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ON SOME ROOT BEHAVIORS OF CERTAIN SUMS OF POLYNOMIALS

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Abstract. It is known that no two of the roots of the polynomial equation

$$\prod_{l=1}^{n}(x-r_l)+\prod_{l=1}^{n}(x+r_l)=0, \tag{1}$$

where $0<r_1\le r_2\le\cdots\le r_n$, can be equal and all of its roots lie on the imaginary axis. In this paper we show that for $0<h<r_k$, the roots of

$$(x-r_k+h)\prod_{\substack{l=1\\ l\neq k}}^{n}(x-r_l)+(x+r_k-h)\prod_{\substack{l=1\\ l\neq k}}^{n}(x+r_l)=0$$

and the roots of (1) in the upper half-plane lie alternatively on the imaginary axis.

1. Introduction

There is an extensive literature concerning roots of sums of polynomials. Many papers and books ([6], [7], [8]) have been written about these polynomials. Perhaps the most immediate question of sums of polynomials, $A+B=C$, is "given bounds for the roots of A and B, what bounds can be given for the roots of C?". By Fell [2], if all roots of A and B lie in $[-1,1]$ with A, B monic and $\deg A=\deg B=n$, then no root of C can have modulus exceeding $\cot(\pi/2n)$, the largest root of $(x+1)^n+(x-1)^n$. This suggests to study polynomials having a form something like $A(x)+B(x)$ where all roots of $A(x)$ are negative and all roots of $B(x)$ are positive.

All (conjugate) roots of the polynomial equation

$$\prod_{l=1}^{n}(x-r_l)+\prod_{l=1}^{n}(x+r_l)=0, \tag{2}$$

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where $0 < r_1 \leq r_2 \leq \cdots \leq r_n$, lie on the imaginary axis. In fact, if z is a root, then

$$\prod_{l=1}^{n}(z - r_l) = -\prod_{l=1}^{n}(z + r_l).$$

On taking absolute values, one gets that the product of the distances of z from the points $-r_i$ equals the product of the distances of z from the points r_i. Thus, if z is to the left or to the right of the y-axis, one of these distances is bigger. Kim [5] showed that all roots of (2) are simple and the gaps between the roots in the upper half-plane strictly increase as one proceeds upward.

Given a polynomial $p(x)$, all of whose roots are real, if we move some of the roots, the critical points will also change. Some fundamental results in this area are Polynomial Root Dragging Theorem [1], Polynomial Root Squeezing Theorem [3], and Polynomial Root Motion Theorem [4] that is a generalization of both Polynomial Root Dragging Theorem and Polynomial Root Squeezing Theorem. The Polynomial Root Squeezing Theorem explains the change qualitatively: moving two real roots x_i and x_j of $p(x)$ an equal distance toward each other, without passing other roots, will cause each critical point to move toward $(x_i + x_j)/2$, or remain fixed.

We now turn to our polynomial equation (2). Perhaps the first question in the vein of "root squeezing" is the root locations of the polynomial equation

$$(x - r_k + h)\prod_{\substack{l=1 \\ l \neq k}}^{n}(x - r_l) + (x + r_k - h)\prod_{\substack{l=1 \\ l \neq k}}^{n}(x + r_l) = 0 \tag{3}$$

for some $h > 0$. This polynomial is still in the form of (2) so that the roots lie on the imaginary axis and the gaps between the roots in the upper half-plane strictly increase as one proceeds upward. In this paper we prove that if $0 < h < r_k$, the roots of (2) and the roots of (3) in the upper half-plane lie alternatively on the imaginary axis. More precisely,

Theorem 1.1. *Let* $0 < h < r_k$. *If*

$$is_1, is_2, \ldots, is_{[n/2]} \quad (s_1 < s_2 < \cdots < s_{[n/2]})$$

are the roots of (2) *and*

$$is'_1, is'_2, \ldots, is'_{[n/2]} \quad (s'_1 < s'_2 < \cdots < s'_{[n/2]})$$

are the roots of (3) *in the upper half plane, then*

$$s'_1 < s_1 < s'_2 < s_2 < \cdots < s'_{[n/2]} < s_{[n/2]}. \tag{4}$$

In Polynomial Root Squeezing Theorem, the distance of "root squeezing" is rather small so that the squeezed roots do not pass other roots. But in

