LINEAR RANK PRESERVERS ON
INFINITE TRIANGULAR MATRICES

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Abstract. We consider $\mathcal{T}_\infty(F)$ – the space of all infinite upper triangular matrices over a field F. We give a description of all linear maps that satisfy the property: if $\mathrm{rank}(x) = 1$, then $\mathrm{rank}(\phi(x)) = 1$ for all $x \in \mathcal{T}_\infty(F)$. Moreover, we characterize all injective linear maps on $\mathcal{T}_\infty(F)$ such that if $\mathrm{rank}(x) = k$, then $\mathrm{rank}(\phi(x)) = k$.

1. Introduction

Linear preserver problem is an intensively studied branch of the matrix theory. One of its most popular issues is the problem of describing rank preservers.

Let $\mathcal{S}$ be a space of matrices over a field F. We say that $\phi : \mathcal{S} \to \mathcal{S}$ preserves rank k matrices if

$$\mathrm{rank}(x) = k \quad \text{implies} \quad \mathrm{rank}(\phi(x)) = k \quad \text{for all } x \in \mathcal{S}.$$

We always assume that ϕ is linear, i.e.,

$$\phi(\alpha x + \beta y) = \alpha \phi(x) + \beta \phi(y) \quad \text{for all } x, y \in \mathcal{S}, \ \alpha, \beta \in F.$$

The problem of characterizing such maps was considered in many papers. One of the first solutions, for the case of all rank one preservers on $M_{n \times m}(F)$ were given in [13] and [14]. Generalization of that – the maps preserving rank less or equal to one were described in [6], preservers of sets of ranks were studied in [2, 12]. This issue was also generalized in various ways, e.g. there were considered matrices over rings [18], or instead of matrix spaces there were investigated some algebras on Banach spaces [16].

There were also studied some special cases, e.g. when ϕ is nonsingular (like in [3, 10, 18]). In particular, it was proved that the nonsingular maps ϕ preserving rank k ($k \in \mathbb{N}$) are of one of the forms

$$(1) \qquad\qquad \phi(X) = PXQ \,,$$

$$\phi(X) = PX^T Q \,,$$

where P, Q are nonsingular matrices of a proper dimension and X^T is the matrix transposed to X.

We will focus on rank preservers of upper triangular matrices. Such linear maps were investigated in [9, 15], additive linear preservers – in [5, 8], and modular automorphisms – in [7]. In general, the authors have shown that if $\phi : \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ is a linear rank-one preserver (where $\mathcal{T}_n(F)$ denotes the space of $n \times n$ upper triangular matrices over F), then ϕ either sends $\mathcal{T}_n(F)$ (the space of upper triangular matrices) to some rank one subspace, or is of form (1) for some nonsingular upper triangular P, Q.

In this paper we will take into consideration $\mathcal{T}_\infty(F)$ – the space of infinite upper triangular matrices (in which rows and columns are indexed by natural numbers) over a field F, i.e. the space consisting of the matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots \\ 0 & x_{22} & x_{23} & \\ 0 & 0 & x_{33} & \\ \vdots & & & \ddots \end{pmatrix}.$$

Let $k \in \mathbb{N}$, we denote by $\mathcal{R}_k(\mathcal{T}_\infty(F))$ the set of all linear rank-k preservers on $\mathcal{T}_\infty(F)$. We will prove:

Theorem 1.1. *Assume that F is a field and that ϕ is a linear rank-one preserver. Then either*

    (1) *the image of ϕ consists only of matrices of rank one and the zero matrix or*

    (2) *there exist matrices a, b such that $\phi(x) = axb$ for $x \in \mathcal{T}_\infty(F)$.*

Note that we do not claim that the matrices a, b which appear in the second point of Theorem 1.1, are upper triangular or invertible. Some more information about a and b will be given at the end of Section 2.

After proving the above theorem, we will point out some connections between the linear maps preserving rank one and the linear maps preserving rank one idempotents. Next we will move to linear rank-k preservers for arbitrary $k \in \mathbb{N}$. We will show that it holds:

Theorem 1.2. *Let F be any field and let $k \in \mathbb{N}$, $k \geq 2$. Assume that $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is an injection. Then ϕ is a linear rank-k preserver if and only if ϕ is a linear rank-one preserver and $\phi(\mathcal{T}_\infty(F))$ is not rank one subspace of $\mathcal{T}_\infty(F)$.*

2. Proofs of first results

Let us start with explaining the notation. By e_{nm} we mean an infinite matrix, whose rows and columns are indexed by natural numbers, with 1 in the position (n, m) and with 0 in every other position. We write e_∞ for the infinite identity matrix.

