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COMMON SOLUTION TO GENERALIZED MIXED EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM FOR A NONEXPANSIVE SEMIGROUP IN HILBERT SPACE

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Abstract. In this paper, we introduce and study an explicit hybrid relaxed extragradient iterative method to approximate a common solution to generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup in Hilbert space. Further, we prove that the sequence generated by the proposed iterative scheme converges strongly to the common solution to generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. The results presented in this paper are the supplement, improvement and generalization of the previously known results in this area.

1. Introduction

Throughout the paper unless otherwise stated, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H.

Recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$.

A family $\Im := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;
(iv) for all $x \in C$, $s \mapsto T(s)x$ is continuous.

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The set of all the common fixed points of a family $\Im$ is denoted by Fix($\Im$), i.e.,

$$\mathrm{Fix}(\Im) := \{x \in C : T(s)x = x,\ 0 \le s < \infty\} = \bigcap_{0 \le s < \infty} \mathrm{Fix}(T(s)),$$

where $\mathrm{Fix}(T(s))$ is the set of fixed points of $T(s)$.

Recall that a mapping $f : C \to C$ is said to be weakly contractive [15] if

$$\|f(x) - f(y)\| \le \|x - y\| - \psi(\|x - y\|),\ \forall x, y \in C,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0, +\infty)$ and $\psi(0) = 0$. If $\psi(t) = (1 - k)t$, then f is said to be contractive with constant $k > 0$. If $\psi(t) = 0$, then f is said to be nonexpansive.

The fixed point problem (in short, FPP) for a nonexpansive semigroup S is: Find $x \in C$ such that

$$x \in \mathrm{Fix}(S). \tag{1.1}$$

Next, we consider the following generalized mixed equilibrium problem (in short, GMEP): Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle + \phi(y, x) - \phi(x, x) \ge 0,\ \forall y \in C, \tag{1.2}$$

where $F : C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$, $\mathbb{R}$ is the set of all real numbers, be nonlinear bifunctions. The solution set of GMEP(1.2) is denoted by Sol(GMEP(1.2)).

If $A = 0$, then GMEP(1.2) reduces to the *generalized equilibrium problem* (in short, GEP) of finding $x \in C$ such that

$$F(x, y) + \phi(y, x) - \phi(x, x) \ge 0,\ \forall y \in C. \tag{1.3}$$

The solution set of GEP(1.3) is denoted by Sol(GEP(1.3)).

If $A = 0$ and $\phi = 0$, then GMEP(1.2) reduces to the *equilibrium problem* (in short, EP) of finding $x \in C$ such that

$$F(x, y) \ge 0,\ \forall y \in C, \tag{1.4}$$

which has been studied by Blum and Oettli [1].

If $F = 0$ and $\phi = 0$, then GMEP(1.2) reduces to the classical variational inequality problem (in short, VIP) of finding $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0,\ \forall y \in C. \tag{1.5}$$

An operator $B : H \to H$ is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2, \forall x \in H$.

In 2006, Marino and Xu [11] introduced the following implicit and explicit iterative methods based on viscosity approximation method for fixed point problem (FPP) for a nonexpansive self mapping T on H:

$$x_t = t\gamma f(x_t) + (I - tB)Tx_t, \tag{1.6}$$

and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \tag{1.7}$$

where f is a contraction mapping on H with constant $\alpha > 0$, B is a strongly positive self-adjoint and bounded linear operator on H with constant $\bar{\gamma} > 0$ and $\gamma \in (0, \frac{\bar{\gamma}}{\alpha})$. They proved that the net (x_t) and the sequence $\{x_n\}$ generated by (1.6) and (1.7), respectively, both converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \ \forall x \in \mathrm{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where h is the potential function for γf.

Recently, Ceng *et al.* [3] introduced and studied the following explicit iterative scheme for FPP for a nonexpansive mapping T:

$$x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n], \tag{1.8}$$

where P_C is a metric projection on C and $\mu > 0$.

In 2008, Plubtieng and Punpaeng [13] introduced and studied the following implicit iterative scheme to prove a strong convergence theorem for FPP(1.1):

$$x_t = tf(x_t) + (1 - t)\frac{1}{s_t}\int_0^{s_t} T(s)x_t ds, \tag{1.9}$$

where (x_t) is a continuous net and (s_t) is a positive real divergent net.

In 2010, Cianciaruso *et al.* [5] introduced the following implicit and explicit iterative methods for approximating a common solution of EP(1.4) and FPP(1.1) for a nonexpansive semigroup $S = \{T(s) : 0 \leq s < \infty\}$:

$$\begin{cases} F_1(u_t, y) + \dfrac{1}{r_t}\langle y - u_t, u_t - x_t\rangle, \ \forall y \in C, \\ x_t = t\gamma f(x_t) + (I - tB)\dfrac{1}{s_t}\displaystyle\int_0^{s_t} T(s)u_t ds, \end{cases} \tag{1.10}$$

where (s_t) and (r_t) are the continuous nets in $(0, 1)$;
and

$$\begin{cases} F_1(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle, \ \forall y \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n B)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{1.11}$$

where $\{\alpha_n\}$, $\{s_n\}$ and $\{r_n\}$ are the sequences in $(0, 1)$.

Very recently, Xiao *et al.* [15] introduced and studied the following implicit iterative scheme and obtained strong convergence theorem for EP(1.4) and

FPP(1.1):

$$
\left\{
\begin{array}{l}
F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\ y \in C, \\
z_n = (I - \beta_n)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds + \beta_n u_n, \\
x_n = (1 - \alpha_n A)z_n + \alpha_n \gamma f(x_n), \ \forall\ n \geq 1.
\end{array}
\right.
\tag{1.12}
$$

Motivated by the work of Ceng *et al.* [3], Xiao *et al.* [15], Cianciaruso *et al.* [5], Kazmi *et al.* [8, 9, 10], and by the ongoing research in this direction, we suggest and analyze an explicit hybrid relaxed extragradient iterative method for approximating a common solution to generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup in Hilbert space. Further, we prove that the sequence generated by the proposed iterative scheme converges strongly to the common solution to generalized mixed equilibrium problem and fixed point problem for a nonexpansive semigroup. This common solution is the unique solution of a variational inequality problem and is the optimality condition for a minimization problem. The results and method presented here improve and generalize the corresponding results and methods given in [5, 9, 10, 15].

2. Preliminaries

We recall some concepts and results which are needed in sequel.

The symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively, I denotes the identity operator on H.

For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$
\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C. \tag{2.1}
$$

The mapping P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive and satisfies

$$
\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H. \tag{2.2}
$$

Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$
\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall y \in C. \tag{2.3}
$$

This implies that

$$
\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H, \ \forall y \in C. \tag{2.4}
$$

In a real Hilbert space H, it is well known that

$$
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.5}
$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

It is also known that every Hilbert space H satisfies:

(1) Opial's condition [12], i.e., for any sequence $\{x^n\}$ with $x^n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x^n - x\| < \liminf_{n\to\infty} \|x^n - y\| \tag{2.6}$$

holds for every $y \in H$ with $y \neq x$;

(2)

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x, y \in H; \tag{2.7}$$

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) *monotone*, if

$$\langle Tx - Ay, x - y\rangle \geq 0, \ \forall x, y \in H;$$

(ii) *α-strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ \geq \alpha\|x - y\|^2, \ \forall x, y \in H;$$

(iii) *β-Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta\|x - y\|, \ \forall x, y \in H.$$

Lemma 2.1 ([7])**.** *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E and let T be a nonexpansive mapping from C into itself with* $\mathrm{Fix}(T) \neq \emptyset$*. Then* $\mathrm{Fix}(T)$ *is closed and convex.*

Definition 2.2. Let E be a nonempty subset of a Hausdorff topological vector space X and *conv*(E) denote the convex hull of E. Then a multivalued mapping $G : E \to 2^X$ is said to be a KKM map if, for every finite subset $\{x_1, x_2, x_3, \dots, x_n\} \subseteq E$, $conv(x_1, x_2, x_3, \dots, x_n) \subseteq \cup_{i=1}^n G(x_i)$.

Lemma 2.2 ([6])**.** *Let E be a nonempty subset of a Hausdorff topological vector space X and let $G : E \to 2^X$ be a KKM map. If $G(x)$ is closed for all $x \in E$ and is compact for at least one $x \in E$, then $\cap_{x\in E} G(x) \neq \emptyset$.*

Lemma 2.3 ([2], Demiclosed principle)**.** *Let H be a real Hilbert space, C be a closed and convex subset of H and let $S : C \to H$ be a nonexpansive mapping. Then $I - S$ is demiclosed at $y \in H$, i.e., for any sequence $\{x^n\}$ in C such that $x^n \rightharpoonup \bar{x} \in C$ and $(I - S)x^n \to y$, we have $(I - S)\bar{x} = y$.*

Lemma 2.4 ([11])**.** *Assume B is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.5 ([14])**.** *Let C be a nonempty bounded closed and convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. Then for any $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.6 ([16])**.** *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} = (1-\gamma_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| \leq \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([4])**.** *Let* $\{\lambda_n\}$ *and* $\{\beta_n\}$ *be two sequences of nonnegative real numbers and let* $\{\alpha_n\}$ *be a sequence of positive real numbers satisfying the conditions* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *such that either* $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$ *or* $\sum_{n=0}^{\infty} \beta_n < \infty$. *Let the recursive inequality*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \psi(\lambda_n) + \beta_n,\ n = 0, 1, 2, 3, \ldots,$$

be given, where $\psi(\lambda)$ *is a continuous and strict increasing function for all* $\lambda \geq 0$ *with* $\psi(0) = 0$. *Then* λ_n *converges to zero, as* $n \to \infty$.

