

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR SCHRÖDINGER-KIRCHHOFF-TYPE EQUATIONS

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ABSTRACT. In this paper, we consider the following Schrödinger-Kirchhoff-type equations

$$\left[a + b \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right) \right] [-\Delta u + V(x)u] = f(x, u), \text{ in } \mathbb{R}^N.$$

Under certain assumptions on V and f , some new criteria on the existence and multiplicity of nontrivial solutions are established by the Morse theory with local linking and the genus properties in critical point theory. Some results from the previously literature are significantly extended and complemented.

1. Introduction and main results

In this paper, we investigate the existence and multiplicity of solutions to the following Schrödinger-Kirchhoff-type equations

$$(1.1) \quad \left[a + b \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right) \right] [-\Delta u + V(x)u] = f(x, u), \text{ in } \mathbb{R}^N,$$

where $N \geq 2$, a, b are positive constants, and the potential V satisfies the following condition:

$$(V) \quad V(x) \in C(\mathbb{R}^N, \mathbb{R}) \text{ satisfies } \inf_{\mathbb{R}^N} V(x) \geq \alpha > 0.$$

Problem (1.1) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = h(x, u),$$

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which was presented by Kirchhoff [10] to describe the transversal oscillations of a stretched string, where u denotes the displacement, h is the external force, b represents the initial tension, and a is related to the intrinsic properties of the string.

In recent years, the following Kirchhoff type problem

$$(1.2) \quad - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \right) \Delta u + \lambda V(x)u = f(x, u), \quad \text{in } \mathbb{R}^N$$

has been widely studied by many authors since Lions [13] proposed an abstract framework. More precisely, Wu in [23] studied the existence of nontrivial solutions and infinitely many high energy solutions of problem (1.2) by using a symmetric mountain pass theorem. Liu and He in [14] also investigated the existence of infinitely many high energy solutions of problem (1.2) under superlinear case by variant version of fountain theorem. In [22], Sun and Wu investigated the existence and the non-existence of nontrivial solutions of problem (1.2) by using variational methods and explored the concentration of solutions. When $N = 3$, Li and Ye considered problem (1.2) with pure power nonlinearities $f(x, u) = |u|^{p-1}u$ in \mathbb{R}^3 . By using a monotonicity trick and a new version of global compactness lemma, they get that the problem has a positive ground state solution that the result can be viewed as a partial extension of [9] where the authors studied the existence and concentration behavior of positive solutions of problem (1.2). In addition, other interesting results on the related Kirchhoff equations can be found in [1, 6, 7, 8, 16, 18, 24].

It is worth mentioning that in [12] Li et al. studied the following autonomous Kirchhoff type problem

$$(1.3) \quad \left(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b \int_{\mathbb{R}^N} u^2 dx \right) [-\Delta u + bu] = f(u), \quad \text{in } \mathbb{R}^N,$$

where $N \geq 3$, a, b are positive constants and $\lambda \geq 0$ is a parameter. They proved that problem (1.3) has at least one positive solution under the following condition:

(H_1) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $|f(u)| \leq C(|u| + |u|^{p-1})$ for all $t \in \mathbb{R}_+ = [0, \infty)$ and some $p \in (2, 2^*)$, where $2^* = 2N/(N-2)$ for $N \geq 3$.

Obviously, a question shows up that what will happen if $p \in (1, 2)$. This is what we interested in the present paper. More precisely, the case we consider in the present paper is that $f(x, u)$ satisfies the following condition.

($f1$) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist constants $1 < \delta_1 < \delta_2 < \dots < \delta_m < 2$, and functions $a_i \in L^{\frac{2}{2-\delta_i}}(\mathbb{R}^N, \mathbb{R}_+)$, ($i = 1, 2, \dots, m$) such that

$$|f(x, u)| \leq \sum_{i=1}^m \delta_i a_i(x) |u|^{\delta_i-1}.$$

The Morse theory and genus properties in critical point theory are the useful tools in looking for the solutions for the variational problem (see, for example [4, 5, 15, 17, 21]). However, to the best of our knowledge, there is no existed works

dealing with problem (1.1) by combining Morse theory and genus properties up to now. Motivated by the above facts and the main purpose of this paper is to study the existence and multiplicity of nontrivial solutions of problem (1.1). The proofs are based on combining Morse theory with local linking method and the genus properties in critical point theory.

Now, we are ready to state the main results of this paper.

