A LINK BETWEEN ORDERED TREES AND
GREEN-RED TREES

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ABSTRACT. The r-ary number sequences given by

$$(\mathfrak{b}_n^{(r)})_{n \geq 0} = \frac{1}{(r-1)n+1}\binom{rn}{n}$$

are analogs of the sequence of the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. Their history goes back at least to Lambert [8] in 1758 and they are of considerable interest in sequential testing. Usually, the sequences are considered separately and the generalizations can go in several directions. Here we link the various r first by introducing a new combinatorial structure related to GR trees and then algebraically as well. This GR transition generalizes to give r-ary analogs of many sequences of combinatorial interest. It also lets us find infinite numbers of combinatorially defined sequences that lie between the Catalan numbers and the Ternary numbers, or more generally, between $\mathfrak{b}_n^{(r)}$ and $\mathfrak{b}_n^{(r+1)}$.

1. Introduction

The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are well known to occur in enumeration of a great variety of combinatorial objects. Some good resources for the Catalan numbers are provided by Stanley [15, 14], which include more than 200 combinatorial interpretations.

One of the important generalizations of the Catalan numbers are the r-ary numbers $\mathfrak{b}_n^{(r)} = \frac{1}{(r-1)n+1}\binom{rn}{n}$ as r varies. Heubach, Li and Mansour [7] provided a list of currently known and new combinatorial structures enumerated by the r-ary numbers. In particular, the r-ary number $\mathfrak{b}_n^{(r)}$ counts the r-ary trees with rn edges defined as ordered trees in which every vertex has either 0 or

r children. In terms of generating functions this means that the generating function $\mathfrak{B}_r(z) = \sum_{n\geq 0} \mathfrak{b}_n^{(r)} z^n$ satisfies the functional equation:

$$(1) \qquad\qquad \mathfrak{B}_r(z) = 1 + z\mathfrak{B}_r^r(z).$$

Sometimes, $\mathfrak{B}_r(z)$ is called the *generalized binomial series* and was investigated as early as 1758 by J. Lambert [8]. It can be shown [6, pp. 200–203, 358–363] that the following identity is valid for all real numbers m:

$$(2) \qquad \mathfrak{B}_r^m(z) := \sum_{n\geq 0} \mathfrak{b}_{n,m}^{(r)} = \sum_{n\geq 0} \frac{m}{nr+m}\binom{nr+m}{n} z^n.$$

For brevity, we shorten $f(z)$ to f and define the coefficient extraction operator $[z^n]$ by $[z^n]f = f_n$ where $f = \sum_{n\geq 0} f_n z^n$.

It is convenient to consider the array called the *r-ary triangle* as an infinite lower triangular array which we denote by $\Delta_r = [d_{i,j}^{(r)}]_{i,j\geq 0}$ where

$$d_{i,j}^{(r)} = [z^i]z^j\mathfrak{B}_r^{j+1} = [z^{i-j}]\mathfrak{B}_r^{j+1}.$$

It follows from (2) that

$$d_{i,j}^{(r)} = \frac{j+1}{(i-j)r+j+1}\binom{(i-j)r+j+1}{i-j}.$$

Here is an example:

$$\Delta_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 12 & 7 & 3 & 1 & 0 & 0 & \cdots \\ 55 & 30 & 12 & 4 & 1 & 0 \\ 273 & 143 & 55 & 18 & 5 & 1 \\ & & & \cdots & & \end{bmatrix}.$$

The formation rule of the array leads to Riordan arrays as introduced by Shapiro, Getu, Woan and Woodson [11] (see also Sprugnoli [13]). A *Riordan array* is an infinite lower triangular array such that the generating function of the jth column for $j = 0, 1, 2, \ldots$ is $g(z)f(z)^j$ where $g = 1 + g_1 z + g_2 z^2 + \cdots$ and $f = f_1 z + f_2 z^2 + \cdots$, $f_1 \neq 0$. All computations are in the ring of formal power series. As usual, we use the pair form notation $(g(z), f(z))$ or simply (g, f) for the Riordan array. It is easy to show that given a Riordan array $M = (g, f)$ and column vector $\mathbf{c} = [c_0, c_1, \ldots]^T$ with the generating function $\phi(z)$ then $M\mathbf{c}$ gives a column vector whose generating function is $g\cdot(\phi\circ f)$ where $\circ$ denotes the composition of functions. We call this property the *fundamental theorem* of Riordan arrays (FTRA). One may express the FTRA as $(g, f)\phi = g \cdot (\phi \circ f)$. This leads to the matrix multiplication of Riordan arrays which can be described in terms of generating functions as $(g, f)(G, F) = (g\cdot(G\circ f), F\circ f)$. The set of all Riordan arrays forms the *Riordan group*. The identity element

is $(1, z)$ and the inverse of (g, f) is $(1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of f.