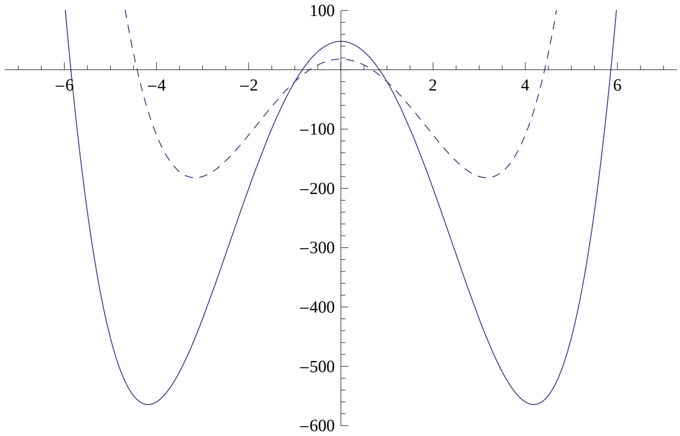
Theorem 1.1, the result holds for any h with $0 < h < r_k$, that is, h can be "large". For example, we let

$$p(x) = (x-1)(x-2)(x-3)(x-4) + (x+1)(x+2)(x+3)(x+4)$$

and with choosing $r_k = 4$ and $h = 2.5$, let

$$q(x) = (x-1)(x-2)(x-3)(x-1.5) + (x+1)(x+2)(x+3)(x+1.5).$$

The next figure demonstrates the graphs of $y = p(it)$ (solid curve) and $y = q(it)$ (dashed curve) for $-7 \le t \le 7$, which show that the roots of $p(x)$ and the roots of $q(x)$ in the upper half-plane lie alternatively on the imaginary axis.



Also by interchanging the roles of (2) and (3) we may obtain the similar result when h is negative.

Remark. When we consider $h < 0$ instead of $0 < h < r_k$ in Theorem 1.1, (4) is replaced by

$$s_1 < s'_1 < s_2 < s'_2 < \cdots < s_{[n/2]} < s'_{[n/2]}.$$

In Section 2, we provide the proof of Theorem 1.1.

2. Proof

To prove Theorem 1.1, we will need the following lemma. This can be seen in the proof[1] of Theorem 2.3 in [5].

Lemma 2.1. *With the same notations used in Theorem 1.1, suppose that s is an imaginary root of* (2) *in the upper half plane. Let, for each i, $\alpha_i = \angle s(-r_i)o$, where o denotes the origin. Then,*

$$\frac{2}{\pi}\sum_{l=1}^{n} \alpha_l = n-1, n-3, n-5, \ldots, c, \tag{5}$$

[1] In the proof of Theorem 2.3 in [5], it is stated that if n is odd, then $c = 0$, which is a typo. As in Lemma 2.1 in this paper, $c = 2$ if n is odd.

$c = 1$ if n is even, $c = 2$ if n is odd. In this case, c, $c+2$, ..., $n-1$ correspond to is_1, is_2, ..., $is_{[n/2]}$, respectively.

In what follows, we write the left sides of (2) and (3) as $p(x)$ and $q(x)$, respectively.

Proof of Theorem 1.1. Let

$$is_1, is_2, \ldots, is_{[n/2]} \quad (s_1 < s_2 < \cdots < s_{[n/2]})$$

be the roots of (2) on the upper half plane. Since

$$\begin{aligned} q(x) &= (x - r_k + h)\prod_{\substack{l=1\\l\neq k}}^{n}(x - r_l) + (x + r_k - h)\prod_{\substack{l=1\\l\neq k}}^{n}(x + r_l) \\ &= p(x) + h\left[\prod_{\substack{l=1\\l\neq k}}^{n}(x - r_l) - \prod_{\substack{l=1\\l\neq k}}^{n}(x + r_l)\right] \\ &= p(x) + h\left[\frac{\prod_{l=1}^{n}(x - r_l)}{x - r_k} - \frac{\prod_{l=1}^{n}(x + r_l)}{x + r_k}\right] \\ &= p(x) + h\left[\frac{\prod_{l=1}^{n}(x - r_l)}{x - r_k} - \frac{p(x) - \prod_{l=1}^{n}(x - r_l)}{(x + r_k)}\right], \end{aligned}$$

we have

$$\begin{aligned} q(is) &= h\left[\frac{\prod_{l=1}^{n}(is - r_l)}{is - r_k} + \frac{\prod_{l=1}^{n}(is - r_l)}{is + r_k}\right] \\ &= h\prod_{l=1}^{n}(is - r_l)\left(\frac{1}{is - r_k} + \frac{1}{is + r_k}\right) \\ &= \frac{2ihs}{-s^2 - r_k^2}\prod_{l=1}^{n}(is - r_l) \end{aligned} \tag{6}$$

for any root is of (2). We first consider the case n even, and write $n = 2u$ for some integer u. Then for an arbitrary positive number t,

$$\begin{aligned} q(it) &= (it - r_k + h)\prod_{\substack{l=1\\l\neq k}}^{n}(it - r_l) + (it + r_k - h)\prod_{\substack{l=1\\l\neq k}}^{n}(it + r_l) \\ &= (r_k - h - it)\prod_{\substack{l=1\\l\neq k}}^{n}(r_l - it) + (it + r_k - h)\prod_{\substack{l=1\\l\neq k}}^{n}(it + r_l). \end{aligned}$$