3. Existence of solution of GEP(1.3)

First, we have the following assumptions.

Assumption 3.1. Let F and ϕ satisfy the following conditions:

(1) $F(x,x) = 0,\ \forall x, y \in C$;
(2) F is monotone, i.e.,

$$F(x,y) + F(y,x) \leq 0,\ \forall x, y \in C;$$

(3) For each $y \in C$, $x \to F(x,y)$ is weakly upper semicontinuous;
(4) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;
(5) $\phi(\cdot,\cdot)$ is weakly continuous and $\phi(\cdot, y)$ is convex;
(6) ϕ is skew-symmetric, i.e.,

$$\phi(x,x) - \phi(x,y) + \phi(y,y) - \phi(y,x) \geq 0,\ \forall x, y \in C.$$

Now, we define $T_r : H \to C$ as follows:

$$T_r(z) = \{x \in C : F(x,y) + \phi(y,x) - \phi(x,x) + \frac{1}{r}\langle y-x, x-z\rangle \geq 0,\ \forall y \in C\}, \tag{3.1}$$

where r is a positive real number.

Now, we prove some properties of the mapping T_r which lead the existence and uniqueness of solution to GEP(1.3).

Theorem 3.1. *Let* H *be a real Hilbert space; let* C *be a nonempty, closed and convex subset of* H. *Let* $F, \phi : C \times C \to \mathbb{R}$ *be nonlinear mappings satisfying Assumption 3.1. Let for each* $z \in H$ *and for each* $x \in C$, *there exist a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$ *such that for any* $y \in C \setminus D_x$,

$$F(y, z_x) + \phi(z_x, y) - \phi(y,y) + \frac{1}{r}\langle z_x - y, y - z\rangle < 0.$$

Let the mapping T_r *be defined by* (3.1). *Then the following conclusions hold:*

(i) $T_r(z)$ *is nonempty for each* $z \in H$;
(ii) T_r *is single-valued*;
(iii) T_r *is firmly nonexpansive mapping, i.e., for all* $z_1, z_2 \in H$,

$$\|T_r(z_1) - T_r(z_2)\|^2 \leq \langle T_r(z_1) - T_r(z_2), z_1 - z_2\rangle;$$

(iv) Fix(T_r)= *Sol*(GEP(1.3));
(v) *Sol*(GEP(1.3)) *is closed and convex.*

Proof. (i). Let x_0 be any given point in C. It is sufficient to show the existence and uniqueness of solution of the following auxiliary problem (in short, AP) of GEP(1.3). Find $\bar{x} \in C$ such that

$$r[F(\bar{x}, y) + \phi(y, \bar{x}) - \phi(\bar{x}, \bar{x})] + \langle y - \bar{x}, \bar{x} - x_0\rangle \geq 0, \ \forall y \in C.$$

For each fixed $y \in C$, we define

$$G(y) = \{x \in C : r[F(x, y) + \phi(y, x) - \phi(x, x)] + \langle y - x, x - x_0\rangle \geq 0\}.$$

We observe that for each $y \in C$, $G(y)$ is nonempty since $y \in G(y)$.

We prove that G is a KKM map. Suppose that there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ of C and $\alpha_i \geq 0$, for all $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^n \alpha_i y_i \notin G(y_i)$, $\forall\ i$. Then we have

$$r[F(\hat{x}, y_i) + \phi(y_i, \hat{x}) - \phi(\hat{x}, \hat{x})] + \langle y_i - \hat{x}, \hat{x} - x_0\rangle \leq 0, \ \forall\ i.$$

Therefore,

$$\sum_{i=1}^n \alpha_i r[F(\hat{x}, y_i) + \phi(y_i, \hat{x}) - \phi(\hat{x}, \hat{x})] + \langle y_i - \hat{x}, \hat{x} - x_0\rangle \leq 0, \ \forall\ i.$$

By making use of Assumption 3.1, we have

$$0 = r[F(\hat{x}, \hat{x}) + \phi(\hat{x}, \hat{x}) - \phi(\hat{x}, \hat{x})] + \langle \hat{x} - \hat{x}, \hat{x} - x_0\rangle \leq 0,$$

which is a contradiction. Hence, G is a KKM map.

Note that $\overline{G(y)^w}$ (the weak closure of $G(y)$) is a weakly closed subset of C for each $y \in C$. Moreover, for each $x_0 \in C$, there exist a bounded subset $D_{x_0} \subseteq C$ and $z_{x_0} \in C$ such that, for any $x \in C \setminus D_{x_0}$,

$$r[F(\hat{x}, z_{x_0}) + \phi(z_{x_0}, x) - \phi(x, x)] + \langle z_{x_0} - x, x - x_0\rangle < 0,$$

which implies that

$$G(z_{x_0}) = \{x \in C : r[F(\hat{x}, z_{x_0}) + \phi(z_{x_0}, x) - \phi(x, x)] + \langle z_{x_0} - x, x - x_0\rangle \geq 0\} \subseteq D_{x_0},$$

and hence $\overline{G(z_{x_0})^w}$ is weakly compact. Thus, it follows from Lemma 2.2 that $\cap_{y \in C}\overline{G(y)^w} \neq \emptyset$.

Let $\bar{x} \in \cap_{y \in C}\overline{G(y)^w}$. Now, we prove that $\overline{G(y)^w} = G(y)$ for each $y \in C$, i.e., $G(y)$ is weakly closed. Let $x \in \overline{G(y)^w}$ and $\{x_m\}$ be a sequence in $G(y)$ such that $x_m \rightharpoonup x \in C$. Then

$$r[F(x_m, y) + \phi(y, x_m) - \phi(x_m, x_m)] + \langle y - x_m, x_m - z\rangle \geq 0.$$

Since ϕ is weakly continuous and F is upper semicontinuous then

$$\begin{aligned}0 &\leq \limsup_{m\to\infty}\{r[F(x_m,y)+\phi(y,x_m)-\phi(x_m,x_m)]+\langle y-x_m,x_m-z\rangle\}\\ &\leq r[\limsup_{m\to\infty}F(x_m,y)+\limsup_{m\to\infty}\phi(y,x_m)-\liminf_{m\to\infty}\phi(x_m,x_m)]\\ &\quad+\limsup_{m\to\infty}\langle y-x_m,x_m-z\rangle\\ &\leq r[F(x,y)+\phi(y,x)-\phi(x,x)]+\langle y-x,x-z\rangle.\end{aligned}$$

This implies that $x \in G(y)$. Hence, $G(y)$ is weakly closed. Consequently, $\bar{x} \in \cap_{y\in C} G(y)$. Therefore, $\bar{x} \in C$ is a solution of AP. Thus $T_r(z)$ is nonempty for each $z \in H$.

(ii) Since, for each $z \in H$, $T_r(z) \neq \emptyset$, let $x_1, x_2 \in T_r(z)$ and hence

$$F(x_1,y)+\phi(y,x_1)-\phi(x_1,x_1)+\frac{1}{r}\langle y-x_1,x_1-z\rangle \geq 0,\ \forall y\in C, \tag{3.2}$$

and

$$F(x_2,y)+\phi(y,x_2)-\phi(x_2,x_2)+\frac{1}{r}\langle y-x_2,x_2-z\rangle \geq 0,\ \forall y\in C. \tag{3.3}$$

Taking $y = x_2$ in (3.2) and $y = x_1$ in (3.3) then on adding, we have

$$\begin{aligned}F(x_1,x_2)+F(x_2,x_1)+\phi(x_2,x_1)-\phi(x_1,x_1)+\phi(x_1,x_2)-\phi(x_2,x_2)\\ +\frac{1}{r}\langle x_2-x_1,x_1-x_2\rangle \geq 0.\end{aligned}$$

Since F is monotone and ϕ is skew-symmetric, we have

$$\frac{1}{r}\langle x_2-x_1,x_1-x_2\rangle \geq 0.$$

Since $r > 0$, we have

$$\begin{aligned}\langle x_2-x_1,x_1-x_2\rangle &\geq 0,\\ -\langle x_1-x_2,x_1-x_2\rangle &\geq 0,\\ \|x_1-x_2\| &\leq 0,\end{aligned}$$

which implies $x_1 = x_2$. Thus T_r is single-valued.

(iii) For any $z_1, z_2 \in H$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$. Then

$$F(x_1,y)+\phi(y,x_1)-\phi(x_1,x_1)+\frac{1}{r}\langle y-x_1,x_1-z_1\rangle \geq 0,\ \forall y\in C, \tag{3.4}$$

and

$$F(x_2,y)+\phi(y,x_2)-\phi(x_2,x_2)+\frac{1}{r}\langle y-x_2,x_2-z_2\rangle \geq 0,\ \forall y\in C. \tag{3.5}$$

Taking $y = x_2$ in (3.4) and $y = x_1$ in (3.5) then on adding, we have

$$\begin{aligned}F(x_1,x_2)+F(x_2,x_1)+\phi(x_2,x_1)-\phi(x_1,x_1)+\phi(x_1,x_2)-\phi(x_2,x_2)\\ +\frac{1}{r}\langle x_2-x_1,x_1-x_2-z_1+z_2\rangle \geq 0.\end{aligned}$$

By using the monotonocity of F and the property of ϕ, we have

$$\frac{1}{r}\langle x_2 - x_1, x_1 - x_2 - z_1 + z_2\rangle \geq 0.$$

Since $r > 0$, therefore

$$\begin{aligned}
&\langle x_2 - x_1, x_1 - x_2 - z_1 + z_2\rangle \geq 0\\
&\langle x_2 - x_1, x_1 - x_2\rangle + \langle x_2 - x_1, z_2 - z_1\rangle \geq 0\\
&\langle x_1 - x_2, x_1 - x_2\rangle \leq \langle x_1 - x_2, z_1 - z_2\rangle,
\end{aligned}$$

i.e.,

$$\|x_1 - x_2\|^2 \leq \langle T_r(z_1) - T_r(z_2), z_1 - z_2\rangle. \tag{3.6}$$

Thus T_r is firmly nonexpansive-type mapping.