Since the generating function of the jth column of Δ_r is $z^j \mathfrak{B}_r^{j+1}$, we conclude that Δ_r is a Riordan array given by $(\mathfrak{B}_r, z\mathfrak{B}_r)$. We note that $\Delta_1 = (1/(1 - z), z/(1-z))$ and $\Delta_2 = (C, zC)$ are the Pascal triangle and the Catalan triangle, respectively.

The purpose of this paper is to link the r-ary triangles as r varies by giving a new combinatorial setting by means of some edge-colored trees. Specifically in Section 2, we define an edge-coloring of ordered trees, and observe a frame structure of the edge-colored tree which we call an r-ary core. In Section 3, various combinatorial sequences are developed giving an infinite number of sequences interpolating between $C_n := \mathfrak{b}_n^{(2)} = \frac{1}{n+1}\binom{2n}{n}$ and $\mathfrak{b}_{n,2}^{(3)} = \frac{1}{n+1}\binom{3n+1}{n}$ or $\mathfrak{b}_n^{(3)} = \frac{1}{2n+1}\binom{3n}{n}$ as an example. More generally, we explore the road from $C := \mathfrak{B}_2$ to $\mathfrak{B}_{k+1}^k$ by means of edge-colored trees. Finally in Section 4, we use the same technique to set up r-ary versions of many combinatorial sequences expressible as weighted trees. Two examples are the Motzkin trees [2, 4], and the Fibonacci trees [3].

2. The r-ary core and the link theorems

In this section, we introduce the r-ary core of an edge-colored tree which leads to combinatorial proofs of the link theorems. We begin with the definition of *green-red trees*.

A *green-red tree* or *GR tree* is an ordered tree where each edge is green or red subject to two conditions:

(i) All edges connected to the root are green.

(ii) All the edges emanating out from any non-root vertex are either green or red but all the red edges are to the right of all the green edges. They can be all red or all green.

Throughout Sections 2 and 3, we assume that the GR variation is applied to the usual ordered trees. In the related generating functions, an edge contributes a z.

There are many situations where GR trees might occur. These would be where we want to allow two states but only one toggle per generation. For instance, a family could originate in a home country but then move to a new country. The children might remain in the new country or might return home. In a different context we could be thinking about a reversible mutation [10, pp. 83–86].

We first observe the construction of GR trees. Every GR tree is formed from a framework called the *Catalan core*. The Catalan core of a GR tree is the largest subtree, including the root, with all green edges. At each non-root vertex, v, in the core there is a subtree that is also a GR tree except that now all the new edges coming out of v are red. Each of the subtrees can be trivial. Let T be the generating function for the number of GR trees. If the tree has n

edges of which k are in the core then the remaining $n - k$ edges are included in these k subtrees. Since there are C_k Catalan cores with k non root vertices, expressed as generating functions this term contributes $z^k C_k T^k$ and summing over k yields

$$T = \sum_{k \geq 0} C_k(zT)^k = C \circ zT.$$

It then follows from $C = 1 + zC^2$ that

$$T = C \circ zT = 1 + zT \cdot (C \circ zT)^2 = 1 + zT^3.$$

Since $\mathfrak{B}_3 = 1 + z\mathfrak{B}_3^3$ it follows that $T = \mathfrak{B}_3 = \sum_{n \geq 0} \frac{1}{2n+1}\binom{3n}{n} z^n$.