Theorem 1.1. *Let the condition (V) and (f1) be satisfied and the following condition holds.*

(f2) *there exist an open set $\Omega \subset \mathbb{R}^N$ and three constants $\zeta, \eta > 0, \kappa \in (1, 2)$ such that*

$$F(x, u) \geq \eta|u|^\kappa, \quad \forall (x, u) \in \Omega \times [-\zeta, \zeta],$$

where and in the sequel $F(x, u) = \int_0^u f(x, s)ds$.

Then problem (1.1) possesses at least one nontrivial solution.

Theorem 1.2. *Let the condition (V) and (f1) be satisfied and the following condition holds.*

(f3) *there exist $1 < \tau < 2, 0 < c_1 < c_2 < \frac{a}{\varsigma\gamma_2}$ such that*

$$c_1|u|^\tau \leq F(x, u) \leq c_2|u|^2 \quad \text{for } |u| \text{ small,}$$

where $\varsigma \geq 2, \gamma_2$ is Sobolev constant.

Then problem (1.1) possesses at least two nontrivial solutions.

Theorem 1.3. *Let all the conditions in Theorem 1.1 be satisfied. In addition, the following condition holds.*

(f4) $f(x, -u) = -f(x, u), \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}$.

Then problem (1.1) possesses infinitely many nontrivial solutions.

It is easy to see that (f2) is satisfied if the following condition holds.

(f2') *there exist an open set $\Omega \subset \mathbb{R}^3$ and three constants $\zeta, \eta > 0, \kappa \in (1, 2)$ such that*

$$uf(x, u) \geq \eta|u|^\kappa, \quad \forall (x, u) \in \Omega \times [-\zeta, \zeta].$$

Therefore, by Theorems 1.1 and 1.2, we have the following corollaries.

Corollary 1.1. *In Theorem 1.1 and Theorem 1.3, if assumption (f2) is replaced by (f2'), then the conclusions still hold.*

Corollary 1.2. *Suppose that V satisfies (V) and the following conditions hold.*

(f5) $F(x, u) = q(x)G(u)$, where $G \in C^1(\mathbb{R} \times \mathbb{R})$ and $q \in C(\mathbb{R}^N \times \mathbb{R}) \cap L^{\frac{2}{2-\kappa_1}}(\mathbb{R}^N \times \mathbb{R})$, $\kappa_1 \in (1, 2)$ is a constant, such that $q(x_0) > 0$ for some $x_0 \in \mathbb{R}^N$;

(f6) *There exist constants $m, M > 0$ and $\kappa_0 \in (1, 2)$ such that*

$$m|u|^{\kappa_0} \leq G(u) \leq M|u|^{\kappa_1}, \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

Then problem (1.1) possesses at least one nontrivial solution.

Corollary 1.3. *Suppose that all the conditions in Corollary 1.1 hold. In addition, the following condition satisfies.*

$$(f7) \quad G(-u) = G(u), \quad \forall u \in \mathbb{R}.$$

Then problem (1.1) possesses infinitely many nontrivial solutions.

The sequel of this paper is organized as follows. In Section 2, some preliminary results are presented. We give the proof of our main results in Section 3. Finally, one example is given to illustrate our results.

2. Preliminaries

Throughout this paper, we work in the following Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)|u|^2] dx < +\infty \right\},$$

which equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x)uv] dx, \quad u, v \in E$$

and the associated norm

$$\|u\| = \left(\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] dx \right)^{\frac{1}{2}}, \quad u \in E.$$

As usual, for $1 \leq p < +\infty$, we let

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}, \quad u \in L^p(\mathbb{R}^N)$$

and

$$\|u\|_\infty = \text{ess sup } |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Evidently, E is continuously embedded into $L^p(\mathbb{R}^N)$ for $2 \leq p \leq 2^*$ under the condition (V) , that is, there exists $\gamma_p > 0$ such that

$$(2.1) \quad \|u\|_p \leq \gamma_p \|u\|, \quad \forall u \in E, \quad p \in [2, 2^*].$$

Definition 2.1. A function $u \in E$ is said to be a (weak) solution of (1.1) if for any $v \in E$, there holds

$$(2.2) \quad a\langle u, v \rangle + b\|u\|^2 \langle u, v \rangle = \int_{\mathbb{R}^N} f(x, u)v dx.$$

Lemma 2.1. *Assume that (V) and $(f1)$ hold. Then the functional $J(u) : E \rightarrow \mathbb{R}$ defined by*

$$(2.3) \quad J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\mathbb{R}^N} F(x, u) dx$$

is well defined and of class $C^1(E, \mathbb{R})$ and

$$(2.4) \quad \langle J'(u), v \rangle = a\langle u, v \rangle + b\|u\|^2 \langle u, v \rangle - \int_{\mathbb{R}^N} f(x, u)v dx.$$

Furthermore, the critical points of $J(u)$ in E are the solutions of (1.1).