(iv) Let $x \in \text{Fix}(T_r)$. Then we have

$$F(x, y) + \phi(y, x) - \phi(x, x) + \frac{1}{r}\langle y - x, x - x\rangle \geq 0,\ \forall y \in C,$$

and so

$$F(x, y) + \phi(y, x) - \phi(x, x) \geq 0,\ \forall y \in C.$$

Thus $x \in \text{Sol(GEP(1.3))}$.

Let $x \in \text{Sol(GEP(1.3))}$. Then we have

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and so

$$F(x, y) + \phi(y, x) - \phi(x, x) + \frac{1}{r}\langle y - x, x - x\rangle \geq 0,\ \forall y \in C.$$

Hence $x \in \text{Fix}(T_r)$. Thus $\text{Fix}(T_r) = \text{Sol(GEP(1.3))}$.

(v) Since T_r is firmly nonexpansive, T_r is also nonexpansive. Hence, it follows from Lemma 2.1 that $\text{Sol(GEP(1.3))} = \text{Fix}(T_r)$ is closed and convex. □

Next, we prove the following lemma.

Lemma 3.1. *Let F and ϕ satisfy Assumption 3.1 and let the mapping T_r be defined by* (3.1). *Let $x_1, x_2 \in H$ and $r_1, r_2 > 0$. Then*

$$\|T_{r_2}(x_2) - T_{r_1}(x_1)\| \leq \|x_2 - x_1\| + \frac{|r_2 - r_1|}{r_2}\|T_{r_2}(x_2) - x_2\|.$$

Proof. For any $z_1, z_2 \in H$ and $r_1, r_2 > 0$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$ for some $x_1, x_2 \in C$, we have

$$F(x_1, y) + \phi(y, x_1) - \phi(x_1, x_1) + \frac{1}{r_1}\langle y - x_1, x_1 - z_1\rangle \geq 0,\ \forall y \in C, \tag{3.7}$$

and

$$F(x_2, y) + \phi(y, x_2) - \phi(x_2, x_2) + \frac{1}{r_2}\langle y - x_2, x_2 - z_2\rangle \geq 0,\ \forall y \in C. \tag{3.8}$$

Taking $y = x_2$ in (3.7) and $y = x_1$ in (3.8), then on adding, we have

$$F(x_1,x_2)+F(x_2,x_1)+\phi(x_2,x_1)-\phi(x_1,x_1)+\phi(x_1,x_2)-\phi(x_2,x_2) + \langle x_2-x_1, \frac{x_1-z_1}{r_1}\rangle + \langle x_1-x_2, \frac{x_2-z_2}{r_2}\rangle \geq 0.$$

Using monotonocity of F and skew symmetricity of ϕ, we have

$$\langle x_2-x_1, \frac{x_1-z_1}{r_1} - \frac{x_2-z_2}{r_2}\rangle \geq 0$$

which implies that

$$\langle x_2-x_1, x_1-x_2+x_2-z_1-\frac{r_1}{r_2}(x_2-z_2)\rangle \geq 0,$$

and so

$$\|x_2-x_1\|^2 \leq \langle x_2-x_1, x_2-z_2+z_2-z_1-\frac{r_1}{r_2}(x_2-z_2)\rangle,$$
$$\|x_2-x_1\|^2 \leq \langle x_2-x_1, z_2-z_1+(1-\frac{r_1}{r_2})(x_2-z_2)\rangle,$$
$$\|x_2-x_1\|^2 \leq \|x_2-x_1\|[\|z_2-z_1\|+\frac{|r_2-r_1|}{r_2}\|x_2-z_2\|].$$

This completes the proof. □

4. Explicit hybrid relaxed extregradient iterative method

We prove the strong convergence of the sequences generated by an explicit hybrid relaxed extregradient iterative scheme for solving GMEP(1.2).

Theorem 4.1. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F, \phi : C \times C \to \mathbb{R}$ be nonlinear bifunctions satisfying Assumption 3.1. Let f be a weakly contractive mapping with a function ψ on H; let $A : H \to H$ be α-inverse strongly monotone operator; let $B : H \to H$ be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$, and $\Im = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on C. Assume that $\Gamma := \mathrm{Fix}(\Im) \cap \mathrm{Sol}(\mathrm{GMEP}(1.2)) \neq \emptyset$. For any $0 < \gamma \leq \overline{\gamma}$, let the sequence $\{x^n\}$ generated by the following iterative schemes:*

$$\begin{aligned} x^0 &\in C,\\ y^n &= T_{r^n}(x^n - r^n A(x^n)),\\ u^n &= T_{r^n}(y^n - r^n A(y^n)),\\ z^n &= \beta^n u^n + (1-\beta^n)\frac{1}{t_n}\int_0^{t_n} T(s)u^n ds,\\ x^{n+1} &= P_C[\alpha^n \gamma f(x^n) + (I-\alpha^n B)z^n], \end{aligned}$$

where $\{\alpha^n\}$, $\{\beta^n\}$ are the sequences in $(0,1)$ and $\{r^n\}$, $\{t^n\}$ are sequences of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha^n = 0$, $\sum_{n=0}^{\infty} \alpha^n = \infty$, $\sum_{n=0}^{\infty} |\alpha^n - \alpha^{n-1}| < \infty$;

(ii) $\lim_{n\to\infty}\beta^n = 0,\ \sum_{n=0}^{\infty}|\beta^n - \beta^{n-1}| < \infty$;
(iii) $\sum_{n=0}^{\infty}\frac{|t^n - t^{n-1}|}{t^n} < \infty$;
(iv) $0 < \liminf_{n\to\infty} r^n \le \limsup_{n\to\infty} r^n < 2\alpha,\ \sum_{n=0}^{\infty}|r^{n+1} - r^n| < \infty$;

Then the sequence $\{x^n\}$ *converges strongly to* $z^* \in \Gamma$ *which uniquely solves the following variational inequality*

$$\langle (B-\gamma f)z^*, z^* - z\rangle \le 0 \quad \textit{for any} \quad z\in\Gamma. \tag{4.1}$$

Proof. Since $\alpha^n \to 0$ as $n\to\infty$, we may assume, with no loss of generality, that $\alpha^n < \|B\|^{-1},\ \forall n \ge 1$. Then, $\alpha^n < \frac{1}{\bar{\gamma}},\ \forall n\ge 1$.

Let $z\in\Gamma$. Then $z = T_{r^n}(z - r^n A(z))$.

Now, we estimate

$$\begin{aligned}
&\|z^n - z\| \\
&= \|\beta^n u^n + (1-\beta^n)\frac{1}{t_n}\int_0^{t_n} T(s)u^n ds - z\| \\
&= \|\beta^n u^n + (1-\beta^n)\frac{1}{t_n}\int_0^{t_n} T(s)u^n ds - \beta^n z - (1-\beta^n)z\| \\
&\le \beta^n\|u^n - z\| + (1-\beta^n)\|\frac{1}{t_n}\int_0^{t_n} T(s)u^n ds - z\| \\
&= \beta^n\|u^n - z\| + (1-\beta^n)\|\frac{1}{t_n}\int_0^{t_n} T(s)u^n ds - \frac{1}{t_n}\int_0^{t_n} T(s)z ds\| \\
&\le \beta^n\|u^n - z\| + (1-\beta^n)\frac{1}{t_n}\int_0^{t_n}\|T(s)u^n - T(s)z\| ds \\
&\le \beta^n\|u^n - z\| + (1-\beta^n)\frac{1}{t_n}\int_0^{t_n}\|u^n - z\| ds \\
&\le \beta^n\|u^n - z\| + (1-\beta^n)\|u^n - z\| \\
&\le \|u^n - z\|,
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\|y^n - z\|^2 \\
&= \|T_{r^n}(x^n - r^n A(x^n)) - z\|^2 \\
&= \|T_{r^n}(x^n - r^n A(x^n)) - T_{r^n}(z - r^n A(z))\|^2 \\
&= \|(x^n - z) - r^n(A(x^n) - A(z))\|^2 \\
&\le \|x^n - z\|^2 - 2r^n\langle A(x^n) - A(z), x^n - z\rangle + (r^n)^2\|A(x^n) - A(z)\|^2 \\
&\le \|x^n - z\|^2 - 2r^n\alpha\|A(x^n) - A(z)\|^2 + (r^n)^2\|A(x^n) - A(z)\|^2 \\
&\le \|x^n - z\|^2 - r^n(2\alpha - r^n)\|A(x^n) - A(z)\|^2 \\
&\le \|x^n - z\|^2 - (r^n)^2(\frac{2\alpha}{r^n} - 1)\|A(x^n) - A(z)\|^2,
\end{aligned} \tag{4.3}$$

or

$$\|y^n - z\| \leq \|x^n - z\|. \tag{4.4}$$

Further, we estimate

$$\begin{aligned}
\|u^n - z\|^2 \quad (4.5)&\\
&= \|T_{r^n}(y^n - r^n A(y^n)) - z\|^2\\
&= \|T_{r^n}(y^n - r^n A(y^n)) - T_{r^n}(z - r^n A(z))\|^2\\
&= \|(y^n - z) - r^n(A(y^n) - A(z))\|^2\\
&\leq \|x^n - z\|^2 - 2r^n\langle A(y^n) - A(z), y^n - z\rangle + (r^n)^2\|A(y^n) - A(z)\|^2\\
&\leq \|y^n - z\|^2 - 2r^n\alpha\|A(y^n) - A(z)\|^2 + (r^n)^2\|A(y^n) - A(z)\|^2\\
&\leq \|y^n - z\|^2 - (r^n)^2(\frac{2\alpha}{r^n} - 1)\|A(y^n) - A(z)\|^2\\
&\leq \|y^n - z\|^2.
\end{aligned}$$