If x is a matrix that has nonzero entries only in the positions (i, j), where $i \in I$, $j \in J$, then we will write

$$x = \sum_{i \in I, j \in J} x_{ij} e_{ij}.$$

By $M_{1 \times \infty}(F)$ we denote the infinite dimensional space $F^{\mathbb{N}}$, i.e., the space consisting of vectors of form

(2) $\qquad (x_1, x_2, x_3, \ldots) \qquad$ with $x_n \in F$ for $n \in \mathbb{N}$.

The symbol $M_{\infty \times 1}^{fin}(F)$ stands for the space over F which consists of vectors of form

(3) $\qquad (x_1, x_2, x_3, \ldots)^T, \qquad x_n \in F, \quad |\{x_n \neq 0\}| < \infty,$

where x^T denotes the vector transposed to x.

In $M_{1 \times \infty}(F)$, $M_{\infty \times 1}^{fin}(F)$ we introduce the following subspaces: $U_\infty^k(F)$ – that contains vectors of form (2) such that $x_i = 0$ for $i < k$, $V_\infty^k(F)$ – that contains vectors of form (3) such that $x_i = 0$ for $i > k$. Moreover, we put $U_\infty^0(F) = M_{1 \times \infty}(F)$, $V_\infty^0(F) = M_{\infty \times 1}^{fin}(F)$.

By e_k we denote the vector of form (3) with $x_k = 1$ and $x_i = 0$ for $i \neq k$, and by f_k – the vector e_k^T. Notice that we have $e_n f_m = e_{nm}$.

If U is a subspace of $\mathcal{T}_\infty(F)$ (or $\mathcal{T}_n(F)$) such that for every $x \in U$ we have either $\mathrm{rank}(x) = 1$ or $x = 0$, then we will call U a rank one subspace.

We also introduce $\mathcal{T}_\infty^{fin}(F)$ – the subspace of all infinite matrices with finite support (i.e., having only finite number of nonzero entries).

Moreover, for any field F, F^* stands for $F \setminus \{0\}$.

Before we prove our main result let us look at two examples.

Example 2.1. For $n \in \mathbb{N}$, define $\phi_{n\to}$ by the formula

$$\phi_{n\to}(\sum_{i \leq j} x_{ij} e_{ij}) = \sum_{i \leq j} x_{ij} e_{i, j+n}.$$

In particular

$$\phi_{1\to}\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{23} & x_{24} & \\ 0 & 0 & x_{33} & x_{34} & \\ 0 & 0 & 0 & x_{44} & \\ & & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{11} & x_{12} & x_{13} & \cdots \\ 0 & 0 & x_{22} & x_{23} & \\ 0 & 0 & 0 & x_{33} & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Clearly, $\phi_{n\to}$ preserves rank one.

Example 2.2. Again let $n \in \mathbb{N}$ and

$$\phi_{n\searrow}(\sum_{i \leq j} x_{ij} e_{ij}) = \sum_{i \leq j} x_{ij} e_{i+n, j+n}.$$

This time for $n = 1$:

$$\phi_{n \searrow} \left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{23} & x_{24} & \\ 0 & 0 & x_{33} & x_{34} & \\ 0 & 0 & 0 & x_{44} & \\ & & & & \ddots \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & x_{11} & x_{12} & x_{13} & \\ 0 & 0 & x_{22} & x_{23} & \\ 0 & 0 & 0 & x_{33} & \\ & & & & \ddots \end{pmatrix}.$$

Examples 2.1 and 2.2 show that there exist maps preserving not only rank one, but also every rank and which are not given by a formula $\phi(X) = PXQ$, where P, Q are nonsingular triangular matrices.

Let us now present three remarks.