Proof. It follows from (f1) that

$$(2.5) \quad |F(x, u)| \leq \sum_{i=1}^m a_i(x) |u|^{\delta_i}.$$

By (V), (2.5) and the Hölder inequality, for any $u \in E$, we get that

$$(2.6) \quad \begin{aligned} \int_{\mathbb{R}^N} |F(x, u)| dx &\leq \sum_{i=1}^m \int_{\mathbb{R}^N} a_i(x) |u|^{\delta_i} dx \\ &\leq \sum_{i=1}^m \int_{\mathbb{R}^N} \left(\frac{V(x)}{\alpha}\right)^{\frac{\delta_i}{2}} a_i(x) |u|^{\delta_i} dx \\ &\leq \sum_{i=1}^m \alpha^{-\frac{\delta_i}{2}} \|a_i(x)\|_{\frac{2}{2-\delta_i}} \left(\int_{\mathbb{R}^N} V(x) |u|^2 dx\right)^{\frac{\delta_i}{2}} \\ &\leq C_1 \sum_{i=1}^m \|u\|^{\delta_i}. \end{aligned}$$

Thus, $J(u)$ is well defined on E by (2.3) and (2.6).

Now, we show that (2.4) holds. By (f1), for any $u, v \in E$, $l \in (0, 1)$, $\theta(x) : \mathbb{R}^N \rightarrow (0, 1)$ and the Hölder inequality, we obtain that

$$(2.7) \quad \begin{aligned} &\int_{\mathbb{R}^N} \max_{l \in (0,1)} |f(x, u(x) + l\theta(x)v(x))v(x)| dx \\ &= \int_{\mathbb{R}^N} \max_{l \in (0,1)} |f(x, u(x) + l\theta(x)v(x))| |v(x)| dx \\ &\leq \sum_{i=1}^m \delta_i \int_{\mathbb{R}^N} a_i(x) |u(x) + \theta(x)v(x)|^{\delta_i-1} |v(x)| dx \\ &\leq C_2 \sum_{i=1}^m \left(\int_{\mathbb{R}^N} |a_i|^{\frac{2}{2-\delta_i}} dx\right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} V(x) |u(x)|^2 dx\right)^{\frac{\delta_i-1}{2}} \left(\int_{\mathbb{R}^N} V(x) |v(x)|^2 dx\right)^{\frac{1}{2}} \\ &\quad + C_2 \sum_{i=1}^m \left(\int_{\mathbb{R}^N} |a_i|^{\frac{2}{2-\delta_i}} dx\right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} V(x) |v(x)|^2 dx\right)^{\frac{\delta_i}{2}} \\ &\leq C_2 \sum_{i=1}^m \|a_i\|_{\frac{2}{2-\delta_i}} (\|u\|^{\delta_i-1} + \|v\|^{\delta_i-1}) \|v\| < +\infty. \end{aligned}$$

Then by (2.3), (2.7) and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} \langle J'(u), v \rangle &= \lim_{l \rightarrow 0^+} \frac{J(u + lv) - J(u)}{l} \\ &= \left[a + b \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) |u|^2) dx \right) \right] \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x) uv] dx \end{aligned}$$

$$\begin{aligned}
& - \lim_{l \rightarrow 0^+} \frac{1}{l} \int_{\mathbb{R}^N} [F(x, u(x) + lv(x)) - F(x, u(x))] dx \\
= & \left[a + b \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right) \right] \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x)uv] dx \\
& - \lim_{l \rightarrow 0^+} \int_{\mathbb{R}^N} f(x, u(x) + \theta(x)lv(x))v(x) dx \\
= & \left[a + b \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \right) \right] \int_{\mathbb{R}^N} [\nabla u \nabla v + V(x)uv] dx \\
& - \int_{\mathbb{R}^N} f(x, u(x))v(x) dx,
\end{aligned}$$

which implies that (2.4) holds. Moreover, by a standard argument, it is easy to show that the critical points of $J(u)$ in E are the solutions of problem (1.1), see for example [3].

