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ON IDEMPOTENTS IN RELATION WITH REGULARITY

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Abstract. We make a study of two generalizations of regular rings, concentrating our attention on the structure of idempotents. A ring R is said to be *right attaching-idempotent* if for $a \in R$ there exists $0 \neq b \in R$ such that ab is an idempotent. Next R is said to be *generalized regular* if for $0 \neq a \in R$ there exist nonzero $b \in R$ such that ab is a nonzero idempotent. It is first checked that generalized regular is left-right symmetric but right attaching-idempotent is not. The generalized regularity is shown to be a Morita invariant property. More structural properties of these two concepts are also investigated.

Throughout this paper all rings are associative with identity unless otherwise specified. Let R be a ring. $X(R)$ denotes the set of all nonzero nonunits in R, and $G(R)$ denotes the group of all units in R. Let $C_r(R)$ (resp., $C_l(R)$) denote the set of all right (resp. left) regular elements in R. $J(R)$, $N_*(R)$, and $N^*(R)$ denote the Jacobson radical, prime radical, and upper nilradical (i.e., the sum of all nil ideals) of R, respectively. $N(R)$ denotes the set of all nilpotent elements in R. $|S|$ denotes the cardinality of a subset S of R. Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$) and use e_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Use $\mathbb{Q}$ to denote the field of rational numbers. $R[x]$ ($R[[x]]$) denotes the polynomial ring (power series ring) with an indeterminate x over R. $\prod$ ($\oplus$) denotes the direct product (sum) of rings.

A ring R (possibly without identity) is usually called *reduced* if $N(R) = 0$. A ring (possibly without identity) is usually called *Abelian* if every idempotent is central. It is easily checked that reduced rings are Abelian.

A ring R (possibly without identity) is usually called *von Neumann regular* (simply, *regular*) (resp., *unit-regular*) if for every $x \in R$ there exists $y \in R$ (resp., $u \in G(R)$) such that $xyx = x$ (resp. $xux = x$) in [3]. It is shown that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent in [3, Theorem 1.1].

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1. Right attaching-idempotent rings

A ring R will be called *right* (resp., *left*) *attaching-idempotent* if for $a \in R$ there exists $0 \neq b \in R$ (resp., $0 \neq c \in R$) such that ab (resp., ca) is an idempotent. R will be called *attaching-idempotent* if it is both left and right attaching-idempotent.

We first observe that the attaching-idempotent property is not left-right symmetric by the following.

Example 1.1. (1) Let $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q}[x] \\ 0 & \mathbb{Q}[x] \end{smallmatrix}\right)$ be the subring of $U_2(\mathbb{Q}[x])$. Let $0 \neq a = (a_{ij}) \in R$. If $a_{11} = 0$, then $ae_{11} = 0$. If $a_{11} \neq 0$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} = e_{11}.$$

Thus R is a right attaching-idempotent ring.

Next we claim that R is not left attaching-idempotent. Consider the element $\left(\begin{smallmatrix} 0 & x \\ 0 & x^2 \end{smallmatrix}\right)$ in R, and assume on the contrary that $\left(\begin{smallmatrix} s & t \\ 0 & u \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & x \\ 0 & x^2 \end{smallmatrix}\right)$ is an idempotent for $\left(\begin{smallmatrix} s & t \\ 0 & u \end{smallmatrix}\right) \in R$. Then $\left(\begin{smallmatrix} 0 & sx+tx^2 \\ 0 & ux^2 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 0 & sx+tx^2 \\ 0 & ux^2 \end{smallmatrix}\right)$. So we must have $sx + tx^2 = 0$ and $u^2x^4 = ux^2$, yielding $u = 0$. Next we get $s = t = 0$ since the degree of sx is 1 and the degree of tx^2 is ≥ 2 if s, t are both nonzero, a contradiction. Therefore R is not left attaching-idempotent.

(2) Let $S = \left(\begin{smallmatrix} \mathbb{Q}[x] & \mathbb{Q}[x] \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ be the subring of $U_2(\mathbb{Q}[x])$. Then S is not right attaching-idempotent but left attaching-idempotent by a similar computation.

In the following we see basic properties of left or right attaching-idempotent rings.

Lemma 1.2. (1) *The class of right attaching-idempotent rings is closed under homomorphic images and direct products.*

(2) *If every element of a ring R is either a unit or a zero-divisor, then R is attaching-idempotent.*

(3) *If R is a right attaching-idempotent ring, then $J(R)$ is contained in $R \backslash C_r(R)$.*

Proof. (1) is shown by definition.

(2) Let $a \in R$ be a zero-divisor. Then $ab = 0 = ca$ for some nonzero $b, c \in R$. Thus R is attaching-idempotent.

(3) Let R be a right attaching-idempotent ring. Then, for $a \in J(R)$, there exists $0 \neq b \in R$ such that $(ab)^2 = ab$. But since $ab \in J(R)$, we must have $ab = 0$. This implies $a \in R \backslash C_r(R)$. □

Recall that regular rings are attaching-idempotent, but this implication is irreversible by the following. This is also shown by Example 2.1 to follow, but we see two sorts of constructions applicable to related situations in the following example.

Example 1.3. (1) Every element of a finite ring is either a unit or a zero-divisor by Lemma 1.4(3) to follow. So a finite ring R with $J(R) \neq 0$ is attaching-idempotent but not regular. For example, $\mathrm{Mat}_n(\mathbb{Z}_4)$ $(n \geq 1)$ is attaching-idempotent but not regular because the Jacobson radical is $\mathrm{Mat}_n(2\mathbb{Z}_4) \neq 0$.