If we introduce three colors, say green, red and blue, again ordered from left to right, (green, then red, then blue), similar ideas apply. Let Q be the generating function for the number of GRB trees. The ternary core of a GRB tree is the largest GR subtree including the root. See Figure 1 for example.
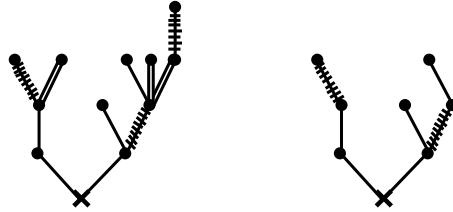

FIGURE 1. A GRB tree and its ternary core.
(——— : Green edge, ⧟⧟⧟⧟ : Red edge, ═══ : Blue edge)

Each non root vertex, v in the ternary core, is a possible root for a GRB subtree but with all blue edges from v. In Riordan array terms this can be written as $(1, zQ)T = Q$:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 4 & 2 & 1 & 0 \\
0 & 22 & 9 & 3 & 1 \\
& & \cdots &
\end{bmatrix}
\begin{bmatrix}
1 \\ 1 \\ 3 \\ 12 \\ 55 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 1 \\ 4 \\ 22 \\ 140 \\ \vdots
\end{bmatrix}.
$$

By the FTRA together with $T = 1 + zT^3$, we have

$$Q = T \circ zQ = 1 + zQ \cdot (T \circ zQ)^3 = 1 + zQ^4$$

which implies $Q = \mathfrak{B}_4$.

The extension to p colors works along the same lines. More generally, we define a $GR_1 \cdots R_{p-1}$-tree or simply p-colored tree ($p \geq 2$) to be an ordered tree whose edges are colored from p colors $G, R_1, \ldots, R_{p-1}$ ordered from left to right, while all edges connected to the root are colored by green G. Let T_p be the generating function for the number of p-colored trees, $p \geq 2$. The

$(p-k)$-*ary core*[1] of an p-colored tree $(p > k \geq 1)$ is defined as the largest $(p-k)$-colored subtree including the root. Clearly, every p-colored tree has an $(p-1)$-ary core. Conversely attachment of p-colored trees but with all R_{p-1} edges at the root to each non root vertex of a $(p-1)$-ary core gives a p-colored tree. Thus we have $(1, zT_p)T_{p-1} = T_p$. Since we have shown that $T_2 = \mathfrak{B}_3$, it follows inductively from (1) that $T_p = 1 + zT_p^{p+1}$, which implies $T_p = \mathfrak{B}_{p+1}$. This argument provides a combinatorial proof of the following theorem.

Theorem 2.1 (Little link theorem, [6]). *Let $\mathfrak{B}_r$ be the generating function for r-ary numbers. For any $r \in \mathbb{Z}$, we have*

(3) $$\mathfrak{B}_r \circ z\mathfrak{B}_{r+1} = \mathfrak{B}_{r+1}.$$

We note that the little link theorem can be also proved combinatorially by using the concept of supertrees [5, pp. 412–414]. The GR version stays in two dimensions and seems easier to visualize.

By the FTRA, the equation (3) is equivalent to $(1, z\mathfrak{B}_{r+1})\mathfrak{B}_r = \mathfrak{B}_{r+1}$. More generally, we have:

Theorem 2.2 (Big link theorem). *Let $\mathfrak{B}_r$ be the generating function for r-ary numbers. For any $r, m \in \mathbb{Z}$, we have*

(4) $$\mathfrak{B}_r \circ z\mathfrak{B}_{r+m}^m = \mathfrak{B}_{r+m}.$$

Proof. By (1),

$$\mathfrak{B}_{r+m} = 1 + z\mathfrak{B}_{r+m}^{r+m} = 1 + z\mathfrak{B}_{r+m}^m \mathfrak{B}_{r+m}^r.$$

Let $Y = \mathfrak{B}_{r+m}$ and $\hat{z} = z\mathfrak{B}_{r+m}^m$ so that $Y = 1 + \hat{z}Y^r$. Since we are in the ring of formal power series, we have $Y = \mathfrak{B}_r(\hat{z})$, which implies (4). $\square$

A combinatorial proof of Theorem 2.2 will be given in the next section by generalizing slightly the road from C to T and then generalizing to more colors. For notational convenience we still use C, T, and Q for T_2, T_3, and T_4 respectively.

We end this section by noting an inverse identity obtained from the little link theorem.

Corollary 2.3. *The compositional inverse of $\mathfrak{B}_r$ can be expressed as $(2-r)$-ary numbers as follows*:

$$\overline{z\mathfrak{B}_r} = z\mathfrak{B}_{2-r}(-z).$$

Proof. Let $\mathcal{F}_r = z\mathfrak{B}_r$. It follows from (3) that $\mathfrak{B}_{r-1} \circ \mathcal{F}_r = \mathcal{F}_r/z$. Replacing z by $\overline{\mathcal{F}_r}$ and then using $1/\mathfrak{B}_r(z) = \mathfrak{B}_{1-r}(-z)$ [6], we obtain

$$\overline{\mathcal{F}_r} = \frac{z}{\mathfrak{B}_{r-1}(z)} = z\mathfrak{B}_{2-r}(-z),$$

as required. $\square$

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[1]We note that r-ary core and r-ary trees have different meanings.