In what follows, we show that $J'(u)$ is continuous. According to (2.3), it suffices to show that

$$(2.8) \quad \Phi' \triangleq \int_{\mathbb{R}^N} f(x, u)v dx$$

is continuous. Let $u_k \rightarrow u$ in E . Then

$$(2.9) \quad u_k \rightarrow u, \quad \text{in } L^2(\mathbb{R}^N), \quad u_k \rightarrow u \quad \text{a.e. in } \mathbb{R}^N.$$

We show that

$$(2.10) \quad \int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2 dx \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

To prove (2.10), arguing by contradiction, suppose that there exists a constant $\varepsilon_0 > 0$ and a subsequence (also denotes $\{u_k\}$) such that

$$(2.11) \quad \int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2 dx \geq \varepsilon_0, \quad \text{as } k \rightarrow +\infty.$$

Since $u_k \rightarrow u$ in $L^2(\mathbb{R}^N)$, passing to a subsequence if necessary we can assume that $\sum_{i=1}^{\infty} \|u_k - u\|_2^2 < +\infty$. Set $w(x) = [\sum_{i=1}^{\infty} |u_k - u|^2]^{\frac{1}{2}}$, $x \in \mathbb{R}^N$, then $w(x) \in L^2(\mathbb{R}^N)$. For all $k \in \mathbb{N}$ and by (f1), we have

$$\begin{aligned}
(2.12) \quad |f(x, u_k(x)) - f(x, u(x))|^2 & \leq 2|f(x, u_k(x))|^2 + 2|f(x, u(x))|^2 \\
& \leq C_3 \sum_{i=1}^m |a_i(x)|^2 [|u_k(x)|^{2\delta_i-2} + |u(x)|^{2\delta_i-2}] \\
& \leq C_3 \sum_{i=1}^m |a_i(x)|^2 [|w(x)|^{2\delta_i-2} + |u(x)|^{2\delta_i-2}] \\
& := g(x), \quad \text{a.e. in } \mathbb{R}^N
\end{aligned}$$

and

$$(2.13) \quad \int_{\mathbb{R}^N} g(x)dx = C_3 \sum_{i=1}^m \int_{\mathbb{R}^N} |a_i(x)|^2 [|w(x)|^{2\delta_i-2} + |u(x)|^{2\delta_i-2}] dx \\ \leq C_3 \sum_{i=1}^m \|a_i(x)\|_{\frac{2}{2-\delta_i}}^2 [\|w(x)\|_2^{2\delta_i-2} + \|u(x)\|_2^{2\delta_i-2}] < +\infty.$$

Then by (2.9), (2.12), (2.13) and Lebesgue's dominated convergence theorem, we have

$$(2.14) \quad \int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2 dx \rightarrow 0, \quad \text{as } k \rightarrow +\infty,$$

which contradicts (2.11). So, (2.10) holds.

By (2.8), (2.14) and the Hölder inequality, for all given $v \in E$, we have

$$|\langle \Phi'(u_k) - \Phi'(u), v \rangle| = \left| \int_{\mathbb{R}^N} [f(x, u_k) - f(x, u)] v dx \right| \\ \leq C_4 \|v\| \left(\int_{\mathbb{R}^N} |f(x, u_k) - f(x, u)|^2 dx \right)^{\frac{1}{2}} \rightarrow 0,$$

as $k \rightarrow +\infty$, which implies the continuity of Φ' . Hence, $J \in C^1(E, \mathbb{R})$. \square

In what follows, we collect some definitions and propositions which are very useful throughout the present paper and we will use in the next section.

Let E be a Banach space, $J(u) \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. Set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$

$$K_c = \{u \in E : J(u) = c, J'(u) = 0\}, \quad J^c = \{u \in E : J(u) \leq c\}.$$

Definition 2.2 (Chang [4]). Let u be an isolated critical point of J with $J(u) = c$, for $c \in \mathbb{R}$, and let U be a neighborhood of u , containing the unique critical point. We call

$$C_q(J, u) := H_q(J^c \cap U, J^c \cap U \setminus \{u\}), \quad q = 0, 1, 2, \dots,$$

the q th critical group of J at u , where $J^c := \{u \in E : J(u) \leq c\}$, $H_q(\cdot, \cdot)$ stands for the q th singular relative homology group with integer coefficients.

We say that u is a homological nontrivial critical point of J if at least one of its critical groups is nontrivial.

Proposition 2.1 (Bartsch and Liu [2]). *Let 0 be a critical point of J with $J(0) = 0$. Assume that J has a local linking at 0 with respect to $E = E_1 \oplus E_2$, $k = \dim E_1 < \infty$, that is, there exists $\rho > 0$ small such that*

$$J(u) \leq 0, \quad u \in E_1, \|u\| \leq \rho \quad \text{and} \quad J(u) > 0, \quad u \in E_2, 0 < \|u\| \leq \rho.$$

Then $C_k(J, 0) \not\cong 0$, that is, 0 is a homological nontrivial critical point of J .