(2) There exists an infinite noncommutative attaching-idempotent ring but not regular. Let $K = \mathbb{Z}_2$ and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Next let I be the ideal of A generated by

$$a^2 - a, b^2, \text{ and } ba.$$

Set $R = A/I$ and identify every element of A with its image in R for simplicity. Then we have the relations

$$a^2 = a \text{ and } b^2 = ba = 0.$$

Consider the ideal of R generated by b, J say. Then $J^2 = 0$ since $b^2 = ba = 0$, and moreover we have

$$\frac{R}{J} \cong \frac{K\langle a\rangle}{(a^2 - a)} \cong K + Ka,$$

where $(a^2 - a)$ is the ideal of $K\langle a\rangle(= K[a])$ generated by $a^2 - a$. Note that every element of $K + Ka$ is an idempotent as can be seen by the computation $0^2 = 0, 1^2 = 1, a^2 = a, (1 + a)^2 = 1 + a$. This yields $J = J(R)$.

Every element of R is of the form

$$k_0 + k_1a + k_2b + k_3ab$$

with $k_i \in K$. Let $f = k_0 + k_1a + k_2b + k_3ab$ and $g = a$. Then

$$fg = (k_0 + k_1a + k_2b + k_3ab)a = (k_0 + k_1a)a \in K + Ka$$

is an idempotent since $(k_0 + k_1a)a$ is an idempotent. Thus R is right attaching-idempotent.

For the case of $k_0 = 1$, we have $a(1+a+b+ab) = 0$, $(1+a+b+ab)(1+a+b) = 1 + a$, $(1 + a)(1 + a + ab) = 1 + a$, $(1 + a)(1 + a) = 1 + a$, $(1 + b)(1 + b) = 1$, $(1 + ab)(1 + ab) = 1$, and $(1 + b + ab)(1 + b + ab) = 1$. For the case of $k_0 = 0$, we have $b(k_1a + k_2b + k_3ab) = 0$. Thus R is also left attaching-idempotent. But $brb = 0$ for all $b \in R$, so R is not regular.

Following the literature, a ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is shown that the class of directly finite rings contains rings that satisfy either the ascending or the descending chain condition for principal right ideals generated by idempotents, in [7, Theorem 1]. So right or left Artinian rings are directly finite. We will use this fact freely.

A ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup in [5]. It is shown that a ring is locally finite if and only if every finite subset generates a finite subring (possibly without identity), in [5, Theorem 2.2(1)]. The class of locally finite rings is easily shown to contain finite rings and algebraic closures of finite fields.

Lemma 1.4. (1) *Locally finite rings are directly finite.*

(2) *Let R be a locally finite ring and $a \in R$. Then some power of a is an idempotent.*

(3) *Every element of a locally finite ring is either a unit or a zero-divisor.*

Proof. Let R be a locally finite ring.

(1) Assume that $ab = 1$ for $a, b \in R$. Consider the subring of R generated by a, b, S say. Then S is finite and so directly finite, entailing $ba = 1$.

(2) Since R is locally finite, a^k is an idempotent for some $k \geq 1$ by the proof of [6, Proposition 16].

(3) Let R be a locally finite ring and $0 \neq a \in R$ be an arbitrary nonunit. Then a^k is an idempotent for some $k \geq 1$ by (2). If $a^\ell(1 - a^k) = 0$ for all positive integers ℓ, then $a(1 - a^k) = 0 \neq 1 - a^k$, and so a is a zero-divisor. Since $a^k(1 - a^k) = 0$, we can suppose that there exists the smallest positive integer k_0 $(1 \leq k_0 \leq k)$ so that $a^{k_0}(1 - a^k) = 0 \neq a^{k_0-1}(1 - a^k)$. So a is a zero-divisor. $\square$

Proposition 1.5. (1) *Locally finite rings are attaching-idempotent.*

(2) *Local rings with nil Jacobson radical are attaching-idempotent.*

Proof. (1) is shown by Lemmas 1.2(2) and 1.4(3).

(2) Let R be a local ring with nil Jacobson radical. Then every element is either a unit or nilpotent, so R is attaching-idempotent. $\square$

The condition "with nil Jacobson radical" in Proposition 1.5(2) is not superfluous as can be seen by the local ring $F[[x]]$ over a field. In fact, each of xb and cx cannot be an idempotent for any nonzero $b, c \in F[[x]]$.

Given a right attaching-idempotent ring R, it is natural to ask whether eRe is also a right attaching-idempotent ring for any idempotent $e \in R$. However the answer is negative by the following argument. Consider the right attaching-idempotent ring $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}[x] \\ 0 & \mathbb{Q}[x] \end{pmatrix}$ in Example 1.1(1). Letting $e = e_{22} \in R$, $eRe \cong \mathbb{Q}[x]$ is neither left nor right attaching-idempotent.

Lemma 1.6. *Let R be a ring and $e^2 = e \in R$. If eRe and $(1-e)R(1-e)$ are both right* (*resp., left*) *attaching-idempotent, then R is right* (*resp., left*) *attaching-idempotent.*

Proof. By the Pierce decomposition of R, $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Note that $eR(1-e)$ and $(1-e)Re$ are clearly right (resp. left) attaching-idempotent. Since eRe and $(1-e)R(1-e)$ are right (resp., left) attaching-idempotent, R is right (resp., left) attaching-idempotent by Lemma 1.2(1). $\square$

Using Lemma 1.6, an inductive argument gives immediately the following.

