have $\varnothing \neq V =$

$\bigcap\{J_\lambda(x) : \lambda \in \Lambda\} \subseteq \bigcap\{D_\lambda : \lambda \in \Lambda\} = D$. Let $J = \{s \in S : s(x) \in V\}$, then J is a left ideal with $J(x) \subseteq D$: For $s \in S$ and $t \in J$, $t(x) \in V$ and $s(t(x)) = st(x) \in sV = \bigcap\{sJ_\lambda(x) : \lambda \in \Lambda\} \subseteq \bigcap\{J_\lambda(x) : \lambda \in \Lambda\} = V$, therefore $st \in J$. The rest follows from Zorn's lemma.

We show that A_0 contains a non-empty closed S-invariant subset, and hence A_0 is S-invariant. Let $x \in A_0$ be fixed and $\mathcal{C}'$ be the collection of all finite intersection of sets in $\mathcal{C}$. For each $\alpha \in \mathcal{C}'$, $\alpha = B_1 \cap \cdots \cap B_n$, where $B_i \in \mathcal{C}$, choose left ideals J_i such that $J_i(x) \subset B_i$ and let $a_\alpha \in \cap\{J_i : i = 1, \ldots, n\}$, this last intersection is non-empty by the reversibility of S and Remark 3.2(ii). Then $Sa_\alpha(x) \subset \alpha$, and if z is a cluster point of the net $\{a_\alpha(x) : \alpha \in \mathcal{C}'\}$ where $\mathcal{C}'$ is directed by inclusion, then $S(z)$ is a closed S-invariant subset of A_0 (notice that the action is separately continuous so $S(z)$ must be closed in K). Now a second application of Zorn's lemma shows that there is a subset $M \subseteq S(z)$ minimal with respect to being non-empty, closed and S-invariant, note that M is compact. We claim that M is S-preserved, i.e., $M = sM$ for all $s \in S$. Since S is left reversible, a straightforward induction argument shows if $\{s_1, \ldots, s_n\}$ is any finite subset of S, then there exists a finite subset $\{t_1, \ldots, t_n\}$ of S such that $s_1t_1 = \cdots = s_nt_n$. Hence

$$\bigcap_{i=1}^{n} s_iM \supseteq \bigcap_{i=1}^{n} s_i(t_iM) = s_1t_1M \neq \varnothing.$$

Thus the family $\{sM : s \in S\}$ has the finite intersection property, so $F = \cap\{sM : s \in S\}$ is a non-empty compact subset of M. We now show that F is S-preserved. Let $a \in S$ be arbitrary since $aS \subseteq S$, we have

$$aF = \bigcap_{s\in S} asM = \bigcap_{t\in aS} tM \supseteq \bigcap_{t\in S} tM = F.$$

Therefore $aF \supseteq F$. For the other inclusion, let $x \in F$, since $F \subseteq cM$ for some $c \in S$, there is $y \in M$ such that $x = cy$, $ax = acy$. Let $b \in S$ be arbitrary, again from left reversibility of S, there is $d \in S$ such that $ac = bd$. Hence $ax = a(cy) = b(dy) \in bM$ and $aF \subseteq F$. Now the minimality of M forces $F = M$, hence M is S-preserved. If M is a singleton we are done, otherwise if $\delta(M) = diam(M) > 0$, we extract a contradiction by DeMarr's Lemma [6]. From DeMarr's Lemma M has normal structure, i.e.,

$$\exists u \in \overline{co}(M), r_0 = \sup\{\|m - u\| : m \in M\} < \delta(M).$$

For each $\varepsilon > 0, B \in \mathcal{C}$, let

$$K_{\varepsilon,B} = B \bigcap (\bigcap\nolimits_{m\in M} B[m, r_0 + \varepsilon]).$$

Note that $K_{\varepsilon,B}$ is a non-empty ($u \in K_{\varepsilon,B}$), compact and convex set. We proceed to show that

$$K_0 = \cap\{K_{\varepsilon,B} : \varepsilon > 0, B \in \mathcal{C}\}$$

also has property $(*)$. Let $x \in K_0$, $\varepsilon > 0$, $B \in \mathcal{C}$ be fixed. Let $I \subset S$ be a left ideal such that $I(x) \subseteq B$. Take $L = Sc \subseteq S$ where c is an arbitrary element of I. For each $m \in M$, $s \in S$, the semi-asymptotic non-expansiveness of the action gives $\|sc(m) - sc(x)\| \leq \|m - x\| \leq r_0 + \varepsilon$, so $sc(x) \in B[sc(m), r_0 + \varepsilon]$. But $scM = M$ for any $s \in S$, hence for the left ideal $L = Sc$ we see that $L(x) \subseteq K_{\varepsilon,B}$ and $(*)$ holds, contradicting the minimality of A_0. □

Next we give a partial affirmative answer to Lau's question in the special case of semi-asymptotic non-expansive actions of amenable discrete semigroups.