Noticing that $(g, f)^{-1} = (1/(g \circ \overline{f}), \overline{f})$ we obtain

$$\Delta_r^{-1} = \left(1/(\mathfrak{B}_r \circ \overline{z\mathfrak{B}_r}), \overline{z\mathfrak{B}_r}\right) = (\mathfrak{B}_{2-r}(-z), z\mathfrak{B}_{2-r}(-z)).$$

3. Modified GR coloring and intermediate generating functions

In this section, we modify the conditions on GR trees to get a variety of intermediate generating functions. At this point there are two tracks. If we allow both colors at the root we go from C to T^2. If we have only one edge color at the root we go from C to T. We will take the route from C to T^2 and then briefly treat the other very similar route.

3.1. Horizontal transition from C to T^2

Consider GR trees but with the condition (i) removed. That is the root may have red edges with the condition (ii). If there are all red edges we start with the Catalan numbers $1, 1, 2, 5, 14, 42, \ldots$ [A000108]. The A number refers to that item in the OEIS [12]. Our next step is to allow at most one green edge at each vertex. In this case the generating function satisfies the equation

$$G = 1 + 2zG + 2z^2G^2 + 2z^3G^3 + \cdots = 1 + \frac{2zG}{1 - zG} = \frac{1 + zG}{1 - zG}$$

and so G is the generating function for the big Schröder numbers:

$$G = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}$$
$$= 1 + 2z + 6z^2 + 22z^3 + 90z^4 + 394z^5 + \cdots. \quad \text{[A006318]}$$

With some small changes of language, these are the "oldest child trees" of [1]. If we allow no green edges at the root then we would get the small Schröder numbers $1, 1, 3, 11, 45, 197, \ldots$ [A001003].

The next step would allow at most two green edges at each vertex. The generating function G for such trees satisfies

$$G = 1 + 2zG + 3z^2G^2 \cdot \frac{1}{1 - zG} = \frac{1 + zG + z^2G^2}{1 - zG}.$$

This yields a quadratic equation and

$$G = \frac{1 - z - \sqrt{1 - 6z - 3z^2}}{2z(1 + z)}$$
$$= 1 + 2z + 7z^2 + 29z^3 + 133z^4 + 650z^5 + \cdots. \quad \text{[A064641]}$$

What if we allow at most k green edges at each vertex? The generating function, G_k, or simply G satisfies the equation

$$G = 1 + 2zG + 3z^2G^2 + \cdots + (k+1)(zG)^k \cdot \frac{1}{1 - zG}.$$

This simplifies to

$$(5) \qquad\qquad G = \frac{1 - (zG)^{k+1}}{(1 - zG)^2}.$$

Another characterization of Riordan arrays is useful in analyzing functional equations like (5). Following [9] we have that a Riordan array

$$(g, f) = [r_{n,m}]_{n,m \geq 0}$$

is associated with two sequences $(a_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ such that each entry can be expressed as a linear combination of the entries in the preceding row:

$$(6) \qquad r_{n+1,0} = \sum_{j \geq 0} z_j r_{n,j} \quad \text{and} \quad r_{n+1,m+1} = \sum_{j \geq 0} a_j r_{n,m+j}.$$

The sequences are called the A- and Z-sequences of the Riordan array (g, f), respectively. If A and Z are the generating functions for the two sequences respectively, (6) is equivalent to

$$(7) \qquad\qquad g = \frac{1}{1 - z(Z \circ f)} \quad \text{and} \quad f = z(A \circ f).$$

Let $f = zG$ in (5). Then

$$\frac{f}{z} = \frac{1 - f^{k+1}}{(1 - f)^2}.$$

Replacing z by the compositional inverse $\bar{f}$, we have $z/\bar{f} = (1 - z^{k+1})/(1-z)^2$ so that $\bar{f} = z\frac{(1-z)^2}{1-z^{k+1}}$. By the second equation in (7), $A(z)\bar{f}(z) = z$ and so

$$(8) \qquad A = A(z) = \frac{1 - z^{k+1}}{(1 - z)^2} = (1 + z + z^2 + \cdots + z^k)\frac{1}{1 - z}.$$

In other words, the A-sequence of the Riordan array $\mathbb{G}_k = (G_k, zG_k)$, is the sequence of partial sums of the sequence $1, 1, \ldots, 1, 0, \ldots$ where the number of 1's is $k + 1$.