This implies that $q(it)$ is a real number because $z + \bar{z}$ is always real for any complex number z. Consequently, (6) is a real number whose sign depends on the argument of $\prod_{l=1}^{n}(is-r_l)$. But

$$\arg\prod_{l=1}^{n}(is-r_l)=\sum_{l=1}^{n}\arg(is-r_l)=\sum_{l=1}^{n}(\pi-\alpha_l)=2u\pi-\sum_{l=1}^{n}\alpha_l. \tag{7}$$

By Lemma 2.1, we see that

$$2u\pi-\sum_{l=1}^{n}\alpha_l=\begin{cases} 2u\pi-\frac{1\pi}{2} & \text{if } s=s_1,\\ 2u\pi-\frac{3\pi}{2} & \text{if } s=s_2,\\ 2u\pi-\frac{5\pi}{2} & \text{if } s=s_3,\\ \quad\vdots & \\ 2u\pi-\frac{(2u-1)\pi}{2} & \text{if } s=s_u,\end{cases}$$

and so

$$\prod_{l=1}^{n}(is-r_l)$$

lies on the imaginary axis in the lower half plane when $s=s_1,s_3,s_5,\ldots$, and lies on the imaginary axis in the upper half plane when $s=s_2,s_4,s_6,\ldots$. This means that (6) has the negative sign when $s=s_1,s_3,s_5,\ldots$, and has the positive sign when $s=s_2,s_4,s_6,\ldots$. That is, the signs of $q(is_m)$ and $q(is_{m+1})$, where $1\le m\le u-1$, are different. So $q(x)$ has at least $u-1$ roots and all their roots lie between adjacent two roots of $p(x)$ on the upper-half plane. Also we see that

$$q(0)=(-r_k+h)\prod_{\substack{l=1\\ l\neq k}}^{n}(-r_l)+(r_k-h)\prod_{\substack{l=1\\ l\neq k}}^{n}r_l=2(r_k-h)\prod_{\substack{l=1\\ l\neq k}}^{n}r_l>0,$$

and so there is at least one root of $q(x)$ between 0 and is_1. Combining these, we can conclude that we have exactly u zeros

$$is_1',is_2',\ldots,is_u'$$

which satisfy

$$0<s_1'<s_1<s_2'<s_2<\cdots<s_u'<s_u.$$

We next consider the case n odd and say $n=2u+1$. Then for arbitrary positive number t,

$$\begin{aligned} q(it)&=(it-r_k+h)\prod_{\substack{l=1\\ l\neq k}}^{n}(it-r_l)+(it+r_k-h)\prod_{\substack{l=1\\ l\neq k}}^{n}(it+r_l)\\ &=-(r_k-h-it)\prod_{\substack{l=1\\ l\neq k}}^{n}(r_l-it)+(it+r_k-h)\prod_{\substack{l=1\\ l\neq k}}^{n}(it+r_l).\end{aligned}$$

So, $q(it)$ must be zero or a purely imaginary number because $z-\bar{z}$ is always zero or purely imaginary for any complex number z. Consequently, (6) is zero or a purely imaginary number whose location depends on the sign of $\prod_{l=1}^{n}(is-r_l)$.

Now define a new function $u(t)$ by $u(it) := -iq(it)$. Then $u(it)$ is always real. Following above process in case n even, we have

$$\arg \prod_{l=1}^{n}(is - r_l) = \sum_{l=1}^{n} \arg(is - r_l) = \sum_{l=1}^{n} \pi - \alpha_l = (2u+1)\pi - \sum_{l=1}^{n} \alpha_l \tag{8}$$

and by Lemma 2.1 again,

$$(2u+1)\pi - \sum_{l=1}^{n} \alpha_l = \begin{cases} (2u+1)\pi - \frac{2\pi}{2} & \text{if } s = s_1, \\ (2u+1)\pi - \frac{4\pi}{2} & \text{if } s = s_2, \\ (2u+1)\pi - \frac{6\pi}{2} & \text{if } s = s_3, \\ \quad \vdots & \\ (2u+1)\pi - \frac{2u\pi}{2} & \text{if } s = s_u. \end{cases}$$