Proposition 1.7. *Let R be a ring and $e_1, \ldots, e_n \in R$ be orthogonal idempotents such that $e_1 + \cdots + e_n = 1$. If each e_iRe_i is right* (*resp., left*) *attaching-idempotent, then R is right* (*resp., left*) *attaching-idempotent.*

The following two results are direct consequences of Proposition 1.7.

Corollary 1.8. *If R is a right (resp., left) attaching-idempotent ring, then $\mathrm{Mat}_n(R)$ is right (resp., left) attaching-idempotent.*

Corollary 1.9. *Let R be a ring and $M_1, \dots, M_n$ be R-modules, $M = M_1 + \cdots + M_n$ say. If each $\mathrm{End}_R M_i$ is right (resp., left) attaching-idempotent for each i, then $\mathrm{End}_R M$ is right (resp., left) attaching-idempotent, where $\mathrm{End}_R M_i$ and $\mathrm{End}_R M$ mean the endomorphism ring of M_i and M respectively.*

By Proposition 1.7, we note that

(1) if P is a finitely projective module over a right (resp., left) attaching-idempotent ring R, then $\mathrm{End}_R P$ is a right (resp., left) attaching-idempotent ring;

(2) for each $n \geq 1$, a ring R is right (resp., left) attaching-idempotent if and only if the ring of all $n \times n$ upper (or lower) triangular matrices over R is right (resp., left) attaching-idempotent.

A proper ideal I in a ring R is called *right attaching-idempotent* if for $0 \neq a \in I$ there exists $0 \neq b \in I$ such that ab is an idempotent, i.e., I is right attaching-idempotent as a ring without identity. It is shown in [3, Lemma 1.3] that for ideals $J \subseteq K$ of a ring R, K is regular if and only if both J and K/J are regular. For a right attaching-idempotent ring R, this result is dependent on ideals of R by the following.

Example 1.10. (1) Consider the ring $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q}[x] \\ 0 & \mathbb{Q}[x] \end{smallmatrix}\right)$ as given in Example 1.1. We note that all proper nonzero ideals of R are

$$I_1 = \begin{pmatrix} 0 & \mathbb{Q}[x] \\ 0 & \mathbb{Q}[x] \end{pmatrix},\ I_2 = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}[x] \\ 0 & 0 \end{pmatrix}.$$

We also note that both I_1 and $R/I_1 \simeq \mathbb{Q}$ are right attaching-idempotent. On the other hand, I_2 is right attaching-idempotent but $R/I_2 \simeq \mathbb{Q}[x]$ is not right attaching-idempotent.

(2) Consider the ring $S = \mathbb{Z} \times \mathbb{Q}$ and an ideal $I = \mathbb{Z} \times \{0\}$ of S. Then I is not an (right) attaching-idempotent ideal of S but both S and S/I are (right) attaching-idempotent.

We next extend the construction of the rings in Example 1.1 to a more general case.

Proposition 1.11. *Let A, B be rings and ${}_AM_B$ be an (A, B)-bimodule.*

(1) *If A is right attaching-idempotent, then so is $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$.*

(2) *If B is left attaching-idempotent, then so is $R = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$.*

Proof. (1) Let A be right attaching-idempotent and $\left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right) \in R$. If $a = 0$, then $\left(\begin{smallmatrix} 0 & m \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0$. Assume $a \neq 0$. Since A is right attaching-idempotent, aa_1 is an idempotent for some $0 \neq a_1 \in A$. This yields $\left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^2$.

The proof of (2) is similar. □

Let A be an algebra, with or without identity, over a commutative ring S. Following Dorroh [2], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. One may conjecture that the class of right attaching-idempotent rings is closed under Dorroh extensions, but the following erases the possibility.

Example 1.12. Let $A = \left(\begin{smallmatrix} \mathbb{Q}(x) & \mathbb{Q}(x) \\ 0 & 0 \end{smallmatrix}\right)$ be the subring of $U_2(\mathbb{Q}(x))$ and R be the Dorroh extension of A by $\mathbb{Z}$, where $\mathbb{Q}(x)$ is the quotient field of $\mathbb{Q}[x]$. Let $a = (\left(\begin{smallmatrix} x & x \\ 0 & 0 \end{smallmatrix}\right), 2) \in R$. Suppose $(ab)^2 = ab$ for $0 \neq b = (\left(\begin{smallmatrix} e & f \\ 0 & 0 \end{smallmatrix}\right), m) \in R$. Then $m = 0$ from $(2m)^2 = 2m$, entailing $ab = (\left(\begin{smallmatrix} xe+2e & xf+2f \\ 0 & 0 \end{smallmatrix}\right), 0)$. Next we get $e = 0$ from $(xe + 2e)^2 = xe + 2e$, entailing $b = (\left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right), 0)$. Note $f \neq 0$ since $b \neq 0$. Consequently $ab = (\left(\begin{smallmatrix} 0 & xf+2f \\ 0 & 0 \end{smallmatrix}\right), 0)$. Since $xf + 2f \neq 0$, ab is not an idempotent. Thus R is not right attaching-idempotent.

2. Generalized regular rings

A ring R (possibly without identity) will be called *generalized regular* if for $0 \neq a \in R$ there exists $0 \neq b \in R$ such that ab is a nonzero idempotent. It is obvious that regular rings are clearly generalized regular, and generalized regular rings are right (left) attaching-idempotent. Each of these implications is strict by the following.