Thus using (6) we can compute the coefficients for G_k recursively as the left most column of $\mathbb{G}_k = [g_{n,m}]_{n,m \geq 0}$ with

$$g_{n+1,m} = g_{n,m-1} + 2g_{n,m} + \cdots + kg_{n,m+k-2} + (k+1)\sum_{j \geq 0} g_{n,m+k-1+j}$$

for $m \geq 1$, where $g_{n,-1} = 0$ and $g_{0,0} = 1$.

We illustrate for $k = 3$:

$$\mathbb{G}_3 = [g_{n,m}]_{n,m \geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 7 & 4 & 1 & 0 & 0 \\ 30 & 18 & 6 & 1 & 0 \\ 142 & 88 & 33 & 8 & 1 \\ & & \cdots & & \end{bmatrix}.$$

We can interpret not only the left most column but also all the numbers in this triangle combinatorially. For the mth column we have a forest of $m + 1$ trees each following the same GR rule. These $m + 1$ trees are connected by a path of simple edges (informally we think of these edges as irrigation pipes).

For instance in the third column with 4 edges and $m = 2$ we have 33 possibilities as shown in Figure 2:
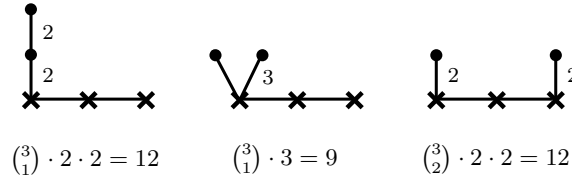

$$\binom{3}{1} \cdot 2 \cdot 2 = 12 \qquad \binom{3}{1} \cdot 3 = 9 \qquad \binom{3}{2} \cdot 2 \cdot 2 = 12$$

FIGURE 2. Forests of 3 GR-trees.

Finally, let us consider the case $k \to \infty$. Since the power series $A(z)$ in (8) converges to $\frac{1}{(1-z)^2}$ for $|z| < 1$, it follows from $A(z)\bar{f}(z) = z$ that $G = \frac{1}{(1-zG)^2}$ where $G = \frac{f}{z} = \frac{1}{z}\lim_{k\to\infty} zG_k$. Further, since $T = 1 + zT^3 = \frac{1}{1-zT^2}$ we have

$$G = T^2 = \sum_{n\geq 0} \frac{1}{n+1}\binom{3n+1}{n}z^n$$
$$= 1 + 2z + 7z^2 + 30z^3 + 143z^4 + 728z^5 + \cdots. \quad [\text{A006013}]$$

Here is a table showing the radii of convergence for the G_k as k increases:

k	exact	approximate
0	$1/4$	.250000000
1	$3 - 2\sqrt{2}$	.171572875
2	$\frac{2\sqrt{3}-3}{3}$	.154700538
3	$\sqrt{\frac{5\sqrt{5}-11}{8}}$	.150141553
4	α	.148782875
∞	$4/27$	.148148148

where α is the positive real root of $125z^5 - 300z^3 + 440z^2 - 32z - 4 = 0$.

3.2. Vertical transition from C to T^2

Instead of the horizontal transition from Catalan to ternary we could proceed vertically. The first step would be to allow GR coloring only for the edges connected to the root. This gives us sequence starting $1, 2, 5, 14, 42, \ldots$ and the generating function

$$1 + 2zC + 3(zC)^2 + 4(zC)^3 + \cdots = \frac{1}{(1-zC)^2} = C^2.$$

If we also allow GR coloring at height one the sequence starts $1, 2, 7, 26, 99, \ldots$ with the generating function

$$1 + 2zC^2 + 3\left(zC^2\right)^2 + \cdots = \frac{1}{\left(1 - zC^2\right)^2} = \left(\frac{B}{C}\right)^2 = \left(\frac{B+1}{2}\right)^2$$

$$= \frac{1}{4}\left(B^2 + 2B + 1\right)$$

$$= 1 + \frac{1}{4}\sum_{n \geq 1}\left[4^n + 2\binom{2n}{n}\right]z^n$$

$$= 1 + 2z + 7z^2 + 26z^3 + 99z^4 + \cdots \quad [\text{A114121}]$$

where $B = \frac{1}{\sqrt{1-4z}} = \sum_{n \geq 0}\binom{2n}{n}z^n$ is the generating function for the central binomial coefficients. The sequence A114121 appears in other combinatorial contexts. For instance consider all ordered trees with root degree one and $n+1$ edges. The sum of heights of all leaves gives the same sequence.