Hence

$$\|u^n - z\| \leq \|x^n - z\|. \tag{4.6}$$

Now, we estimate

(4.7)

$$\begin{aligned}
&\|x^{n+1} - z\|\\
&= \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - z\|\\
&= \|(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - z\|\\
&= \|(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - \alpha^n\gamma f(z) - (I - \alpha^n B)z - \alpha^n B(z) + \alpha^n\gamma f(z)\|\\
&\leq \|I - \alpha^n B\|\|z^n - z\| + \alpha^n\gamma\|f(x^n) - f(z)\| + \alpha^n\|\gamma f(z) - B(z)\|\\
&\leq (1 - \alpha^n\overline{\gamma})\|x^n - z\| + \alpha^n\gamma\|x^n - z\| - \alpha^n\gamma\phi(\|x^n - z\|) + \alpha^n\|\gamma f(z) - B(z)\|\\
&\leq (1 - \alpha^n(\overline{\gamma} - \gamma))\|x^n - z\| + \alpha^n\|\gamma f(z) - B(z)\|.
\end{aligned}$$

By induction,

$$\|x^{n+1} - z\| \leq \max\left\{\|x_0 - z\|,\ \frac{\|\gamma f(z) - B(z)\|}{\overline{\gamma} - \gamma}\right\},\ n \geq 0.$$

Thus $\{x^n\}$ is bounded and hence $\{y^n\}$, $\{z^n\}$, $\{u^n\}$, $\{f(x^n)\}$, $\{B(z^n)\}$ are bounded.

Next, we estimate

(4.8)

$$\begin{aligned}
&\|x^{n+1} - x^n\|\\
&= \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - P_C(\alpha^{n-1}\gamma f(x^{n-1}) + (I - \alpha^{n-1}B)z^{n-1})\|\\
&\leq \|(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - (\alpha^{n-1}\gamma f(x^{n-1}) + (I - \alpha^{n-1}B)z^{n-1})\|\\
&\leq \|\alpha^n\gamma f(x^n) - \alpha^n\gamma f(x^{n-1}) + \alpha^n\gamma f(x^{n-1}) - \alpha^{n-1}\gamma f(x^{n-1})
\end{aligned}$$

$$+ (I - \alpha^n B)z^n - (I - \alpha^n B)z^{n-1} + (I - \alpha^n B)z^{n-1} - (I - \alpha^{n-1} B)z^{n-1}\|$$
$$\leq \alpha^n \gamma \|f(x^n) - f(x^{n-1})\| + \gamma |\alpha^n - \alpha^{n-1}| \|f(x^{n-1})\|$$
$$+ \|I - \alpha^n B\| \|z^n - z^{n-1}\| + |\alpha^n - \alpha^{n-1}| \|B(z^{n-1})\|$$
$$\leq \alpha^n \gamma \|x^n - x^{n-1}\| - \alpha^n \gamma \psi(\|x^n - x^{n-1}\|) + \gamma |\alpha^n - \alpha^{n-1}| \|f(x^{n-1})\|$$
$$+ (I - \alpha^n \overline{\gamma}) \|z^n - z^{n-1}\| + |\alpha^n - \alpha^{n-1}| \|B(z^{n-1})\|.$$

Let $w^n = \frac{1}{t^n} \int_0^{t^n} T(s)u^n ds$. Now, we estimate

$$\begin{aligned}
&\|z^n - z^{n-1}\| \\
&= \|\beta^n u^n + (1 - \beta^n) w^n - \beta^{n-1} u^{n-1} - (1 - \beta^{n-1}) w^{n-1}\| \\
&= \|\beta^n u^n + \beta^n u^{n-1} - \beta^n u^{n-1} + (1 - \beta^n) w^n - (1 - \beta^n) w^{n-1} \\
&\quad + (1 - \beta^n) w^{n-1} - \beta^{n-1} u^{n-1} - (1 - \beta^{n-1}) w^{n-1}\| \\
&\leq |\beta^n - \beta^{n-1}| \|u^{n-1}\| + \beta^n \|u^n - u^{n-1}\| + |\beta^n - \beta^{n-1}| \|w^{n-1}\| \\
&\quad + (1 - \beta^n) \|w^n - w^{n-1}\| \\
&\leq (1 - \beta^n) \|w^n - w^{n-1}\| + \beta^n \|u^n - u^{n-1}\| \\
&\quad + |\beta^n - \beta^{n-1}| (\|w^{n-1}\| + \|u^{n-1}\|).
\end{aligned} \tag{4.9}$$

Further, we estimate

$$\|w^n - w^{n-1}\|$$
$$= \|\frac{1}{t^n} \int_0^{t^n} T(s)u^n ds - \frac{1}{t^{n-1}} \int_0^{t^{n-1}} T(s)u^{n-1} ds\|$$
$$= \|\frac{1}{t^n} \int_0^{t^n} T(s)u^n ds - \frac{1}{t^n} \int_0^{t^n} T(s)u^{n-1} ds + \frac{1}{t^n} \int_0^{t^n} T(s)u^{n-1} ds$$
$$- \frac{1}{t^n} \int_0^{t^{n-1}} T(s)u^{n-1} ds + \frac{1}{t^n} \int_0^{t^{n-1}} T(s)u^{n-1} ds - \frac{1}{t^{n-1}} \int_0^{t^{n-1}} T(s)u^{n-1} ds\|$$
$$= \|\frac{1}{t^n} \int_0^{t^n} [T(s)u^n - T(s)u^{n-1}] ds + (\frac{1}{t^n} - \frac{1}{t^{n-1}}) \int_0^{t^{n-1}} T(s)u^{n-1} ds$$
$$+ \frac{1}{t^n} \int_0^{t^n} T(s)u^{n-1} ds - \frac{1}{t^n} \int_0^{t^{n-1}} T(s)u^{n-1} ds\|$$
$$= \|\frac{1}{t^n} \int_0^{t^n} [T(s)u^n - T(s)u^{n-1}] ds + (\frac{1}{t^n} - \frac{1}{t^{n-1}}) \int_0^{t^{n-1}} T(s)u^{n-1} ds$$
$$+ \frac{1}{t^n} \int_{t^{n-1}}^{t^n} T(s)u^{n-1} ds\|.$$

If $p \in \text{Fix}(\Im)$, then

$$\|w^n - w^{n-1}\| \tag{4.10}$$

$$
\begin{aligned}
&= \|\frac{1}{t^n}\int_0^{t^n}[T(s)u^n - T(s)u^{n-1}]ds \\
&\quad + (\frac{1}{t^n} - \frac{1}{t^{n-1}})\int_0^{t^{n-1}}[T(s)u^{n-1} - T(s)p]ds \\
&\quad + \frac{1}{t^n}\int_{t^{n-1}}^{t^n}[T(s)u^{n-1} - T(s)p]ds\| \\
&\le \|u^n - u^{n-1}\| + (\frac{2|t^n - t^{n-1}|}{t^n})\|u^{n-1} - p\|.
\end{aligned}
$$

Thus, from (4.9), we have

$$
\begin{aligned}
\text{(4.11)} \qquad & \|z^n - z^{n-1}\| \\
&\le \|u^n - u^{n-1}\| + (1-\beta^n)(\frac{2|t^n - t^{n-1}|}{t^n})\|u^{n-1} - p\| \\
&\quad + |\beta^n - \beta^{n-1}|(\|w^{n-1}\| + \|u^{n-1}\|) \\
&\le \|u^n - u^{n-1}\| + (\frac{2|t^n - t^{n-1}|}{t^n})\|u^{n-1} - p\| \\
&\quad + |\beta^n - \beta^{n-1}|(\|w^{n-1}\| + \|u^{n-1}\|).
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
\text{(4.12)} \qquad & \|u^{n+1} - u^n\| \\
&= \|T_{r^{n+1}}(y^{n+1} - r^{n+1}A(y^{n+1})) - T_{r^n}(y^n - r^nA(y^n))\| \\
&\le \|y^{n+1} - r^{n+1}A(y^{n+1}) - (y^n - r^nA(y^n))\| \\
&\quad + |1 - \frac{r^n}{r^{n+1}}|\|u^{n+1} - (y^{n+1} - r^{n+1}A(y^{n+1}))\| \text{ (using Lemma 3.1)} \\
&\le \|y^{n+1} - y^n - r^{n+1}(A(y^{n+1}) - A(y^n))\| + |r^{n+1} - r^n|\|A(y^n)\| \\
&\quad + |1 - \frac{r^n}{r^{n+1}}|\|u^{n+1} - y^{n+1} + r^{n+1}A(y^{n+1})\| \\
&\le \|y^{n+1} - y^n\| + |r^{n+1} - r^n|\|A(y^n)\| \\
&\quad + |1 - \frac{r^n}{r^{n+1}}|\|u^{n+1} - y^{n+1}\| + |r^{n+1} - r^n|\|A(y^{n+1})\| \\
&\le \|y^{n+1} - y^n\| + |r^{n+1} - r^n|(\|A(y^{n+1})\| + \|A(y^n)\|) \\
&\quad + |1 - \frac{r^n}{r^{n+1}}|\|u^{n+1} - y^{n+1}\|.
\end{aligned}
$$