Definition 2.3 ([19]). For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^N \setminus \{0\})$ and n is the smallest integer with this property.

We say that $J \in C^1(E, \mathbb{R})$ satisfies *(PS)*-condition if any sequence $\{u_n\}$ in E such that

$$(2.15) \quad J(u_n) \rightarrow c, \quad J'(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

has a convergent subsequence.

Proposition 2.2. *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the *(PS)*-condition. If J is bounded from below, then $c = \inf_E J$ is a critical value of J .*

Proposition 2.3 (Zhang and Li [25]). *Assume that J satisfies the *(PS)*-condition and is bounded from below. If J has a critical point that is homological nontrivial and is not the minimizer of J . Then J has at least three critical points.*

Proposition 2.4 ([20]). *Let J be an even C^1 functional on E and satisfy the *(PS)*-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\}, \quad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} J(u).$$

- (1) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of J ;*
- (2) *If there exists $r \in \mathbb{N}$ such that $c_n = c_{n+1} = \dots = c_{n+r} = c \in \mathbb{R}$, and $c \neq J(0)$, then $\gamma(K_c) \geq r + 1$.*

3. Proof of main results

We begin this section by verifying the follow compactness lemma which shows that the functional J satisfies *(PS)*-condition and is bounded from below.

Lemma 3.1. *Assume that the conditions (V) and (f1) hold, then J is bounded from below and satisfies the *(PS)*-condition.*

Proof. First, we show that J is bounded from below. By (2.3), (f1) and the Hölder inequality, one has

(3.1)

$$\begin{aligned} J(u) &= \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \int_{\mathbb{R}^N} F(x, u) dx \\ &\geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \sum_{i=1}^m \int_{\mathbb{R}^N} a_i(x) |u|^{\delta_i} dx \\ &\geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - C_5 \sum_{i=1}^m \left(\int_{\mathbb{R}^N} |a_i(x)|^{\frac{2}{2-\delta_i}} dx \right)^{\frac{2-\delta_i}{2}} \left(\int_{\mathbb{R}^N} V(x) |u|^2 dx \right)^{\frac{\delta_i}{2}} \end{aligned}$$

$$\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_5 \sum_{i=1}^m \|a_i\|_{\frac{2}{2-\delta_i}} \|u\|^{\delta_i}.$$

Since $1 < \delta_i < 2$ ($i = 1, 2, \dots, m$), (3.1) implies that $J(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$. Consequently, J is bounded from below.

Next, we prove that J satisfies the (PS) -condition. Assume that $\{u_n\}_{n \in \mathbb{N}} \subset E$ is a sequence such that (2.15) holds. Then by (3.1), there exists a constant $M > 0$ such that

$$(3.2) \quad \|u_n\| \leq M, \quad \forall n \in \mathbb{N}.$$

Going if necessary to a subsequence we can assume that $u_n \rightharpoonup u_0$ in E . Hence, by Rellich embedding theorem, we have

$$(3.3) \quad u_n \rightarrow u_0, \text{ in } L^s_{loc}(\mathbb{R}^N), \quad s \in [2, 2^*].$$

By (f1), for any given $\varepsilon > 0$, we can choose $\rho > 0$ such that

$$(3.4) \quad \left(\int_{|x| \geq \rho} |a_i|^{\frac{2}{2-\delta_i}} dx \right)^{\frac{2-\delta_i}{2}} < \varepsilon.$$

It follows from (2.12), (2.13), (3.2), (3.3) and the Hölder inequality that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} (3.5) \quad & \int_{|x| \leq \rho} |f(x, u_n) - f(x, u_0)| |u_n - u_0| dx \\ & \leq \left(\int_{|x| \leq \rho} |f(x, u_n) - f(x, u_0)|^2 dx \right)^{\frac{1}{2}} \left(\int_{|x| \leq \rho} |u_n - u_0|^2 dx \right)^{\frac{1}{2}} \\ & \leq \varepsilon \left(\int_{|x| \leq \rho} |f(x, u_n) - f(x, u_0)|^2 dx \right)^{\frac{1}{2}} \\ & \leq C_6 \varepsilon \left[\sum_{i=1}^m (\|u_n\|_2^{2\delta_i-2} + \|u_0\|_2^{2\delta_i-2}) \right]^{\frac{1}{2}} \\ & \leq C_7 \varepsilon \left[\sum_{i=1}^m (M^{2\delta_i-2} + \|u_0\|_2^{2\delta_i-2}) \right]^{\frac{1}{2}} \\ & \leq C_8 \varepsilon \end{aligned}$$

for $n \geq n_0$.