Example 2.1. (1) Let F be a field and define

$$R = \{(a_i) \in \prod_{i=1}^{\infty} E_i \mid \text{ there exists } m \geq 1 \text{ such that } a_j \in U_2(F) \text{ for all } j \geq m\},$$

where $E_i = \mathrm{Mat}_2(F)$ for all i. Consider the element

$$a = (a_i) = (e_{12}, e_{12}, \ldots, e_{12}, \ldots) \in R, \text{ i.e., } a_i = e_{12} \text{ for all } i.$$

For any $b = (b_i) \in R$ there exists $n \geq 1$ such that $b_k \in U_2(F)$ for all $k \geq n$, and so $c_k = 0$ for all $k \geq n$, where $(c_i) = aba$. Thus $a \notin aRa$, concluding that R is not regular. But R is generalized regular by Theorem 2.4 to follow since $\mathrm{Mat}_2(F)$ is regular.

(2) Let $S = \mathbb{Z} \times \mathbb{Q}$. Then S is (right, left) attaching-idempotent, but is clearly not generalized regular.

The Jacobson radical of a regular ring is known to be zero. We see that generalized regular rings are also semiprimitive in the following.

Lemma 2.2. (1) *The generalized regularity is left-right symmetric.*

(2) *Generalized regular rings are semiprimitive.*

(3) *Let R be a ring. For $0 \neq a \in R$ there exists unique $b \in R$ with $0 \neq (ab)^2 = ab$ if and only if R is a division ring.*

Proof. (1) Let R be a generalized regular ring and $0 \neq a \in R$. Then there exists $0 \neq b \in R$ such that $0 \neq ab = (ab)^2$. This yields $(baba)^2 = babababa = baba \in Ra$, obtaining $baba \neq 0$ and $bab \neq 0$ from $a(baba)b = ab \neq 0$. The converse is proved similarly.

(2) Let R be a generalized regular ring and take $0 \neq a \in J(R)$ on the contrary. Then $(ab)^2 = ab \neq 0$ for some $b \in R$. But $ab \in J(R)$, a contradiction. Thus $J(R)$ must be zero.

(3) Assume that given $0 \neq a \in R$ there exists unique $b \in R$ with $0 \neq (ab)^2 = ab$. We first show that R is a domain. Let $c, d \in R\backslash\{0\}$ satisfy $cd = 0$. By assumption, there exists unique $e \in R$ with $(ce)^2 = ce$. Then we also have $(c(d+e))^2 = c(d+e)$, but the uniqueness of e forces $d + e = e$. This yields $d = 0$, a contradiction. This result implies $ab = 1$ since 1 is the only nonzero idempotent in a domain. We also get $ba = 1$ since R is Abelian. The converse is clear. □

$R = \mathrm{Mat}_2(D)$ is clearly (generalized) regular over a division ring D. For $e_{11} \in R$ there exist $b_1 = e_{11}, b_2 = e_{11} + e_{22} \in R$ such that $ab_1 = ab_2 = e_{11}$ is an idempotent, noting $b_1 \neq b_2$.

The following is similar to [3, Theorem 1.1].

Proposition 2.3. *For a ring R the following conditions are equivalent*:

(1) *R is generalized regular ring*;

(2) *Every nonzero* (*principal*) *right ideal contains a nonzero idempotent*;

(3) *Every nonzero* (*principal*) *left ideal contains a nonzero idempotent.*

Proof. (1)⇒(2). Let R be a generalized regular ring and $aR \neq 0$ for $a \in R$. Then $0 \neq ab = (ab)^2$ for some $b \in R$. Note $ab \in aR$.

(2)⇒(1). Let $0 \neq c \in R$. By the condition, cR contains a nonzero idempotent. This must be of the form cd with $d \in R$.

The proof of the equivalence of (1) and (3) is done by the left version of the preceding one and help of Lemma 2.2(1). □

Following Nicholson [9], a ring R is said to be an I_0-ring if for $a \notin J(R)$ there exists $0 \neq x \in R$ such that $xax = x$. So, by Lemma 2.2(2), a generalized regular ring is just a semiprimitive I_0-ring, letting $x = bab$ in the definition of generalized regular rings. Thus Proposition 2.3 follows also from [9, Lemma 1.1].

The following theorem will play an important role in this note.

Theorem 2.4. (1) *Let R_i be a regular ring and S_i be a subring of R_i for all $i \in I$, where I is a countably infinite set. Define*

$$R = \{(a_i) \in \prod_{i=1}^{\infty} R_i \mid \text{ there exists } m \geq 1 \text{ such that } a_j \in S_j \text{ for all } j \geq m\}.$$

Then R is a generalized regular ring.

(2) *Let A be a regular ring and B be a subring of A. Define*

$$R = \{(a_i) \in \prod_{i=1}^{\infty} R_i \mid \text{there exists } m \geq 1 \text{ such that } a_m \in B \text{ and}$$
$$a_m = a_{m+1} = \cdots\},$$

where I is a countably infinite set and $R_i = A$ for all $i \in I$. Then R is a generalized regular ring.

Proof. Let $0 \neq a = (a_i) \in R$ and say that $k \geq 1$ is the smallest integer such that $a_k \neq 0$. Since R_k is regular, there exists $\alpha \in R_k$ such that $a_k \alpha a_k = a_k$. Consider the sequence

$$b = (b_i) = (0, \ldots, 0, \alpha, 0, \ldots) \in R, \text{ i.e., } b_k = \alpha \text{ and } b_i = 0 \text{ for all } i \neq k.$$

Then ab, $ab = (c_i)$ say, is a nonzero idempotent since $c_k = a_k\alpha = (a_k\alpha)^2 \neq 0$ and $c_i = 0$ for all $i \neq k$. Thus R is generalized regular, noting $0 \neq (ab)^2 = ab$. The proof of (2) is almost same. □