Using (4.12) in (4.11), we have

$$
\begin{aligned}
\text{(4.13)} \qquad & \|z^n - z^{n-1}\| \\
&\le \|y^n - y^{n-1}\| + |r^n - r^{n-1}|(\|A(y^n)\| + \|A(y^{n-1})\|) \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|u^n - y^n\| + (\frac{2|t^n - t^{n-1}|}{t^n})\|u^{n-1} - p\|
\end{aligned}
$$

$$+ |\beta^n - \beta^{n-1}|(\|w^{n-1}\| + \|u^{n-1}\|).$$

Now, we estimate

$$\begin{aligned}
(4.14)\quad & \|y^n - y^{n-1}\| \\
&= \|T_{r^n}(x^n - r^n A(x^n)) - T_{r^{n-1}}(x^{n-1} - r^{n-1}A(x^{n-1}))\| \\
&\leq \|x^n - r^n A(x^n) - (x^{n-1} - r^{n-1}A(x^{n-1}))\| \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|y^n - (x^n - r^n A(x^n))\| \\
&\leq \|(x^n - x^{n-1}) - r^n A(x^n) + r^n A(x^{n-1}) - r^n A(x^{n-1}) + r^{n-1}A(x^{n-1})\| \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|(y^n - (x^n) + r^n A(x^n))\| \\
&\leq \|(x^n - x^{n-1}) - r^n A(x^n) + r^n A(x^{n-1})\| + |r^n - r^{n-1}|\|A(x^{n-1})\| \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|x^n - y^n\| + |r^n - r^{n-1}|\|A(x^n)\| \\
&\leq \|(x^n - x^{n-1})\| + |r^n - r^{n-1}|(\|A(x^{n-1})\| + \|A(x^n)\|) \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|x^n - y^n\|,
\end{aligned}$$

and

$$\begin{aligned}
(4.15)\qquad \|u^n - y^n\| &\leq \|T_{r^n}(y^n - r^n A(y^n)) - T_{r^n}(x^n - r^n A(x^n))\| \\
&\leq \|y^n - r^n A(y^n) - (x^n - r^n A(x^n))\| \\
&\leq \|(y^n - x^n) - r^n(A(y^n) - A(x^n))\| \\
&\leq \|x^n - y^n\|.
\end{aligned}$$

Using (4.13) in (4.8), we have

$$\begin{aligned}
(4.16)\quad & \|x^{n+1} - x^n\| \\
&\leq \alpha^n\gamma\|x^n - x^{n-1}\| - \alpha^n\gamma\psi(\|x^n - x^{n-1}\|) \\
&\quad + \gamma|\alpha^n - \alpha^{n-1}|\|f(x^{n-1})\| + |\alpha^n - \alpha^{n-1}|\|B(z^{n-1})\| \\
&\quad + (1 - \alpha^n\overline{\gamma})\{\|y^n - y^{n-1}\| + |r^n - r^{n-1}|(\|A(y^n)\| + \|A(y^{n-1})\|) \\
&\quad + |1 - \frac{r^{n-1}}{r^n}|\|u^n - y^n\| + \frac{2|t^n - t^{n-1}|}{t^n}\|u^{n-1} - p\| \\
&\quad + |\beta^n - \beta^{n-1}|\|w^{n-1}\| + \|u^{n-1}\|\} \\
&\leq \alpha^n\gamma\|x^n - x^{n-1}\| - \alpha^n\gamma\psi(\|x^n - x^{n-1}\|) \\
&\quad + |\alpha^n - \alpha^{n-1}|(\gamma\|f(x^{n-1})\| + \|B(z^{n-1})\|) + (1 - \alpha^n\overline{\gamma})\{\|x^n - x^{n-1}\| \\
&\quad + |r^n - r^{n-1}|(\|A(x^n)\| + \|A(x^{n-1})\|) + |1 - \frac{r^{n-1}}{r^n}|\|x^n - y^n\| \\
&\quad + |r^n - r^{n-1}|(\|A(y^n)\| + \|A(y^{n-1})\|) + \frac{2|t^n - t^{n-1}|}{t^n}\|u^{n-1} - p\|
\end{aligned}$$

$$+ |\beta^n - \beta^{n-1}| \|w^{n-1}\| + \|u^{n-1}\|\}.$$

Since $\liminf_{n\to\infty} r^n > 0$, $\exists\, b > 0$ such that $r^n > b$. Hence (4.16) reduces to

$$\begin{aligned}\|x^{n+1} - x^n\| \leq \|x^n - x^{n-1}\| &- \alpha^n \gamma \psi(\|x^n - x^{n-1}\|) + |\alpha^n - \alpha^{n-1}|M \\ &+ |r^n - r^{n-1}|M + \frac{2|r^n - r^{n-1}|}{b} M + \frac{2|t^n - t^{n-1}|}{t^n} M \\ &+ |\beta^n - \beta^{n-1}|M,\end{aligned}$$

where

$$\begin{aligned}M = \max\{ &\sup_{n\geq 1}\{\gamma\|f(x^{n-1})\| + \|B(z^{n-1})\|\}, \sup_{n\geq 1}\{\|A(x^n)\| + \|A(x^{n-1})\|\}, \\ &\sup_{n\geq 1}\{\|A(y^n)\| + \|A(y^{n-1})\|\}, \sup_{n\geq 1}\{\|w^{n-1}\| + \|u^{n-1}\|\}, \\ &\sup_{n\geq 1}\{\|u^{n-1} - p\|\}\}.\end{aligned}$$

Thus,

$$\|x^{n+1} - x^n\| \leq \alpha^n \gamma \|x^n - x^{n-1}\| - \alpha^n \gamma \psi(\|x^n - x^{n-1}\|) + \xi^n,$$

where

$$\xi^n := \left[|\alpha^n - \alpha^{n-1}| + |r^n - r^{n-1}| + \tfrac{2|r^n - r^{n-1}|}{b} + \tfrac{2|t^n - t^{n-1}|}{t^n} + |\beta^n - \beta^{n-1}| \right] M$$

and hence by conditions (i)-(iv), we observe that $\Sigma_{n=0}^{\infty} \xi^n < \infty$. By Lemma 2.7, we have

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0. \tag{4.17}$$

Next, we show $\lim_{n\to\infty} \|T(s)x^n - x^n\| = 0$.

First, we estimate

$$\begin{aligned}&\|x^{n+1} - w^n\| \\ &= \|P_C(\alpha^n \gamma f(x^n) + (I - \alpha^n B)z^n) - w^n\| \\ &\leq \|P_C(\alpha^n \gamma f(x^n) + (I - \alpha^n B)z^n) - P_C(w^n)\| \\ &\leq \|\alpha^n \gamma f(x^n) + (I - \alpha^n B)z^n - w^n\| \\ &\leq \|\alpha^n \gamma f(x^n) + (I - \alpha^n B)(\beta^n u^n + (1-\beta^n)\tfrac{1}{t^n}\textstyle\int_0^{t^n} T(s)u^n ds) - \tfrac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| \\ &\leq \alpha^n \gamma \|f(x^n)\| + (1-\alpha^n\overline{\gamma})\beta^n\|u^n\| + ((1-\alpha^n\overline{\gamma})(1-\beta^n) - 1)\|\tfrac{1}{t^n}\textstyle\int_0^{t^n} T(s)u^n ds\| \\ &\leq \alpha^n \gamma \|f(x^n)\| + (1-\alpha^n\overline{\gamma})\beta^n\|u^n\| + (\alpha^n\beta^n\overline{\gamma} - \beta^n - \alpha^n\overline{\gamma})\|\tfrac{1}{t^n}\textstyle\int_0^{t^n} T(s)u^n ds\|.\end{aligned}$$

Since $\lim_{n\to\infty} \alpha^n = 0$, $\lim_{n\to\infty} \beta^n = 0$, $\{x^n\}$, $\{u^n\}$, $\{f(x^n)\}$ and $\{T(s)u^n\}$ are bounded, therefore

$$\lim_{n\to\infty} \|x^{n+1} - w^n\| = 0. \tag{4.18}$$

Thus, by (4.17) and (4.18), we have

$$\|x^n - w^n\| \leq \|x^n - x^{n+1}\| + \|x^{n+1} - w^n\|,$$

$$\lim_{n\to\infty} \|x^n - w^n\| = 0. \tag{4.19}$$

Now, we estimate

$$\begin{aligned}
&\|T(s)x^n - x^n\| \\
&\le \|T(s)x^n - T(s)\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| + \|T(s)\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| \\
&\quad - \|\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| + \|\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds - x^n\| \\
&\le \|x^n - \frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| + \|T(s)\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds \\
&\quad - \frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| + \|\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds - x^n\| \\
&\le 2\|x^n - w^n\| + \|T(s)\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds - \frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\|.
\end{aligned} \tag{4.20}$$

Without loss of generality, we may assume that $\Im$ is nonexpansive semigroup on C, and by Lemma 2.5, we have

$$\lim_{n\to\infty} \sup_{x\in C} \|T(s)\frac{1}{t^n}\int_0^{t^n} T(s)u^n ds - \frac{1}{t^n}\int_0^{t^n} T(s)u^n ds\| = 0. \tag{4.21}$$

Using (4.19) and (4.21) in (4.20), we have

$$\lim_{n\to\infty} \|T(s)x^n - x^n\| = 0. \tag{4.22}$$

Now, we prove that $\lim_{n\to\infty} \|u^n - x^n\| = 0$.