On the other hand, it follows from (2.1), (3.4) and the Hölder inequality that

$$\begin{aligned} (3.6) \quad & \int_{|x| > \rho} |f(x, u_n) - f(x, u_0)| |u_n - u_0| dx \\ & \leq \int_{|x| > \rho} [|f(x, u_n)| + |f(x, u_0)|] (|u_n - u_0|) dx \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=1}^m \delta_i \int_{|x|>\rho} |a_i(x)| [|u_n|^{\delta_i} + |u_0|^{\delta_i} + |u_n|^{\delta_i-1}|u_0| + |u_0|^{\delta_i-1}|u_n|] dx \\
&\leq \varepsilon \sum_{i=1}^m \delta_i (\|u_n\|_2^{\delta_i} + \|u_0\|_2^{\delta_i} + \|u_n\|_2^{\delta_i-1} \|u_0\|_2 + \|u_0\|_2^{\delta_i-1} \|u_n\|_2) \\
&\leq C_9 \varepsilon.
\end{aligned}$$

Since ε is arbitrary, combining (3.5) and (3.6) we get

$$(3.7) \quad \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u_0)| |u_n - u_0| dx \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

It follows from (2.4) that

$$\begin{aligned}
(3.8) \quad &\min\{b\|u_n\|^2 + a, b\|u_0\|^2 + a\} \|u_n - u_0\|^2 \\
&\leq \langle J'(u_n) - J'(u_0), u_n - u_0 \rangle + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_0)](u_n - u_0) dx.
\end{aligned}$$

Obviously,

$$(3.9) \quad \langle J'(u_n) - J'(u_0), u_n - u_0 \rangle \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Combining (3.7), (3.8) and (3.9), we obtain that $u_n \rightarrow u_0$ in E . Hence, J satisfies (PS) -condition. The proof is complete. \square

Now, we are in the position to give the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.1. In view of Lemma 2.1, $J \in C^1(E, \mathbb{R})$. By Lemma 3.1 and Proposition 2.1, we obtain $c = \inf_E J(u)$ is a critical value of J , that is, there exists a critical point $u^* \in E$ such that $J(u^*) = c$.

Now, we show that $u^* \neq 0$. Let $u_0 \in (W_0^{1,2}(\Omega) \cap E) \setminus \{0\}$. Then by (2.3) and (f2), we infer that

$$\begin{aligned}
(3.10) \quad J(tu_0) &= \frac{at^2}{2} \|u_0\|^2 + \frac{\lambda t^4}{4} \|u_0\|^4 - \int_{\mathbb{R}^N} F(x, tu_0) dx \\
&\leq \frac{at^2}{2} \|u_0\|^2 + \frac{\lambda t^4}{4} \|u_0\|^4 - \eta t^\kappa \int_{\Omega} |u_0(x)|^\kappa dx.
\end{aligned}$$

Since $1 < \kappa < 2$, it follows from (3.10) that $J(tu_0) < 0$ for $t > 0$ small enough. Thus, we get that $J(u^*) = c < 0$. Therefore, u^* is a nontrivial critical point of J with $J(u^*) = \inf_E J(u)$ and is a nontrivial solution of problem (1.1). The proof is complete. \square

Proof of Theorem 1.2. Lemma 3.1 shows that J satisfies (PS) condition and is bounded from below. From (2.4) and the definition of $F(x, u)$, it is easy to check that $J(0) = 0$. In what follows, we show that 0 is a homological nontrivial critical point of J . Since E is a Hilbert space, we choose an orthogonal basis $\{e_j\}$ of E and let $E = E^- \oplus E^+$, where

$$(3.11) \quad E^- := \text{span}\{e_1, \dots, e_k\} \quad \text{and} \quad E^+ := (E^-)^\perp.$$

On one hand, for $u \in E^-$, we have from (2.4) and (f3) that

$$\begin{aligned}
 J(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\mathbb{R}^N} F(x, u)dx \\
 (3.12) \quad &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_1 \int_{\mathbb{R}^N} u^\tau dx \\
 &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_1\|u\|_\tau^\tau.
 \end{aligned}$$

Since all norms on a finite dimensional space are equivalent, we deduce from (3.12) that

$$J(u) \leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_1\|u\|^\tau,$$

which implies that $J(u) \leq 0$ if we choose $\|u\|$ small enough since $1 < \tau < 2$.