The Jacobson radical of a generalized regular ring is zero by Lemma 2.2(2). So $J(R) = 0$ for the ring R in Example 2.1. In fact, letting $0 \neq (a_i) \in J(R)$ on the contrary, $a_t \neq 0$ say. Then we get $(b_i) \in J(R)$ with $b_t = 1$ and $b_i = 0$ for all $i \neq t$ since $\mathrm{Mat}_2(F)a_t\mathrm{Mat}_2(F) = \mathrm{Mat}_2(F)$ ($1 = \sum_{j=1}^{m} r_j a_t s_j$ say), via the multiplication

$$(b_i) = \sum_{j=1}^{m} (r(j)_i)(a_i)(s(j)_i),$$

where $r(j)_t = r_j$, $s(j)_t = s_j$ and $r(j)_i = 0$, $s(j)_i = 0$ for all $i \neq t$. It then follows that $1 - (b_j)$ is not a unit, a contradiction. Thus $J(R) = 0$.

For a reduced ring R, it is well-known that $r_1 r_2 \cdots r_n = 0$ implies

$$r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i. We will use this fact without mentioning.

A ring R (possibly without identity) is called *strongly regular* if for every $x \in R$ there exists $y \in R$ such that $x = x^2 y$ in [3]. A ring R (possibly without identity) will be called *strongly generalized regular* if for every $0 \neq x \in R$ there exists $y \in R$ such that $0 \neq xy = x^2 y^2$. Strongly regular rings are clearly strongly generalized regular, but the converse need not hold by help of Theorem 2.5 and Example 2.8 to follow. A ring is strongly regular if and only if it is Abelian regular [3, Theorem 3.5]. We see a similar result for generalized regular rings as follows.

Theorem 2.5. *A ring is strongly generalized regular if and only if it is Abelian generalized regular.*

Proof. Let R be an Abelian generalized regular ring and $0 \neq x \in R$. Then there exists $y \in R$ such that $0 \neq xy = xyxy$. Since xy is central in R, we have $xy = xy(xy) = x(xy)y = x^2y^2$.

Conversely, assume that R is strongly generalized regular and $0 \neq x \in R$. Then $0 \neq xy = x^2y^2$ for some $y \in R$. Here if $x^2 = 0$, then $0 \neq xy = x^2y^2 = 0$, a contradiction. This leads us to conclude that R a reduced ring. From $xy = x^2y^2$, we get $x(xy-1)y = 0$. But since R is reduced, we have $xy(xy-1) = 0$ and so $0 \neq xy = xyxy$. Reduced rings are clearly Abelian, and therefore R is Abelian generalized regular. □

Applying the proof of Theorem 2.5, we can obtain a simpler proof of the necessity of the well-known result that a ring is Abelian regular if and only if it is strongly regular ([3, Theorem 3.5]).

For a regular ring R, the following conditions are equivalent: (1) R is Abelian; and (2) R/P is a division ring for any prime ideal P of R [3, Theorem 3.2]. We find a similar result for generalized regular rings as follows.

Corollary 2.6. *Let R be a generalized regular ring. Then the following conditions are equivalent:*

(1) *R is Abelian;*

(2) *R is reduced;*

(3) *R/P is a domain for any minimal prime ideal P of R.*

Proof. (1)⇒(2) is obtained by Theorem 2.5 and its proof.

(2)⇒(1) is clear, and (3)⇒(2) comes from the elementary fact that $R/N_*(R)$ is a subdirect product of R/P where P runs over all minimal prime ideals of R.

(2)⇒(3). Let R be a reduced ring and let P be any minimal prime ideal of R. Then R/P is a domain by [10, Proposition 1.11]. □

One may compare Corollary 2.6 with [3, Theorem 3.2]. $\mathrm{Mat}_2(F)$, over a field F, is regular but not reduced. Letting $x = e_{12}$, $x^2 = 0$ and so there cannot exist $y \in \mathrm{Mat}_2(F)$ such that $0 \neq xy = x^2y^2$.

We apply Theorem 2.4(2) to compare Corollary 2.6 with [3, Theorem 3.2]. In Theorem 2.4(2), consider the ring R with $A = \mathbb{Q}(x)$, the quotient field of $\mathbb{Z}[x]$, and $B = \mathbb{Z}[x]$. Then R is generalized regular by Theorem 2.4(2). Set $I = \oplus_{i=1}^{\infty} R_i$. Then R/I is isomorphic to $\mathbb{Z}[x]$ which is a domain but not a division ring, noting that I is a prime ideal of R.

Proposition 2.7. (1) *Let R be a generalized regular ring. Then R is strongly generalized regular if and only $ab = ba$ whenever ab is an idempotent for $a, b \in R$.*

(2) *Let R be a regular ring. Then R is strongly regular if and only if $ab = ba$ whenever ab is an idempotent for $a, b \in R$.*

Proof. (1) Suppose that R is strongly generalized regular and that $ab = (ab)^2$ for $a, b \in R$. We use the reducedness of R freely, based on Theorem 2.5 and

Corollary 2.6. From $ab(1-ab)=0$, we get $a(1-ab)b=0$, $ba(1-ab)=0$, and $b(1-ab)a=0$, entailing $ab=aabb$, $ba=baab$, and $ba=(ba)^2$. Thus

$$ab=(ab)(ab)=a(ba)b=(ba)(ab)=ba.$$

The converse is clear.

(2) Strongly regular rings are clearly strongly generalized regular, so the following is an immediate consequence of (1). □

Comparing Proposition 2.7 with [3, Theorem 3.2], one may ask whether reduced generalized regular rings can be regular. However the following provides a negative answer.