Using Lemma 2.4 and (2.7), we estimate

(4.23)

$$\begin{aligned}
&\|x^{n+1} - z\|^2 \\
&= \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - z\|^2 \\
&\le \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n) - P_C(z)\|^2 \\
&\le \|\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n - z\|^2 \\
&\le \|(I - \alpha^n B)(z^n - z) + \alpha^n\gamma f(x^n) - \alpha^n B(z)\|^2 \\
&\le (1 - \alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle \\
&\le (1 - \alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - \gamma f(z) + \gamma f(z) - B(z), x^{n+1} - z\rangle \\
&\le (1 - \alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\gamma\langle f(x^n) - f(z), x^{n+1} - z\rangle \\
&\quad + 2\alpha^n\langle\gamma f(z) - B(z), x^{n+1} - z\rangle \\
&\le (1 - \alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\gamma\|x^{n+1} - z\|\|x^n - z\| \\
&\quad - 2\alpha^n\gamma\psi(\|x^n - z\|)\|x^{n+1} - z\| + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\|
\end{aligned}$$

$$\leq (1-\alpha^n\overline{\gamma})^2\|u^n - z\|^2 + 2\alpha^n\gamma\|x^{n+1}-z\|\|x^n - z\|$$
$$+ 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1}-z\|.$$

Using firmly nonexpansivity of T_{r^n}, we estimate

$$\|u^n - z\|^2$$
$$= \|T^C_{r^n}(y^n - r^nA(y^n)) - T^C_{r^n}(z - r^nA(z))\|^2$$
$$\leq \langle T^C_{r^n}(y^n - r^nA(y^n)) - T^C_{r^n}(z - r^nA(z)), y^n - r^nA(y^n) - (z - r^nA(z))\rangle$$
$$= \langle u^n - z, y^n - z + r^nA(z) - r^nA(y^n)\rangle$$
$$= \langle u^n - z, y^n - z\rangle + r^n\langle u^n - z, A(z) - A(y^n)\rangle$$
$$= \frac{1}{2}(\|u^n - z\|^2 + \|y^n - z\|^2 - \|u^n - y^n\|^2) + r^n\|u^n - z\|\|A(z) - A(y^n)\|.$$

Thus,

$$\|u^n - z\|^2 \leq \|y^n - z\|^2 - \|u^n - y^n\|^2 + r^n\|u^n - z\|\|A(z) - A(y^n)\|, \tag{4.24}$$

$$\|u^n - z\|^2 \leq \|x^n - z\|^2 - \|u^n - y^n\|^2 + r^n\|x^n - z\|\|A(z) - A(y^n)\|. \tag{4.25}$$

(using (4.4) and (4.6))

Now, we estimate

$$\|x^{n+1} - z\|^2$$
$$= \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^nB)z^n) - z\|^2$$
$$\leq \|P_C(\alpha^n\gamma f(x^n) + (I - \alpha^nB)z^n) - P_C(z)\|^2$$
$$\leq \|\alpha^n\gamma f(x^n) + (I - \alpha^nB)z^n - z\|^2$$
$$\leq \|(I - \alpha^nB)(z^n - z) + \alpha^n(\gamma f(x^n) - B(z))\|^2$$
$$\leq (1-\alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle$$
$$\leq (1-\alpha^n\overline{\gamma})^2\|u^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle$$

(using (4.2))

$$\leq \|u^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle$$
$$\leq \|y^n - z\|^2 - r^n(2\alpha - r^n)\|A(y^n) - A(z)\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle.$$

(using (4.5))

Hence,

$$r^n(2\alpha - r^n)\|A(y^n) - A(z)\|^2$$
$$\leq \|x^n - z\|^2 - \|x^{n+1} - z\|^2 + 2\alpha^n\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|$$
$$\leq \|x^{n+1} - x^n\|(\|x^{n+1} - z\| + \|x^n - z\|) + 2\alpha^n\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|$$
$$\leq \|x^{n+1} - x^n\|(\|x^{n+1} - z\| + \|x^n - z\|) + 2\alpha^nM_1,$$

where $M_1 = \sup_{n\geq 1}\{\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|\}$. Using $0 < \liminf_{n\to\infty} r^n \leq \limsup_{n\to\infty} r^n < 2\alpha$, $\alpha^n \to 0$ as $n \to \infty$ and (4.17) in above inequality, we have

$$\lim_{n\to\infty} \|A(y^n) - A(z)\| = 0. \tag{4.26}$$

Again, we have

$$\begin{aligned}
&\|x^{n+1} - z\|^2\\
&\leq \|\alpha^n\gamma f(x^n) + (I - \alpha^n B)z^n - z\|^2\\
&\leq \|(I - \alpha^n B)(z^n - z) + \alpha^n(\gamma f(x^n) - B(z))\|^2\\
&\leq (1 - \alpha^n\overline{\gamma})^2\|z^n - z\|^2 + 2\alpha^n\langle\gamma f(x^n) - B(z), x^{n+1} - z\rangle\\
&\leq \|y^n - z\|^2 + 2\alpha^n\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|\\
&\leq \|x^n - z\|^2 - r^n(2\alpha - r^n)\|A(x^n) - A(z)\|^2 + 2\alpha^n\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|.\\
&\quad \text{(using (4.3))}
\end{aligned}$$

Hence,

$$\begin{aligned}
&r^n(2\alpha - r^n)\|A(x^n) - A(z)\|^2\\
&\leq \|x^{n+1} - x^n\|(\|x^{n+1} - z\| + \|x^n - z\|) + 2\alpha^n\|\gamma f(x^n) - B(z)\|\|x^{n+1} - z\|\\
&\leq \|x^{n+1} - x^n\|(\|x^{n+1} - z\| + \|x^n - z\|) + 2\alpha^n M_1.
\end{aligned}$$

Using $0 < \liminf_{n\to\infty} r^n \leq \limsup_{n\to\infty} r^n < 2\alpha$, $\alpha^n \to 0$ as $n \to \infty$ and (4.17) in above inequality, we have

$$\lim_{n\to\infty} \|A(x^n) - A(z)\| = 0. \tag{4.27}$$

From (4.23), we have

$$\begin{aligned}
&\|x^{n+1} - z\|^2\\
&\leq (1 - \alpha^n\overline{\gamma})^2\|u^n - z\|^2 + 2\alpha^n\gamma\|x^{n+1} - z\|\|x^n - z\|\\
&\quad + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\|\\
&\leq (1 - \alpha^n\overline{\gamma})^2[\|x^n - z\|^2 - \|u^n - y^n\|^2 + 2r^n\|x^n - z\|\|A(z) - A(y^n)\|]\\
&\quad + 2\alpha^n\gamma\|x^{n+1} - z\|\|x^n - z\| + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\|.\\
&\quad \text{(using (4.25))}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(1 - \alpha^n\overline{\gamma})^2\|u^n - y^n\|^2\\
&\leq (1 - \alpha^n\overline{\gamma})^2\|x^n - z\|^2 - \|x^{n+1} - z\|^2\\
&\quad + (1 - \alpha^n\overline{\gamma})^2 2r^n\|x^n - z\|\|A(z) - A(y^n)\|\\
&\quad + 2\alpha^n\gamma\|x^{n+1} - z\|\|x^n - z\| + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\|\\
&\leq \|x^n - z\|^2 - \|x^{n+1} - z\|^2 + (1 - \alpha^n\overline{\gamma})^2 2r^n\|x^n - z\|\|A(z) - A(y^n)\|
\end{aligned}$$

$$
\begin{aligned}
&+ 2\alpha^n\gamma\|x^{n+1}-z\|\|x^n-z\| + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\| \\
&\le \|x^{n+1}-x^n\|(\|x^{n+1}-z\|+\|x^n-z\|) \\
&\quad + (1-\alpha^n\overline{\gamma})^2 2r^n\|x^n-z\|\|A(z)-A(y^n)\| \\
&\quad + 2\alpha^n\gamma\|x^{n+1}-z\|\|x^n-z\| + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\|.
\end{aligned}
$$

Taking $n \to \infty$ and using (4.26), (4.17) and $\alpha^n \to 0$ in above inequality, we have

$$
\lim_{n\to\infty}\|u^n-y^n\| = 0. \tag{4.28}
$$

Using firmly nonexpansivity of T_{r^n}, we estimate

$$
\begin{aligned}
&\|y^n-z\|^2 \\
&= \|T_{r^n}(x^n-r^nA(x^n)) - T_{r^n}(z-r^nA(z))\|^2 \\
&\le \langle T_{r^n}(x^n-r^nA(x^n)) - T_{r^n}(z-r^nA(z)), x^n-r^nA(x^n)-(z-r^nA(z))\rangle \\
&= \langle y^n-z, x^n-z+r^nA(z)-r^nA(x^n)\rangle \\
&= \langle y^n-z, x^n-z\rangle + r^n\langle y^n-z, A(z)-A(x^n)\rangle \\
&= \frac{1}{2}(\|y^n-z\|^2+\|x^n-z\|^2-\|y^n-x^n\|^2) + r^n\|y^n-z\|\|A(z)-A(x^n)\|.
\end{aligned}
$$

Therefore,

$$
\|y^n-z\|^2 \le \|x^n-z\|^2 - \|y^n-x^n\|^2 + r^n\|x^n-z\|\|A(z)-A(x^n)\|. \tag{4.29}
$$