On the other hand, for $u \in E^+$, from (2.1), (2.4) and (f3) we have

$$\begin{aligned}
 J(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_{\mathbb{R}^N} F(x, u)dx \\
 (3.13) \quad &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_2 \int_{\mathbb{R}^N} u^2 dx \\
 &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - c_2\gamma_2^2\|u\|^2 \\
 &\geq C_9\|u\|^2 + \frac{b}{4}\|u\|^4,
 \end{aligned}$$

which means that $J(u) \leq 0$ if we choose $\|u\|$ small enough.

From (2.4) and the definition of E^- , it is easy to see that $J(0) = 0$ and $k = \dim E^- < \infty$. Therefore, combining the above arguments and Proposition 2.1, we have $C_k(J, 0) \not\cong 0$. This means that 0 is an homological nontrivial critical point of J . Moreover, (3.12) shows that 0 is not the minimizer of J . Thus all the conditions of Proposition 2.3 are satisfied and we get that the problem (1.1) has two nontrivial solutions. We complete the proof. \square

Proof of Theorem 1.3. By Lemma 2.1 and Lemma 3.1, we get that $J \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the (PS)-condition. It follows from (2.3) and (f3) that J is even and $J(0) = 0$.

Now, we prove that for any $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that

$$(3.14) \quad \gamma(J^{-\varepsilon}) \geq n.$$

For any $n \in \mathbb{N}$, we take n disjoint open sets Ω_i such that

$$\bigcup_{i=1}^n \Omega_i \subset \Omega.$$

For each $i \in \{1, 2, \dots, n\}$, let $u_i \in (W_0^{1,2}(\Omega_i) \cap E) \setminus \{0\}$ and $\|u_i\| = 1$, and

$$E_n = \text{span}\{u_1, u_2, \dots, u_n\}, \quad S_n = \{u \in E_n : \|u\| = 1\}.$$

Thus, for any $u \in E_n$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, n$ such that

$$(3.15) \quad u(x) = \sum_{i=1}^n \lambda_i u_i \quad \text{for } x \in \mathbb{R}^N.$$

Then we have

$$(3.16) \quad \|u\|_\kappa = \left(\int_{\mathbb{R}^N} |u(x)|^\kappa dx \right)^{\frac{1}{\kappa}} = \left(\sum_{i=1}^n |\lambda_i|^\kappa \int_{\Omega_i} |u_i(x)|^\kappa dx \right)^{\frac{1}{\kappa}}$$

and

$$(3.17) \quad \begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^N} [|\nabla u(x)|^2 + V(x)|u(x)|^2] dx \\ &= \sum_{i=1}^n \lambda_i^2 \int_{\Omega_i} [|\nabla u_i(x)|^2 + V(x)|u_i(x)|^2] dx \\ &= \sum_{i=1}^n \lambda_i^2 \int_{\mathbb{R}^N} [|\nabla u_i(x)|^2 + V(x)|u_i(x)|^2] dx \\ &= \sum_{i=1}^n \lambda_i^2 \|u_i(x)\|^2 \\ &= \sum_{i=1}^n \lambda_i^2. \end{aligned}$$

Since E_n is a finite dimensional normed space, one gets that all norms of E_n are equivalent. Therefore, there is a constant $\vartheta > 0$ such that

$$(3.18) \quad \vartheta \|u\| \leq \|u\|_\kappa \quad \text{for } u \in E_n.$$

By (f3), (3.16), (3.17) and (3.18), we have

$$(3.19) \quad \begin{aligned} J(tu) &= \frac{at^2}{2} \|u\|^2 + \frac{\lambda t^4}{4} \|u\|^4 - \int_{\mathbb{R}^N} F(x, tu) dx \\ &= \frac{at^2}{2} \|u\|^2 + \frac{\lambda t^4}{4} \|u\|^4 - \sum_{i=1}^n \int_{\Omega_i} F(x, t\lambda_i u_i(x)) dx \\ &\leq \frac{at^2}{2} \|u\|^2 + \frac{\lambda t^4}{4} \|u\|^4 - \eta t^\kappa \sum_{i=1}^n |\lambda_i|^\kappa \int_{\Omega_i} |u_i(x)|^\kappa dx \\ &\leq \frac{at^2}{2} \|u\|^2 + \frac{\lambda t^4}{4} \|u\|^4 - \eta t^\kappa \|u\|_\kappa^\kappa \\ &\leq \frac{at^2}{2} \|u\|^2 + \frac{\lambda t^4}{4} \|u\|^4 - \eta \vartheta^\kappa t^\kappa \|u\|^\kappa \\ &= \frac{a}{2} t^2 + \frac{\lambda}{4} t^4 - \eta \vartheta^\kappa t^\kappa, \quad \forall u \in S_n. \end{aligned}$$