We can proceed inductively. If GR coloring is allowed also at height two then the generating functions is

$$\frac{1}{\left(1 - z\frac{B^2}{C^2}\right)^2} = 1 + 2z + 7z^2 + 30z^3 + 135z^4 + \cdots.$$

As expected each iteration leads to one more term agreeing with that of T^2.

If we want to converge to T instead of T^2, then we require that all edges at the root be all red (or all green). Then allowing at most one green at heights one and above gives the small Schröder numbers starting $1, 1, 3, 11, 45, \ldots$ [A001003] and the computations are similar.

3.3. Transition from C to $\mathfrak{B}_{m+1}^m$: A combinatorial proof of the big link theorem

What happens if we have m colors? We can also find the road from C to $\mathfrak{B}_{m+1}^m$ ($m \geq 3$) along similar lines. If we allow all m colors at the root of an m-colored tree, then the generating function G_m counting such m-colored trees satisfies

$$(9) \qquad G_m = 1 + \binom{m}{1}zG_m + \binom{m+1}{2}(zG_m)^2 + \cdots = \frac{1}{(1 - zG_m)^m}.$$

For instance the term $\binom{m+1}{2}(zG_m)^2$ accounts for the case where the root has updegree 2 and we pick two of m colors with repetition allowed. It follows from (1) that $\mathfrak{B}_m = 1/(1 - z\mathfrak{B}_m^{m-1})$ or equivalently $\mathfrak{B}_{m+1} = 1/(1 - z\mathfrak{B}_m^m)$. Taking the m-th power of both sides gives exactly the same equation as (9). Thus $G_m = \mathfrak{B}_{m+1}^m$.

By attaching m-colored trees allowing $m-1$ colors $R_1 \cdots R_{m-1}$ at the root to each non root vertex of ordered trees colored by G, we obtain m-colored

trees. Since such attached m-colored trees have the generating function $\mathfrak{B}_{m+1}^{m-1}$, we have

$$\mathfrak{B}_{m+1} = C \circ (z\mathfrak{B}_{m+1}^{m-1}) = \mathfrak{B}_2 \circ (z\mathfrak{B}_{m+1}^{m-1}).$$

This proves the big link theorem for $r = 2$ and $m \geq 1$. The case where $r \geq 3$ can be proved by starting from the r-ary core of an $(r + m - 1)$-colored tree. We allow only $m - 1$ colors at the root but can attach r more colors at any non root vertex using the same left to right rules.

4. Some other GR trees

By the link theorem, we obtained the equation $T = C \circ zT$ in Section 2. The generating function C can be replaced by many generating functions that count some variety of ordered trees.

Given a generating function A with some conditions, it is known that the analytic solution of the functional equation $G = A \circ zG$ is the generating function for the so-called simply generated trees. In this case, A plays a role of the degree-weight function that defines the weight of a vertex with a fixed degree. See [5] for the details. This connects a class of simply generated trees with the degree-weight function A and a Riordan array with the A-sequence generating function A. If A itself counts some ordered trees what does G mean?

Let A be the generating function that counts some variety of ordered trees, say A-trees. The GR version of an A-tree is recursively obtained from the A-tree by attaching the GR-A trees to each non root vertex of it, but all edges connected to the root of the attached GR-A tree are red.

We briefly describe two examples to give some idea of the scope of this construction.

Example 4.1 (GR-Motzkin trees)**.** A Motzkin tree is one where every vertex has updegree at most 2. Let $A = M$ be the generating function for the Motzkin numbers so that

$$M = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}$$
$$= 1 + z + 2z^2 + 4z^3 + 9z^4 + 21z^5 + \cdots. \quad [\text{A001006}]$$

Then the GR-Motzkin trees have the generating function

$$\hat{M} = M \circ z\hat{M} = 1 + z + 3z^2 + 11z^3 + 46z^4 + \cdots \quad [\text{A006605}]$$

or equivalently $f = z(M \circ f)$ where $f = z\hat{M}$.