From (4.23), we have

$$
\begin{aligned}
&\|x^{n+1}-z\|^2 \\
&\le (1-\alpha^n\overline{\gamma})^2\|u^n-z\|^2 + 2\alpha^n\gamma\|x^{n+1}-z\|\|x^n-z\| \\
&\quad + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\| \\
&\le (1-\alpha^n\overline{\gamma})^2[\|y^n-z\|^2 - \|u^n-y^n\|^2 + 2r^n\|x^n-z\|\|A(z)-A(y^n)\|] \\
&\quad + 2\alpha^n\gamma\|x^{n+1}-z\|\|x^n-z\| + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\| \\
&\le (1-\alpha^n\overline{\gamma})^2[\|x^n-z\|^2 - \|y^n-x^n\|^2 + r^n\|y^n-z\|\|A(x^n)-A(z)\| \\
&\quad - (1-\alpha^n\overline{\gamma})^2\|u^n-y^n\|^2 + (1-\alpha^n\overline{\gamma})^2 2r^n\|x^n-z\|\|A(z)-A(y^n)\| \\
&\quad + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\| + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(1-\alpha^n\overline{\gamma})^2\|y^n-x^n\|^2 \\
&\le (1-\alpha^n\overline{\gamma})^2\|x^n-z\|^2 - \|x^{n+1}-z\|^2 \\
&\quad + (1-\alpha^n\overline{\gamma})^2 r^n\|y^n-z\|\|A(x^n)-A(z)\| - (1-\alpha^n\overline{\gamma})^2\|u^n-y^n\|^2 \\
&\quad + (1-\alpha^n\overline{\gamma})^2 2r^n\|x^n-z\|\|A(z)-A(y^n)\| \\
&\quad + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\| + 2\alpha^n\|\gamma f(z)-B(z)\|\|x^{n+1}-z\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \|x^n - z\|^2 - \|x^{n+1} - z\|^2 + (1-\alpha^n\overline{\gamma})^2 r^n \|y^n - z\|\|A(x^n) - A(z)\| \\
&\quad - (1-\alpha^n\overline{\gamma})^2\|u^n - y^n\|^2 + (1-\alpha^n\overline{\gamma})^2 2r^n\|x^n - z\|\|A(z) - A(y^n)\| \\
&\quad + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\| + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\| \\
&\leq \|x^{n+1} - x^n\|(\|x^{n+1} - z\| + \|x^n - z\|) \\
&\quad + (1-\alpha^n\overline{\gamma})^2 r^n\|y^n - z\|\|A(x^n) - A(z)\| - (1-\alpha^n\overline{\gamma})^2\|u^n - y^n\|^2 \\
&\quad + (1-\alpha^n\overline{\gamma})^2 2r^n\|x^n - z\|\|A(z) - A(y^n)\| \\
&\quad + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\| + 2\alpha^n\|\gamma f(z) - B(z)\|\|x^{n+1} - z\|.
\end{aligned}$$

Taking $n \to \infty$ and using (4.26), (4.27), (4.17), (4.28) and $\alpha^n \to 0$ as $n \to \infty$ in above, we have

$$\lim_{n\to\infty} \|y^n - x^n\| = 0. \tag{4.30}$$

Now,

$$\|u^n - x^n\| \leq \|u^n - y^n\| + \|y^n - x^n\|.$$

Using (4.28) and (4.30) in above inequality, we have

$$\lim_{n\to\infty} \|u^n - x^n\| = 0.$$

Further, it follows from (4.19), $\lim_{n\to\infty} \|u^n - x^n\| = 0$, and $\lim_{n\to\infty} \beta^n = 0$ that

$$\|z^n - u^n\| \leq (1-\beta^n)[\|w^n - x^n\| + \|x^n - u^n\| \to 0 \text{ as } n \to \infty.$$

Since $\{x^n\}$ is bounded, there exists a weakly convergent subsequence $\{x^{n_k}\}$ of $\{x^n\}$, say $x^{n_k} \rightharpoonup \hat{x}$. Then it follows from (4.30) that there exists also a weakly convergent subsequence $\{y^{n_k}\}$ of $\{y^n\}$ such that $y^{n_k} \rightharpoonup \hat{x}$.

Now, we show that $\hat{x} \in \mathrm{Sol(GMEP(1.2))} \cap \mathrm{Fix}(\Im)$. First, we show that $\hat{x} \in \mathrm{Fix}(\Im)$. Assume that $\hat{x} \in \mathrm{Fix}(\Im)$. Since $x^{n_k} \rightharpoonup \hat{x}$ and $T(s)\hat{x} \neq \hat{x}$, from Opial's condition (2.6), we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x^{n_k} - \hat{x}\| &< \liminf_{n\to\infty} \|x^{n_k} - T(s)\hat{x}\| \\
&\leq \liminf_{n\to\infty} \{\|x^{n_k} - T(s)x^{n_k}\| + \|T(s)x^{n_k} - T(s)\hat{x}\|\} \\
&\leq \liminf_{n\to\infty} \|x^{n_k} - \hat{x}\|
\end{aligned}$$

which is a contradiction. Thus, we obtain $\hat{x} \in \mathrm{Fix}(\Im)$.

Next, we show that $\hat{x} \in \mathrm{Sol(GMEP(1.2))}$. The relation $y^n = T_{r^n}(x^n - r^n A(x^n))$ implies that

$$F(y^n, y) + \phi(y, y^n) - \phi(y^n, y^n) + \frac{1}{r^n}\langle y - y^n, y^n - (x^n - r^n A(x^n))\rangle \geq 0, \ \forall y \in C,$$

$$\phi(y, y^n) - \phi(y^n, y^n) + \frac{1}{r^n}\langle y - y^n, y^n - x^n\rangle \geq F(y, y^n) + \langle A(x^n), y^n - y\rangle, \ \forall y \in C,$$

using monotonicity of F.

Hence,

$$\phi(y, y^{n_k}) - \phi(y^{n_k}, y^{n_k}) + \langle y - y^{n_k}, \frac{y^{n_k} - x^{n_k}}{r^{n_k}} \rangle \geq F(y, y^{n_k}) + \langle A(x^{n_k}), y^{n_k} - y \rangle, \ \forall y \in C. \tag{4.31}$$

For t, with $0 < t \leq 1$, let $y_t = ty + (1-t)\hat{x} \in C$. Then from (4.31), we have

$$\begin{aligned}\langle A(y_t), y_t - y^{n_k} \rangle \geq \langle A(y_t) - A(y^{n_k}), y_t - y^{n_k} \rangle - \phi(y_t, y^{n_k}) + \phi(y^{n_k}, y^{n_k}) \\ - \langle y - y^{n_k}, \frac{y^{n_k} - x^{n_k}}{r^{n_k}} \rangle + F(y_t, y^{n_k}) \\ - \langle A(x^{n_k}) - A(y^{n_k}), y_t - y^{n_k} \rangle.\end{aligned}$$

Since A is α-inverse strongly monotone then it is $\frac{1}{\alpha}$-Lipschitz continuous and monotone. Again since $\|y^{n_k} - x^{n_k}\| \to 0$ then $\|A(y^{n_k}) - A(x^{n_k})\| \to 0$. Further, since ϕ is weak continuous and F is weak lower semicontinuous in second argument, then above inequality implies that

$$\phi(y_t, \hat{x}) - \phi(\hat{x}, \hat{x}) \geq F(y_t, \hat{x}) - \langle A(y_t), y_t - \hat{x} \rangle. \tag{4.32}$$

Now, for $t > 0$,

$$\begin{aligned}0 &= F(y_t, y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \text{ (using (4.32))} \\ &\leq tF(y_t, y) + (1-t)[\phi(y_t, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle A(y_t), y_t - \hat{x} \rangle] \\ &\leq tF(y_t, y) + (1-t)t[\phi(y, \hat{x}) - \phi(\hat{x}, \hat{x})] + (1-t)\langle A(y_t), ty + (1-t)\hat{x} - \hat{x} \rangle \\ &= F(y_t, y) + (1-t)[\phi(y, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle A(y_t), y - \hat{x} \rangle].\end{aligned}$$

Letting $t \to 0$ then by Assumption 3.1(3) and Lipschitz continuity of A, we have

$$F(\hat{x}, y) + \phi(y, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle A(\hat{x}), y - \hat{x} \rangle \geq 0, \ \forall y \in C.$$

This implies that $\hat{x} \in \mathrm{Sol}(\mathrm{GMEP}(1.2))$.

We claim that z^* is the unique solution of the variational inequality (4.1).

First, we show the uniqueness of the solution to the variational inequality (4.1) in Γ. Let $\bar{x}, \ \hat{x} \in \Gamma$, then

$$\langle (B - \gamma f)\bar{x}, \bar{x} - \hat{x} \rangle \leq 0,$$
$$\langle (B - \gamma f)\hat{x}, \hat{x} - \bar{x} \rangle \leq 0.$$

Adding the above two inequalities, we have

$$\begin{aligned}0 &\geq \langle B(\bar{x} - \hat{x}), \bar{x} - \hat{x} \rangle - \gamma \langle f(\bar{x}) - f(\hat{x}), \bar{x} - \hat{x} \rangle \\ &\geq \overline{\gamma} \|\bar{x} - \hat{x}\|^2 - \gamma \|\bar{x} - \hat{x}\|^2 + \gamma \psi(\|\bar{x} - \hat{x}\|) \|\bar{x} - \hat{x}\| \\ &= (\overline{\gamma} - \gamma) + \gamma \psi(\|\bar{x} - \hat{x}\|) \|\bar{x} - \hat{x}\|.\end{aligned}$$

Hence,

$$\psi(\|\bar{x} - \hat{x}\|) \leq \frac{\gamma - \overline{\gamma}}{\gamma} \|\bar{x} - \hat{x}\|.$$

Since $\frac{\gamma-\overline{\gamma}}{\gamma} \leq 0$, we have

$$\psi(\|\bar{x}-\hat{x}\|) \leq 0.$$

By the property of ψ, we have $\hat{x} = \bar{x}$.