Lemma 2.1. *Let F be any field. A matrix $x \in \mathcal{T}_\infty(F)$ has rank one if and only if it is of form $x = vu$ for some $v \in M_{\infty \times 1}^{fin}(F)$, $u \in M_{1 \times \infty}(F)$.*

Lemma 2.2. *Let F be a field and let $v_1, v_2 \in M_{\infty \times 1}^{fin}(F)$, $u_1, u_2 \in M_{1 \times \infty}(F)$ and $x = v_1 u_1 + v_2 u_2$. If x has rank one, then either v_1, v_2 are linearly dependent or u_1, u_2 are linearly dependent.*

Lemma 2.3. *Let F be a field and $\phi \in \mathcal{R}_1(\mathcal{T}_\infty(F))$. Assume that for all $n < m < m'$ we have $\phi(e_{nm}) = v_1 u_1$, $\phi(e_{nm'}) = v_2 u_2$ for some $v_1, v_2 \in M_{\infty \times 1}^{fin}(F)$, $u_1, u_1 \in M_{1 \times \infty}(F)$. Then either v_1, v_2 are linearly independent and u_1, u_2 are not, or u_1, u_2 are linearly independent and v_1, v_2 are not.*

Proof. From Lemma 2.2, it follows that either v_1, v_2 or u_1, u_2 are linearly dependent. Assume that the both possibilities hold. Then for some $\alpha, \beta \in F$ we would have $\alpha\phi(e_{nm}) + \beta\phi(e_{nm'}) = 0$, i.e., $\text{rank}(\alpha\phi(e_{nm}) + \beta\phi(e_{nm'})) = 0$, although $\text{rank}(e_{nm} + e_{nm'}) = 1$. Hence, the claim must hold. $\square$

In the same way it can be proved:

Lemma 2.4. *Let F be a field and $\phi \in \mathcal{R}_1(\mathcal{T}_\infty(F))$. Assume that $\phi(e_{nm}) = v_1 u_1$, $\phi(e_{n'm}) = v_2 u_2$ for some $n < n' < m$, $v_1, v_2 \in M_{\infty \times 1}^{fin}(F)$, $u_1, u_1 \in M_{1 \times \infty}(F)$. Then either v_1, v_2 are linearly independent and u_1, u_2 are not, or u_1, u_2 are linearly independent and v_1, v_2 are not.*

The proof of Theorem 1.1 follows from theorems that we will now present.

Theorem 2.1. *Let F be an arbitrary field and let $\phi : \mathcal{T}_\infty^{fin}(F) \to \mathcal{T}_\infty(F)$ be a map preserving rank one matrices. Then either*

(1) *there exist a map $u : \mathbb{N} \times \mathbb{N} \to M_{1 \times \infty}(F)$ and a vector $v \in M_{\infty \times 1}^{fin}(F)$ such that*
   (a) *for all $n \leq m$ we have $vu(n, m) \in \mathcal{T}_\infty(F)$,*
   (b) *for all $n \in \mathbb{N}$ the set $\{u(n, m) : m \geq n\}$ is linearly independent,*
   (c) *for all $m \in \mathbb{N}$ the set $\{u(n, m) : n \leq m\}$ is linearly independent, and for these v, u we have*

$$\phi(\sum_{n \leq m} x_{nm} e_{nm}) = \sum_{n \leq m} x_{nm} vu(n, m),$$

or

(2) *there exists a pair of maps* $v : \mathbb{N} \to M^{fin}_{\infty \times 1}(F)$, $u : \mathbb{N} \to M_{1 \times \infty}(F)$
   satisfying conditions
   (a) $v(n)u(m) \in \mathcal{T}_\infty(F)$ *for all* $n \le m$,
   (b) *the set* $\{v(n)\}_{n \in \mathbb{N}}$ *is linearly independent,*
   (c) *the set* $\{u(m)\}_{m \in \mathbb{N}}$ *is linearly independent*
   for which we have

$$\phi(\sum_{n \le m} x_{nm} e_{nm}) = \sum_{n \le m} x_{nm} v(n) u(m) .$$

Proof. Let us begin with discussion on images of e_{nm} $(n, m \in \mathbb{N})$. All these matrices are of rank one, so for each of them there exist $v(n, m) \in M^{fin}_{\infty \times 1}(F)$, $u(n, m) \in M_{1 \times \infty}(F)$ such that $\phi(e_{nm}) = v(n, m)u(n, m) \in \mathcal{T}_\infty(F) \setminus \{0\}$.