Example 2.8. There exists a reduced generalized regular ring but not regular. Let D be a division ring. Then $D[x]$ is a Noetherian domain by [8, Theorem 2.9]. So there exists a right quotient division ring of $D[x]$, say $D(x)$, by [8, Theorem 1.15 and Corollary 1.14].

We next apply the construction of Example 2.1 to this situation. Define

$$R=\{(a_i)\in\prod_{i=1}^{\infty}E_i \mid \text{ there exists } m\geq 1 \text{ such that } a_j\in D[x] \text{ for all } j\geq m\},$$

where $E_i=D(x)$ for all i. Then clearly R is a reduced ring. Consider the element

$$a=(a_i)=(x,x,\ldots,x,\ldots)\in R, \text{ i.e., } a_i=x \text{ for all } i.$$

For any $b=(b_i)\in R$ there exists $n\geq 1$ such that $b_k\in D[x]$ for all $k\geq n$, and so $c_k=b_kx^2$ for all $k\geq n$, where $(c_i)=aba$. Thus $a\notin aRa$, concluding that R is not regular. But R is generalized regular by Theorem 2.4 since $D(x)$ is a division ring (hence regular).

A ring R is called *unit-regular* if for every $x\in R$ there exists $u\in G(R)$ such that $xux=x$ in [1]. Abelian regular rings are unit-regular by [3, Corollary 4.2]. One can refer to the statements for unit-regular rings obtained by Henriksen [4] to delimit generalizations of unit-regular rings.

A ring R is usually called *directly finite* if $ab=1$ implies $ba=1$ for $a,b\in R$. Abelian rings are clearly directly finite. Abelian regular rings are proved to be unit-regular in [3, Corollary 4.2], by help of module theoretic method. In the following we prove that independently by using a direct computation.

Theorem 2.9. *Abelian regular rings are unit-regular.*

Proof. Let R be an Abelian regular ring and $x\in R$. Then $x=x^2y\neq 0$ for some $y\in R$. Consider an element $1+x-xy$. Then

$$\begin{aligned}&(1+x-xy)(1-xy+xy^2)\\ &=1-xy+xy^2+x-xxy+xxy^2-xy+xyxy-xyxy^2\\ &=1-xy+xy^2+x-x+xy-xy+xy-xy^2=1\end{aligned}$$

by help of Proposition 2.7(2). So $1 - xy + xy^2 \in G(R)$. Now letting $u = 1 - xy + xy^2$, we have

$$xux = x(1 - xy + xy^2)x = (x - xxy + xxy^2)x = (x - x + xy)x = xyx = x,$$

concluding that R is unit-regular. □

Observing the proof of Theorem 2.9, we can see that the method in the proof of Badawi [1, Theorem 2] is also applicable to obtain another simpler proof.

Let R be a strongly generalized regular ring and $xy = xyxy$ for $x, y \in R$. Then $xy = yx$ by Proposition 2.7(1). In this situation, we also have units, induced by x and y, as in the following computation:

$$\begin{aligned}
&(1 - xy + x^2y)(1 - xy + xy^2)\\
&= 1 - xy + xy^2 - xy + xyxy - xyxy^2 + x^2y - x^2yxy + x^2yxy^2\\
&= 1 - xy + xy^2 - xy + xy - xy^2 + x^2y - x^2y + xy = 1.
\end{aligned}$$

Thus, considering the proof of Theorem 2.9, one may ask whether strongly generalized regular rings are unit-regular. However the following provides a negative answer.

Example 2.10. There exists a strongly generalized regular ring but not unit-regular. In Theorem 2.4(2), consider the ring R with $A = \mathbb{Q}(x)$, the quotient field of $\mathbb{Z}[x]$, and $B = \mathbb{Z}[x]$. Then R is generalized regular by Theorem 2.4(2). R is moreover reduced (hence Abelian), and so R is strongly generalized regular by Theorem 2.5.

As in Example 2.10, consider the element $a = (x, x, \ldots, x, \ldots)$ in R. Note that for every unit $u = (u_i)$ in R there exists $m \geq 1$ such that $u_j = 1$ or $u_j = -1$ for all $j \geq m$. So we have

$$aua = (x, x, \ldots, x, \ldots)u(x, x, \ldots, x, \ldots) = (x^2, x^2, \ldots, x^2, \ldots),$$

(b_i) say. This implies that $b_m = b_{m+1} = \cdots = x^2$ or $b_m = b_{m+1} = \cdots = -x^2$, and so aua cannot be a. Therefore R is not unit-regular.

3. Structural properties of generalized regular rings

In this section we study a more structural property of generalized regular rings. We actually show that the generalized regularity is a Morita invariant property. The class of generalized regular rings is not closed under subrings as we see the rings $D[x] \subsetneqq D(x)$ in Example 2.8. But we see a kind of subring which preserves the generalized regularity in the following.

Proposition 3.1. *Let R be a generalized regular ring.*

(1) *Let e, f be idempotents in R such that $eRf \neq 0$ and $fRe \neq 0$. Then for $0 \neq x \in eRf$ there exists $y \in fRe$ such that $0 \neq (xy)^2 = xy$.*

(2) *If $0 \neq e = e^2 \in R$, then eRe is a generalized regular ring.*

Proof. (1) Let $0 \neq x \in eRf$. Since R is generalized regular, $0 \neq (xy)^2 = xy$ for some $y \in R$. Seeing that

$$xy = xyxy = (exf)y(exf)y = (exf)(fye)(exf)y,$$

we can let $y = fye \in fRe$.

(2) is an immediate consequence of (1). □

As follows, we see that the generalized regularity is a Morita invariant property.