So for t small enough, (3.19) implies that there exist $\varepsilon > 0$ and $\tau > 0$ such that

$$(3.20) \quad J(\tau u) < -\varepsilon \quad \text{for } u \in S_n.$$

Let

$$S_n^\tau = \{\tau u : u \in S_n\}, \quad Q = \{(\lambda_1, \lambda_2, \dots, \lambda_n) : \sum_{i=1}^n \lambda_i^2 < \tau^2\}.$$

It follows from (3.20) that $J(u) < -\varepsilon$ for $u \in S_n^\tau$, which, together with the fact that $J \in C^1(E, \mathbb{R})$ and J is even, implies that

$$S_n^\tau \subset J^{-\varepsilon} \in \Sigma.$$

On the other hand, from (3.15) and (3.17), there exists an odd homeomorphism mapping $\psi \in C(S_n^\tau, \partial Q)$. By the monotonicity of genus and Proposition 7.7 in [19], we deduce

$$(3.21) \quad \gamma(J^{-\varepsilon}) \geq \gamma(S_n^\tau) = n.$$

Thus, (3.14) holds.

Set

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} J(u).$$

It follows from (3.21) and the fact that J is bounded from below on E that $-\infty < c_n \leq -\varepsilon < 0$, that is, for any $n \in \mathbb{N}$, c_n is a real negative number.

Therefore, J has infinitely many nontrivial critical points by Proposition 2.2. So, problem (1.1) possesses infinitely many nontrivial solutions. \square

4. Example

In this section, an example is given to illustrate our main results.

Example 4.1. In problem (1.1), let $V(x) = 3 + \sin|x|^2$ and let $\alpha = 1$. Then, condition (V) satisfies. For $\forall(x, u) \in \mathbb{R}^N \times \mathbb{R}$, let

$$f(x, u) = \frac{1 + \sin^2|x|}{1 + |x|^{\frac{3}{2}}} |u|^{\frac{-3}{4}} u + \frac{4}{3e^{|x|}} |u|^{\frac{-2}{3}} u + \frac{3 \cos|x|}{2e^3|x|} |u|^{\frac{-1}{2}} u.$$

Clearly,

$$|f(x, u)| \leq \frac{2}{1 + |x|^{\frac{3}{2}}} |u|^{\frac{1}{4}} + \frac{4}{3e^{|x|}} |u|^{\frac{1}{3}} + \frac{3}{2e^3|x|} |u|^{\frac{1}{2}}$$

and

$$\begin{aligned} F(x, u) &= \frac{1 + \sin^2|x|}{1 + |x|^{\frac{3}{2}}} \frac{4}{5} |u|^{\frac{5}{4}} u + \frac{4}{e^{|x|}} |u|^{\frac{4}{3}} + \frac{\cos|x|}{e^3|x|} |u|^{\frac{3}{2}} \\ &\geq \frac{1}{e^{|x|}} |u|^{\frac{4}{3}}, \quad \forall(x, u) \in B_{\frac{\pi}{2}} \times [-1, 1]. \end{aligned}$$

Here, we can choose a large enough such that $0 < c_1 < c_2 < \frac{a}{\varsigma \gamma_2^2}$ holds, where $c_1 = \eta = \frac{1}{e^{|x|}}$, $c_2 = 6 \max\{a_1, a_2, a_3\}$. Thus (f1), (f2), (f3) and (f4) are satisfied with

$$\delta_1 = \frac{1}{4}, \quad \delta_2 = \frac{1}{3}, \quad \delta_3 = \frac{1}{2},$$

$$a_1 = \frac{8}{5(1 + |x|^{\frac{3}{2}})}, \quad a_2 = \frac{4}{3e^{|x|}}, \quad a_3 = \frac{3}{2e^3|x|},$$

$$\zeta = 1, \quad \Omega = B_{\frac{\pi}{2}}, \quad \eta = \frac{1}{e^{|x|}}, \quad \kappa = \frac{4}{3}.$$

By Theorem 1.1, Theorem 1.2 and Theorem 1.3, problem (1.1) has at least one, two and infinitely many nontrivial solutions.

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