Fix now $n \in \mathbb{N}$ and consider the set $R_n = \{e_{nm} : m \in \mathbb{N}, \ m \ge n\}$. For every three pairwise distinct elements from R_n we have

$$\text{rank}(e_{nm} + e_{nm'}) = 1 \quad \text{and} \quad \text{rank}(e_{nm} + e_{nm''}) = 1.$$

Hence $\text{rank}(\phi(e_{nm}) + \phi(e_{nm'})) = 1$, $\text{rank}(\phi(e_{nm}) + \phi(e_{nm''})) = 1$ and consequently either $\alpha_{nm'}v(n, m') = \alpha_{nm}v(n, m)$ for some $\alpha_{nm'}, \alpha_{nm} \in F^*$ (and then by Lemmas 2.3 and 2.4 $u(n, m), u(n, m')$ are linearly independent) or $\beta_{nm'}u(n, m') = \beta_{nm}u(n, m)$ for some $\beta_{nm}, \beta_{nm'} \in F^*$ (and then $v(n, m)$, $v(n, m')$ are linearly independent), and the same holds for the pair m, m''. Assume that $v(n, m') = v(n, m)$ and $u(n, m'') = u(n, m)$. This situation is
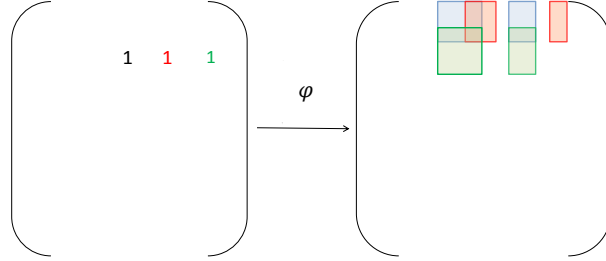

FIGURE 1. Picture to the proof of Theorem 2.1.

symbolically depicted in Figure 1.

Then

$$\phi(e_{nm} + e_{nm'} + e_{nm''})$$
$$= v(n, m) \left[\alpha u(n, m) + \beta u(n, m') \right] + v(n, m'') \cdot \gamma \cdot u(n, m)$$
$$= \left[\alpha' v(n, m) + \beta' v(n, m'') \right] u(n, m) + \gamma' v(n, m) u(n, m') .$$

From the fact that $\text{rank}(e_{nm} + e_{nm'} + e_{nm''}) = 1$, it follows that either $u(n, m)$, $u(n, m')$ are linearly dependent or $v(n, m)$, $v(n, m'')$ are linearly dependent – a contradiction.

Therefore either

$$\alpha_{nm} v(n, m) = \alpha_{nm'} v(n, m') \quad \text{for all } m, m' \geq n$$

or

$$\beta_{nm} u(n, m) = \beta_{nm'} u(n, m') \quad \text{for all } m, m' \geq n.$$

Suppose that we have $\beta_{nm} u(n, m) = \beta_{nm'} u(n, m')$ for all $m, m' \geq n$, and write $u(n, m)$ as $\beta_{nm} u(n)$. Let j_{min} stand for the smallest index j for which we have $(u(n))_j \neq 0$. Notice that since $v(n, m) u(n) \in \mathcal{T}_\infty(F)$ for all $m \geq n$, $(v(n, m))_i = 0$ for all $i > j_{min}$. Then $v(n, m) \in V_\infty^{j_{min}}(F)$ for all $m \geq n$. However $\{v(n, m) : m \in \mathbb{N}\}$ is linearly independent, whereas any subset of $M_{\infty \times 1}^{j_{min}}(F)$ containing more than j_{min} elements, is not – a contradiction. We must have $\alpha_{nm'} v(n, m') = \alpha_{nm} v(n, m)$ for all $n \in \mathbb{N}$, $m, m' \geq n$.

Now we fix $m \in \mathbb{N}$ and consider the set $C_m := \{e_{nm} : n \leq m\}$. By arguments analogous to the given for R_n, we have either

$$\alpha_{n'm} v(n', m) = \alpha_{nm} v(n, m) \quad \text{for } n, n' \leq m$$

or

$$\beta_{n'm} u(n', m) = \beta_{nm} u(n, m) \quad \text{for all } n, n' \leq m.$$

Suppose first that $\alpha_{n'm} v(n', m) = \alpha_{nm} v(n, m)$. Since we also have $\alpha_{nm} v(n, m) = \alpha_{nm'} v(n, m')$, we get that $\alpha_{nm} v(n, m) = \alpha_{n'm'} v(n', m')$ for all $n \leq m$, $n' \leq m'$. Denoting $v(n, m)$ by $\alpha'_{nm} v$ we can write that $\phi(e_{nm}) = \alpha'_{nm} v u(n, m) = v u'(n, m)$, where $\{u'(n, m) : n \leq m\}$ is linearly independent set, as well as $\{u'(n, m) : m \geq n\}$.