Theorem 3.2. (1) *Let $e_1, \ldots, e_n$ be orthogonal idempotents in a ring R such that $e_1 + \cdots + e_n = 1$. Then R is generalized regular if and only if for each $0 \neq x \in e_iRe_j$ there exists $y \in e_jRe_i$ such that $xyxy = xy \neq 0$.*

(2) *If S is a generalized regular ring, then so is $R = \mathrm{Mat}_n(S)$ for $n \geq 2$.*

Proof. (1) We apply the method of proof of [3, Lemma 1.6]. First assume that R is generalized regular, and let $0 \neq x \in e_iRe_j$. Then $0 \neq xy = xyxy$ for some $0 \neq y \in R$. Since $x = xe_j = e_ixe_j$ and $(xy)e_i = (xy)(xy)e_i$, there exists $0 \neq e_jye_i$ such that

$$xe_jye_i = xye_i = (xy)(xy)e_i = (xe_jye_i)(xe_jye_i) \neq 0.$$

Conversely, assume that for any $0 \neq x \in e_iRe_j$ there exists $0 \neq y \in e_jRe_i$ such that $0 \neq xy = xyxy$. We proceed by induction on n. Since the case $n = 1$ is trivial, we begin with the case $n = 2$. First consider an element $0 \neq x \in R$ such that $e_1xe_2 = 0$, entailing $e_1xe_1 + e_2xe_1 + e_2xe_2 \neq 0$. If $e_1xe_1 + e_2xe_2 = 0$, then $x = e_2xe_1$, so the assumption works.

Let $e_1xe_1 + e_2xe_2 \neq 0$. Then there are elements $y \in e_1Re_1$ and $z \in e_2Re_2$ such that

$$(e_1xe_1)y(e_1xe_1)y = (e_1xe_1)y \text{ and } (e_2xe_2)z(e_2xe_2)z = (e_2xe_2)z,$$

noting that $(e_1xe_1)y \neq 0$ or $(e_2xe_2)z \neq 0$. This yields

$$\begin{aligned}
&x(y+z)x(y+z)\\
&= (e_1xe_1 + e_2xe_1 + e_2xe_2)(y+z)(e_1xe_1 + e_2xe_1 + e_2xe_2)(y+z)\\
&= (e_1xe_1y + e_2xe_1y + e_2xe_2z)(e_1xe_1y + e_2xe_1y + e_2xe_2z)\\
&= e_1xe_1y + e_2xe_2z + e_2x(y+z)xe_1y.
\end{aligned}$$

As a result, we see that the element $x' = x(y+z) - x(y+z)x(y+z)$ lies in e_2Re_1 since $(e_1xe_1)y \neq 0$ or $(e_2xe_2)z \neq 0$.

$$\begin{aligned}
&x(y+z) - x(y+z)x(y+z)\\
&= (e_1xe_1y + e_2xe_1y + e_2xe_2z) - (e_1xe_1y + e_2xe_2z + e_2x(y+z)xe_1y)\\
&= e_2xe_1y - e_2x(y+z)xe_1y.
\end{aligned}$$

Here if $x' = 0$, then we are done, noting that $x(y+z) \neq 0$ since $(e_1xe_1)y \neq 0$ or $(e_2xe_2)z \neq 0$. So suppose $x' \neq 0$. Then $x'wx'w = x'w \neq 0$ for some $w \in e_1Re_2$

by assumption. This implies that $xvxv = xv \neq 0$ for some $v \in R$, letting $v = x((y+z) - (y+z)x(y+z))w$.

Now consider the case of $x \in R$ with $e_1xe_2 \neq 0$. Then by assumption, there exists $y \in e_2Re_1$ such that $(e_1xe_2)y(e_1xe_2)y = (e_1xe_2)y = e_1xy \neq 0$. Since $y \in e_2Re_1$, we see that $e_1xye_2 = 0$. By the case above, there exists an element $z \in R$ such that $(xy)z(xy)z = (xy)z \neq 0$, hence $x(yz)x(yz) = x(yz) \neq 0$. Therefore, R is generalized regular.

Finally, let $n > 2$, and assume that the result holds for $n-1$ orthogonal idempotents. Setting $f = e_2 + \cdots + e_n$ and $g = e_1 + e_3 + e_4 + \cdots + e_n$, we thus know that fRf and gRg are generalized regular. Consider any element $0 \neq x \in e_1Rf$. There exists $y \in fRe_1$ such that $(xf)y(xf)y = xfy = xy \neq 0$. But $e_1xye_2 = 0$, so $xy \in gRg$, whence $xyzxyz = xyz$ for some $z \in gRg$. As a result, $xwxw = xw$ for some $w \in R$, hence we obtain $fwe_1 \in fRe_1$ such that $x(fwe_1)x(fwe_1) = x(fwe_1) \neq 0$. Likewise, for any $0 \neq x \in fRe_1$ there is some $t \in e_1Rf$ such that $xtxt = xt \neq 0$. Applying the case of $n = 2$ to the orthogonal idempotents e_1 and f, we conclude that R is generalized regular. Therefore the induction works.

(2) Let S be a generalized regular ring and $e_i = e_{ii} \in R$ for $i = 1, \ldots, n$. Then e_i's are orthogonal idempotents in R such that $e_1 + \cdots + e_n = 1$.