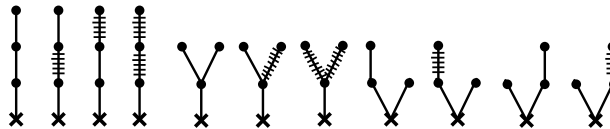To illustrate this for 3 edges we have the 11 cases as shown in Figure 3.

FIGURE 3. GR-Motzkin trees. (——— : Green edge, ⊞⊞⊞⊞ : Red edge)

A vertex in this GR-Motzkin tree could have degree 3 or 4 at a vertex as long as each green subtree and each red subtree have degree at most 2.

Example 4.2 (GR-Fibonacci trees)**.** Let $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$ so $F = \sum_{n \geq 0} F_n z^n = \frac{1}{1-z-z^2}$. A Fibonacci tree is an ordered tree whose only branch point is the root and each branch has length 1 or 2. The figure below illustrates the bijection between Fibonacci trees and compositions of n using parts of size 1 or 2.

$$2 + 1 + 1 + 2 + 2$$
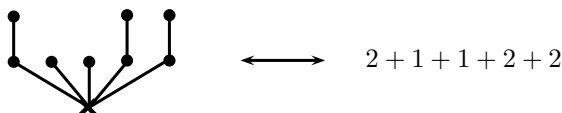
We denote the number of GR-Fibonacci trees with n edges as $\hat{f}_n$, and its generating function as $\hat{F} = \hat{F}(z)$. Let $(1, z\hat{F}) = [\hat{f}_{n,m}]_{n,m \geq 0}$ where $\hat{f}_{0,0} = 1$, $\hat{f}_{n,0} = 0$ and $\hat{f}_{n,1} = \hat{f}_{n-1}$ for $n \geq 1$. Then $\hat{F} = F \circ z\hat{F}$ or equivalently

$$\hat{f}_{n+1,m} = \sum_{j \geq 0} F_j \hat{f}_{n,m-1+j}, \quad m \geq 1.$$

Here are the first few terms

$$(1, z\hat{F})F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 10 & 7 & 3 & 1 & 0 \\ 0 & 38 & 26 & 12 & 4 & 1 \\ & \cdots & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 5 \\ 8 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 10 \\ 38 \\ 154 \\ \vdots \end{bmatrix} = \hat{F}.$$

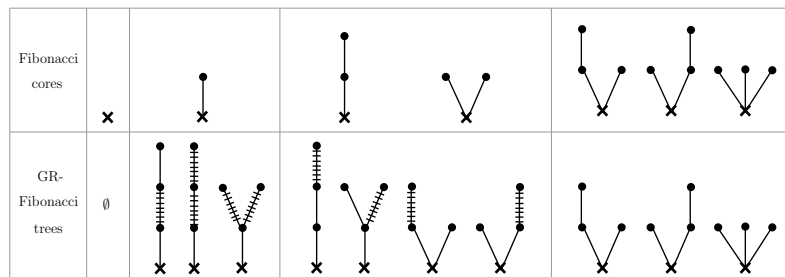Figure 4 illustrates $0 \cdot 1 + 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 = 10$.

FIGURE 4. Fibonacci cores and the generated GR-Fibonacci trees.

The sequence $1, 1, 3, 10, 38, 154, \ldots$ is sequence [A001002]. It also counts the number of ways of inserting nonintersecting diagonals into a regular $(n+2)$-gon so that the resulting figure consists of triangles and quadrilaterals. For instance there are 5 rotations each of and and $5 + 5 = 10$.

As shown in these two examples, an A-tree plays a role of core of a GR-A tree. In general, the construction of GR-A trees naturally yields the following theorem:

Theorem 4.3. *Let A be the generating function that counts ordered trees of some variety. Then GR-A trees are counted by the generating function $A^{\#}$ that satisfies the equation*

$$A^{\#} = A \circ zA^{\#}.$$

We note that the GR-A trees are not more than just applying the GR-coloring rule to A-trees. Once a new color edge appears in an A tree, it changes the whole structure of the tree. For instance, one can verify that a non root vertex of a GR-Motzkin tree with more than 3 edges in Example 4.1 may have updegree as much as 4 since the green subtrees and the red subtrees may have vertices of updegree at most 2.

On the other hand, letting $f = zA^{\#}$ we have $f = z(A \circ f)$. Thus Theorem 4.3 gives us an insight into a combinatorial relationship between a Riordan array $(1, f)$ counting certain class of ordered trees and its A-sequence A by means of GR trees.

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