Corollary 4.2. *Let K be a non-empty, compact convex subset of a Banach space E and S be an amenable discrete semigroup which acts on K separately continuous and left semi-asymptotic non-expansive, then S has a common fixed point in K.*

Proof. It is enough to show that S is both left and right reversible. Let λ be an invariant mean of S, I_1 and I_2 be two non-empty right ideals of S. Suppose the contrary that I_1 and I_2 are disjoint. Since S is discrete, it is a normal topological space, thus there exists $f \in l^\infty(S)$ such that $f \equiv 1$ on I_1 and $f \equiv 0$ on I_2. Now if $a_1 \in I_1$, then $l_{a_1}(f) = 1$. Therefore $\lambda(f) = \lambda(l_{a_1}(f)) = 1$, but if $a_2 \in I_2$, then $l_{a_2}(f) = 0$. Hence $\lambda(f) = \lambda(l_{a_2}(f)) = 0$, which is impossible. Similarly S is right reversible. □

We could infer Takahashi's theorem [16] from the above corollary:

Corollary 4.3. *Let K be a non-empty, compact, convex subset of a Banach space E and S be an amenable discrete semigroup which acts on K separately continuous and non-expansive, then S has a common fixed point in K.*

In Theorem 4.1 we have assumed that the semigroup S is reversible. The following example shows that the right reversibility is not enough.

Remark 4.4. Let S be the left zero discrete semigroup, i.e., a discrete topological space S with the operation $ab = a$ for all a and b in S. Obviously S is right reversible and S is the only left ideal, however S is not reversible. Now the action of dual operators $\{(l_s)^* : s \in S\}$ on compact subsets of Banach space $(l_\infty(S))^*$ is non-expansive hence semi-asymptotic non-expansive, but there is no common fixed point, otherwise [10] ensures that S has a left invariant mean, which is false [5], [7].

Acknowledgement. The authors would like to thank the anonymous reviewers for their instructive comments.

References

[1] D. E. Alspach, *A fixed point free non-expansive map*, Proc. Amer. Math. Soc. **82** (1981), no. 3, 423–424.

[2] L. P. Belluce and W. A. Kirk, *Fixed point theorems for families of contraction mappings*, Pacific J. Math. **18** (1966), 213–217.

[3] ______, *Nonexpansive mappings and fixed points in Banach spaces*, Illinois J. Math. **11** (1967), 474–479.
[4] J. F. Berglund, H. D. Junghen, and P. Milnes, *Analysis on Semigroups*, John Wiley & Sons Inc., New York, 1989.
[5] M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
[6] R. DeMarr, *Common fixed points for commuting contraction mappings*, Pacific J. Math. **13** (1963), 1139–1141.
[7] R. D. Holmes and A. T. M. Lau, *asymptotically non-expansive actions of topological semigroups and fixed points*, Bull. London. Math. Soc. **3** (1971), 343–347.
[8] ______, *Nonexpansive action of topological semigroups and fixed points*, J. London Math. Soc. **5** (1972), 330–336.
[9] R. D. Holmes and P. P. Narayanaswami, *On asymptotically nonexpansive semigroups of mappings*, Canad. Math. Bull. **13** (1970), 209–214.
[10] A. T. M. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain J. Math. **3** (1973), 69–76.
[11] ______, *Normal structure and common fixed point properties for semigroups of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2010** (2010), Art. ID 580956, 14 pp.
[12] A. T. M. Lau and Y. Zhang, *Fixed point properties of semigroups of non-expansive mappings*, J. Funct. Anal. **254** (2008), no. 10, 2534–2554.
[13] ______, *Fixed point properties for semigroups of nonlinear mappings and amenability*, J. Funct. Anal. **263** (2012), no. 10, 2949–2977.
[14] T. Mitchell, *Fixed points of reversible semigroups of nonexpansive mappings*, Kodai Math. Sem. Rep. **21** (1970), 322–323.
[15] A. L. Paterson, *Amenability*, American Mathematical Society, Providence, 1988.
[16] W. Takahashi, *Fixed point theorem for amenable semigroup of nonexpansive mappings*, Kodai Math. Sem. Rep. **21** (1969), 383–386.

Massoud Amini
Department of Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
Tehran 14115-134, Iran
E-mail address: mamini@modares.ac.ir

Alireza Medghalchi
Faculty of Mathematical Sciences and Computer
Kharazmi University
50 Taleghani Avenue, Tehran, Iran
E-mail address: a_medghalchi@khu.ac.ir

Fouad Naderi
Department of Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
Tehran 14115-134, Iran
E-mail address: f.naderi@modares.ac.ir