Let $0 \neq x = (x_{ij}) \in e_iRe_j$, i.e., $x_{ij} \neq 0$ and other entries of x are all zero. Then $e_ixe_j = x_{ij} \neq 0$, a say. Since S is generalized regular, there exists $b \in S$ such that $abab = ab \neq 0$. Let $y = (y_{ij}) \in S$ such that $y_{ji} = b$ and other entries of y are all zero. Then $y \in e_jRe_i$ such that

$$xy = (ae_{ij})(be_{ji}) = (ab)e_{ii} = (abab)e_{ii} = ((ab)e_{ii})((ab)e_{ii}) = xyxy.$$

So $\mathrm{Mat}_n(S)$ is generalized regular by (1). □

The following is obtained by help of Proposition 3.1(2) and Theorem 3.2.

Corollary 3.3. *If P is a finitely generated projective module over a generalized regular ring R, then the endomorphism ring of P over R is also generalized regular.*

The generalized regularity is Morita invariant by Proposition 3.1(2) and Theorem 3.2.

A proper ideal I in a ring R is called *generalized regular* if for $0 \neq a \in I$ there exists $0 \neq b \in I$ such that ab is a nonzero idempotent, i.e., I is generalized regular as a ring without identity. It is shown in [3, Lemma 1.3] that for ideals $J \subseteq K$ of a ring R, K is regular if and only if J and K/J are both regular. So it is natural to ask whether this result is also valid for generalized regular rings. The sufficiency is true as we see later, but the answer for the necessity is negative by the following.

Example 3.4. (1) Consider the ring

$$R = \{(a_i) \in \prod_{i=1}^{\infty} E_i \mid \text{ there exists } m \geq 1 \text{ such that } a_j \in U_2(F) \text{ for all } j \geq m\},$$

where $E_i = \mathrm{Mat}_2(F)$ for all i, in Example 2.1. Set $I = \oplus_{i=1}^{\infty} E_i$. Then I is a proper ideal of R such that

$$\frac{R}{I} \cong \prod_{i=1}^{\infty} F_i \quad \text{where } F_i = U_2(F) \text{ for all } i.$$

R/I is not semiprimitive, and so this ring is not generalized regular by Lemma 2.2(2).

(2) Let R be a ring defined by

$$R = \{(a_i) \in \prod_{i=1}^{\infty} E_i \mid \text{ there exists } m \geq 1 \text{ such that } a_j \in U_2(\mathbb{Z}) \text{ for all } j \geq m\},$$

where $E_i = \mathrm{Mat}_2(\mathbb{Q})$ for all i. Then R is generalized regular by applying the proof of Theorem 2.4. Next set

$$J = \{(a_i) \in R \mid \text{ there exists } m \geq 1 \text{ such that } a_j \in \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \text{ for all } j \geq m\}.$$

Then J is a proper ideal of R such that

$$\frac{R}{J} \cong \prod_{i=1}^{\infty} F_i \quad \text{where } F_i = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix} \text{ for all } i.$$

R/J is not generalized regular as can be seen by the sequence (b_i) with $b_i = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ for all i (in fact, $(b_i)(c_i)$ cannot be a nonzero idempotent for any $(c_i) \in R/J$).

Note that $I^2 = I$ and $J^2 \subsetneq J$ in Example 3.4.

Proposition 3.5. *Let $I \subseteq K$ be ideals of a ring R.*

(1) *If K is generalized regular, then so is I.*

(2) *If I and K/I are both generalized regular, then K is generalized regular.*

Proof. (1) Suppose that K is generalized regular and let $0 \neq a \in I$. Then there exists a nonzero $b \in K$ such that ab is a nonzero idempotent of I. We apply the proof of Lemma 2.2(1). Let $z = bab \in I$. Then $z \neq 0$ and

$$(az)^2 = (az)(az) = (abab)(abab) = (ab)(ab) = az$$

is a nonzero idempotent of I, and so I is generalized regular.

(2) Assume that I and K/I are both generalized regular, and let $0 \neq a \in K$. It suffices to deal with the case of $a \notin I$. Then, from the generalized regularity of K/I, we have that $ab \notin I$ and $c = abab - ab \in I$ for some $b \in K \setminus I$. Since I is generalized regular, there exists a nonzero $p \in I$ such that cp is a nonzero idempotent. Thus $cp = a(bab - b)p$ is a nonzero idempotent for some nonzero $(bab - b)p \in K$, which means that K is generalized regular. $\square$

In the following we see a kind of condition for I under which K/I is generalized regular, where $I \subsetneq K$ are ideals of a ring R and K is generalized regular.

Proposition 3.6. *Let $I \subsetneq K$ be ideals of a ring R and suppose that K is generalized regular. Assume that $aR \cap I = 0$ whenever $a \notin I$ and a is not right invertible in R. Then K/I is generalized regular.*

Proof. Let $\bar{a} = a + I \in K/I$ be nonzero. If a is right invertible in R, then we are done. So let a be not right invertible in R. Since K is generalized regular, there exists $b \in K$ such that $abab = ab \neq 0$. From $a \notin I$, we have $ab \notin I$ by assumption, i.e., $\bar{a}\bar{b} = ab + I$ is nonzero in K/I. Moreover $(\bar{a}\bar{b})^2 = \bar{a}\bar{b}$, so K/I is generalized regular. □

Let $R = \mathbb{Z}_n$ be such that $n = p_1 p_2 \cdots p_k$ and p_i's are distinct primes. Here suppose that $p_{i_1}, p_{i_2}, \ldots, p_{i_{k-1}}$ are distinct and $\{i_1, i_2, \ldots, i_{k-1}\} \subset \{1, 2, \ldots, n\}$. Then $(p_{i_1} p_{i_2} \cdots p_{i_{k-1}})\mathbb{Z}_n$ is such an ideal of R which satisfies the assumption of Proposition 3.6.

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