From this we see that the signs of $u(is_m)$ and $u(is_{m+1})$, where $1 \le m \le u-1$, are different. This implies that $u(x)$ has at least $u-1$ roots and all their roots lie between adjacent two roots of $p(x)$ on the upper-half plane, and so $q(x)$ has such locations. Also $q(0) = 0$ and

$$\begin{aligned}
\frac{d}{dt}u(it) &= \frac{d}{dt}\left[-iq(it)\right] \\
&= -i\frac{d}{dt}\left[(it - r_k + h)\prod_{\substack{l=1 \\ l\neq k}}^{n}(it - r_l) + (it + r_k - h)\prod_{\substack{l=1 \\ l\neq k}}^{n}(it + r_l)\right] \\
&= -i\left[i\prod_{\substack{l=1 \\ l\neq k}}^{n}(it - r_l) + i(it - r_k + h)\prod_{\substack{l=2 \\ l\neq k}}^{n}(it - r_l) + \cdots + \right. \\
&\qquad \left. i(it - r_k + h)\prod_{\substack{l=1 \\ l\neq k}}^{n-1}(it - r_l)\right] \\
&\quad - i\left[i\prod_{\substack{l=1 \\ l\neq k}}^{n}(it + r_l) + i(it + r_k - h)\prod_{\substack{l=2 \\ l\neq k}}^{n}(it + r_l) + \cdots + \right. \\
&\qquad \left. i(it + r_k - h)\prod_{\substack{l=1 \\ l\neq k}}^{n-1}(it + r_l)\right].
\end{aligned}$$

And at $t = 0$, the above equals

$$2(-1)^{n-1}\left[\prod_{\substack{l=1 \\ l\neq k}}^{n} r_l + (r_k - h)\prod_{\substack{l=2 \\ l\neq k}}^{n} r_l + \cdots + (r_k - h)\prod_{\substack{l=1 \\ l\neq k}}^{n-1} r_l\right] > 0$$

since n is odd. So we can conclude that $u(it) > 0$ for all $0 < t < \epsilon$ for some ϵ with $0 < \epsilon < \frac{s_1}{2}$, and we can find at least one root of $q(x)$ between 0 and is_1 since $u(is_1) < 0$. Combining these, we can conclude that we have exactly u zeros

$$is'_1, is'_2, \ldots, is'_u$$

which satisfy

$$0 < s'_1 < s_1 < s'_2 < s_2 < \cdots < s'_u < s_u,$$

which completes the proof. □

3. Examples

In this section we give two examples of Theorem 1.1. The first one is given for even degree polynomial and $h > 0$.

Example 3.1. Consider the polynomial equation

$$(x-1)(x-10)(x-100)(x-1000) + (x+1)(x+10)(x+100)(x+1000) = 0.$$

This has the roots $\pm is_1$ and $\pm is_2$, where

$$s_1 = 2.98672\cdots \quad \text{and} \quad s_2 = 334.81499\cdots.$$

Choose h with $0 < h \leq 1000$. When $h = 1$, replacing 1000 with 999 in above equation gives

$$(x-1)(x-10)(x-100)(x-999) + (x+1)(x+10)(x+100)(x+999) = 0,$$

which has the roots

$$\pm i\, 2.98671\cdots, \quad \pm i\, 334.64918\cdots.$$

Here

$$2.98671\cdots < s_1 < 334.64918\cdots < s_2.$$

Also when $h = 999.5$, replacing 1000 with 0.5 gives

$$(x-1)(x-10)(x-100)(x-0.5) + (x+1)(x+10)(x+100)(x+0.5) = 0,$$

which has the roots

$$\pm i\, 0.65510\cdots \quad \text{and} \quad \pm\, i\, 34.13313\cdots.$$

This change also satisfies

$$0.65510\cdots < s_1 < 34.13313\cdots < s_2.$$

The next is in the case of odd degree polynomial and $h < 0$.

Example 3.2. Consider the polynomial equation

$$\begin{aligned}&(x-1)(x-10)(x-100)(x-1000)(x-10000)\\ &+ (x+1)(x+10)(x+100)(x+1000)(x+10000) = 0.\end{aligned}$$

This has the roots 0, $\pm is_1$ and $\pm is_2$, where

$$s_1 = 31.46722\cdots \quad \text{and} \quad s_2 = 3349.79399\cdots.$$

Choose $h < 0$. When $h = -1$, replacing 100 with 101 in above equation gives

$$\begin{aligned}&(x-1)(x-10)(x-101)(x-1000)(x-10000)\\&+(x+1)(x+10)(x+101)(x+1000)(x+10000)=0,\end{aligned}$$

which has the roots 0,

$$\pm i\,31.60727\cdots, \quad \pm i\,3351.43580\cdots.$$

Here

$$s_1 < 31.60727\cdots < s_2 < 3351.43580\cdots.$$

Also when $h = -99.5$, replacing 100 with 199.5 gives

$$\begin{aligned}&(x-1)(x-10)(x-199.5)(x-1000)(x-10000)\\&+(x+1)(x+10)(x+199.5)(x+1000)(x+10000)=0,\end{aligned}$$

which has the roots 0,

$$\pm i\,42.32898\cdots \quad \text{and} \quad \pm\, i\,3509.40347\cdots.$$

This change also satisfies

$$s_1 < 42.32898\cdots < s_2 < 3509.40347.$$

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