Next, we prove $\langle (\gamma f - B)z^*, z - z^* \rangle \leq 0$ for any $z \in \Gamma$. Let $v^n = \alpha^n \gamma f(x^n) + (I - \alpha^n B)z^n$ then $x^{n+1} = P_C(v^n)$. Now, we estimate

$$\begin{aligned}
\|x^{n+1} - z\|^2 &= \|P_C(v^n) - z\|^2 \\
&\leq \langle v^n - z, x^{n+1} - z\rangle \\
&= \langle \alpha^n \gamma f(x^n) + (I - \alpha^n B)(z^n) - z, x^{n+1} - z\rangle \\
&= \alpha^n \langle \gamma f(x^n) - B(z^n), x^{n+1} - z\rangle + \langle z^n - z, x^{n+1} - z\rangle.
\end{aligned}$$

Since $\alpha^n \in (0,1)$, then above inequality implies that

$$\langle B(z^n) - \gamma f(x^n), x^{n+1} - z\rangle \leq \langle z^n - x^{n+1}, x^{n+1} - z\rangle. \tag{4.33}$$

Now, it follows from $\|z^n - u^n\| \to 0$, $\|x^n - u^n\| \to 0$ as $n \to \infty$ and $x^{n_k} \to z^* \in \Gamma$ that $\|z^{n_k} - u^{n_k}\| \to 0$ as $k \to \infty$. Replacing n in (4.33) with n_k and taking limit $k \to \infty$, we have

$$\langle (B - \gamma f)z^*, z^* - z\rangle \leq 0 \text{ for any } z \in \Gamma, \tag{4.34}$$

which implies that $z^* \in \Gamma$ is unique solution of (4.1), i.e., $z^* = P_\Gamma(I - \gamma f + B)$.

Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(z^*) - B(z^*), x^{n+1} - z^*\rangle \leq 0.$$

Indeed, we can consider a subsequence x^{n_k} of x^n such that

$$\limsup_{n\to\infty} \langle \gamma f(z^*) - B(z^*), x^{n+1} - z^*\rangle = \lim_{k\to\infty} \langle \gamma f(z^*) - B(z^*), x^{n_k+1} - z^*\rangle. \tag{4.35}$$

Since x^n is bounded therefore the subsequence x^{n_k} of x^n converges weakly to z. We have already proved that such $z \in \Gamma$. From (4.34) and (4.35), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \langle \gamma f(z^*) - B(z^*), x^{n+1} - z^*\rangle \\
&= \lim_{k\to\infty} \langle \gamma f(z^*) - B(z^*), x^{n_k+1} - z^*\rangle \\
&= \langle z - z^*, \gamma f(z^*) - B(z^*)\rangle \leq 0.
\end{aligned} \tag{4.36}$$

Finally, we show $x^n \to z^*$.

Let $v^n = \alpha^n \gamma f(x^n) + (I - \alpha^n B)z^n$. We estimate

$$\begin{aligned}
&\|x^{n+1} - z^*\|^2 \\
&= \|P_C(v^n) - z^*\|^2 \\
&\leq \langle P_C(v^n) - v^n, x^{n+1} - z^*\rangle + \alpha^n \langle \gamma f(x^n) - B(z^*), x^{n+1} - z^*\rangle \\
&\quad + \langle (I - \alpha^n B)(z^n - z^*), x^{n+1} - z^*\rangle \\
&\leq \alpha^n \langle \gamma f(x^n) - B(z^*), x^{n+1} - z^*\rangle + \langle (I - \alpha^n B)(z^n - z^*), x^{n+1} - z^*\rangle \\
&\leq \alpha^n \gamma \langle f(x^n) - f(z^*), x^{n+1} - z^*\rangle + \alpha^n \langle \gamma f(z^*) - B(z^*), x^{n+1} - z^*\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\langle (I-\alpha^n B)(z^n-z^*), x^{n+1}-z^*\rangle\\
&\le \alpha^n\gamma\|x^n-z^*\|\|x^{n+1}-z^*\| + \alpha^n\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle\\
&\quad +(1-\alpha^n\overline{\gamma})\|z^n-z^*\|\|x^{n+1}-z^*\|\\
&\le \alpha^n\gamma\|x^n-z^*\|\|x^{n+1}-z^*\| + \alpha^n\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle\\
&\quad +(1-\alpha^n\overline{\gamma})\|x^n-z^*\|\|x^{n+1}-z^*\|\\
&\le (1-\alpha^n(\overline{\gamma}-\gamma))\|x^n-z^*\|\|x^{n+1}-z^*\| + \alpha^n\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle\\
&\le \frac{(1-\alpha^n(\overline{\gamma}-\gamma))}{2}(\|x^n-z^*\|^2+\|x^{n+1}-z^*\|^2)\\
&\quad +\alpha^n\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x^{n+1}-z^*\|^2\\
&\le \frac{(1-\alpha^n(\overline{\gamma}-\gamma))}{(1+\alpha^n(\overline{\gamma}-\gamma))}\|x^n-z^*\|^2 + \frac{2\alpha^n}{(1+\alpha^n(\overline{\gamma}-\gamma))}\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle\\
&\le (1-\alpha^n(\overline{\gamma}-\gamma))\|x^n-z^*\|^2 + \frac{2\alpha^n}{(1+\alpha^n(\overline{\gamma}-\gamma))}\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle\\
&\le (1-a^n)\|x^n-z^*\|^2+\alpha^n b^n,
\end{aligned}
$$

where $a^n=\alpha^n(\overline{\gamma}-\gamma)$ and $b^n=\frac{2}{(1+\alpha^n(\overline{\gamma}-\gamma))}\langle \gamma f(z^*)-B(z^*), x^{n+1}-z^*\rangle$.

Thus, it follows from condition (i) and (4.36) that $\Sigma_{n=0}^{\infty}a^n<\infty$ and $\limsup b^n\le 0$. Therefore by Lemma 2.6, we can conclude that $x^n\to z^*=P_\Gamma(I-\gamma f+B)$. This completes the proof. □

Finally, we have the following consequence of Theorem 4.1, which generalizes Theorem 3.1 due to Xiao [15].

Corollary 4.1. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F: C\times C\to\mathbb{R}$ be nonlinear bifunctions satisfying Assumption 3.1* (1)-(4)*. Let f be a weakly contractive mapping with a function ψ on H, and let $B: H\to H$ be a strongly positive bounded linear operator with coefficient $\overline{\gamma}>0$, and $\Im=\{T(s): s\ge 0\}$ be a nonexpansive semigroup on C. Assume that $\Gamma:=\mathrm{Fix}(\Im)\cap\mathrm{Sol}(\mathrm{EP}(1.4))\neq\emptyset$. For any $0<\gamma\le\overline{\gamma}$, let the sequence $\{x^n\}$ generated by the following iterative schemes*:

$$
\begin{aligned}
x^0&\in C,\\
u^n&=T_r(x^n),\\
z^n&=\beta^n u^n+(1-\beta^n)\frac{1}{t_n}\int_0^{t_n}T(s)u^n ds,\\
x^{n+1}&=P_C[\alpha^n\gamma f(x^n)+(I-\alpha^n B)z^n],
\end{aligned}
$$

where $0<r<2\alpha$; $\{\alpha^n\}$, $\{\beta^n\}$ are the sequences in $(0,1)$ and $\{t^n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha^n = 0$, $\sum_{n=0}^{\infty} \alpha^n = \infty$, $\sum_{n=0}^{\infty} |\alpha^n - \alpha^{n-1}| < \infty$;
(ii) $\lim_{n\to\infty} \beta^n = 0$, $\sum_{n=0}^{\infty} |\beta^n - \beta^{n-1}| < \infty$;
(iii) $\sum_{n=0}^{\infty} \frac{|t^n - t^{n-1}|}{t^n} < \infty$;

Then the sequence $\{x^n\}$ *converges strongly to* $z^* \in \Gamma$ *which uniquely solves the following variational inequality*

$$\langle (B - \gamma f)z^*, z^* - z\rangle \leq 0 \text{ for any } z \in \Gamma.$$

Proof. It is on similar lines of proof of Theorem 4.1, and hence omitted. □

References

[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), no. 1-4, 123–145.
[2] F. E. Browder, *Fixed point theorems for noncompact mappings in Hilbert space*, Proc. Natl. Acad. Sci. (USA) **53** (1965), 1272–1276.
[3] L. C. Ceng, Q. H. Ansari, and J. C. Yao, *Some iterative methods for finding fixed points and solving constrained convex minimization problems*, Nonlinear Anal. **74** (2011), no. 16, 5286–5802.
[4] L. C. Ceng, T. Tanaka, and J. C. Yao, *Iterative construction of fixed points of nonself-mappings in Banach spaces*, J. Comput. Appl. Math. **206** (2007), no. 2, 814–825.
[5] F. Cianciaruso, G. Marino, and L. Muglia, *Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert space*, J. Optim. Theory Appl. **146** (2010), no. 2, 491–509.
[6] K. Fan, *A generalization of Tychonoff's fixed-point theorem*, Math. Ann. **142** (1961), 305–310.
[7] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics, **28**, Cambridge University Press, Cambridge, 1990.
[8] K. R. Kazmi and S. H. Rizvi, *A hybrid extragradient method for approximating the common solutions of a variational inequality, a system of variational inequalities, a mixed equilibrium problem and a fixed point problem*, Appl. Math. Comput. **218** (2012), no. 9, 5439–5452.
[9] ______, *Iterative approximation of a common solution of split generalized equilibrium problem and a fixed point problem for a nonexpansive semigroup*, Math. Sci. **7** (2013), Article 1.
[10] ______, *Implicit iterative method for approximating a common solution of split equilibrium problem and fixed point problem for a nonexpansive semigroup*, Arab J. Math. Sci. **20** (2014), no. 1, 57–75.
[11] G. Marino and H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **318** (2006), no. 1, 43–52.
[12] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), no. 4, 595–597.
[13] S. Plubtieng and R. Punpaeng, *Fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces*, Math. Comput. Modelling **48** (2008), no. 1-2, 279–286.
[14] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), no. 1, 71–83.
[15] X. Xiao, S. Li, L. Li, H. Song, and L. Zhang, *Strong convergence of composite general iterative methods for one-parameter nonexpansive semigroup and equilibrium problems*, J. Inequal. Appl. **2012** (2012), 131, 19 pp.
[16] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. **66** (2002), no. 1, 240–256.

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