Assume now that we have $\beta_{n'm} u(n', m) = \beta_{nm} u(n, m)$ and write $u(n, m)$ as $\beta'_{nm} u(m)$. In this case $\{v(n, m) : n \leq m\} = \{v(n) : n \leq m\}$ is linearly independent, and so is $\{v(n) : n \in \mathbb{N}\}$. Moreover, we have

$$\phi(e_{nm}) = \alpha'_{nm} \beta'_{nm} v(n) u(m) = \gamma_{nm} v(n) u(m).$$

Suppose that we have $\alpha_{nm} v(n, m) = \alpha_{n'm} v(n', m)$ and $\beta_{nm'} u(n, m') = \beta_{n'm'} u(n', m')$. Since

$$\text{rank}(e_{nm} + e_{n'm} + e_{nm'} + e_{n'm'}) = 1,$$

we also have

$$\text{rank}(\phi(e_{nm}) + \phi(e_{n'm}) + \phi(e_{nm'}) + \phi(e_{n'm'})) = 1,$$

so

$$\text{rank}\left[v(n, m)(\alpha u(n, m) + \beta u(n', m) + \gamma u(n, m')) + \delta v(n, m') u(n, m')\right] = 1$$

for some $\alpha, \beta, \gamma, \delta \in F$. As $v(n, m)$, $v(n, m')$ are linearly independent, this means that $u(n, m)$, $u(n', m)$, $u(n, m')$ are linearly independent – a contradiction.

Summing up, we have proved that for every $n, m \in \mathbb{N}$, $n \leq m$, $\phi(e_{nm}) = vu'(n,m)$ in case (1) and $\phi(e_{nm}) = \gamma_{nm}v(n)u(m)$ in case (2).

Notice that from

$$\text{rank}(e_{nm} + e_{nm'} + e_{n'm} + e_{n'm'}) = 1 \quad \text{for } n < n',\ n < m,\ n' < m',$$

it follows that

$$\begin{aligned}
\text{(4)} \quad &\text{rank}\left[(\gamma_{nm}v(n) + \gamma_{n'm}v(n'))u(m) + (\gamma_{nm'}v(n) + \gamma_{n'm'}v(n'))u(m')\right] \\
&= \text{rank}\left[v(n)(\gamma_{nm}u(m) + \gamma_{nm'}u(m')) + v(n')(\gamma_{n'm}u(m) + \gamma_{n'm'}u(m'))\right] \\
&= 1
\end{aligned}$$

for some $\gamma_{nm}, \gamma_{nm'}, \gamma_{n'm}, \gamma_{n'm'} \in F$. From (4) we obtain

$$\frac{\gamma_{nm'}}{\gamma_{nm}} = \frac{\gamma_{n'm'}}{\gamma_{n'm}} \qquad \text{and} \qquad \frac{\gamma_{n'm}}{\gamma_{nm}} = \frac{\gamma_{n'm'}}{\gamma_{nm'}},$$

i.e., the coefficients γ_{nm}, $\gamma_{nm'}$ are equal for $m, m' \neq 1$. Hence we can assume that γ_{nm} depends only on m and write simply $\phi(e_{nm}) = v'(n)u(m)$.

Now notice that from the linearity of ϕ, it follows that for every $x \in \mathcal{T}_\infty^{fin}(F)$ we have $\phi(x) = \sum_{n \leq m} x_{nm}v'(n)u(m)$ or $\phi(x) = \sum_{n \leq m} x_{nm}vu'(n,m)$. With no loss of generality we can consider this first possibility. $\qquad \square$

Theorem 2.2. *Let F be an arbitrary field. If $\phi : \mathcal{T}_\infty^{fin}(F) \to \mathcal{T}_\infty(F)$ preserves rank one matrices and the image of ϕ is not a rank one subspace of $\mathcal{T}_\infty(F)$, then there exist matrices a, b such that $\phi(x) = axb$ for all $x \in \mathcal{T}_\infty^{fin}(F)$.*

Proof. If ϕ satisfies the given assumptions and $\phi(\mathcal{T}_\infty^{fin}(F))$ is not rank one subspace, then ϕ must be of form described in point (2) of Theorem 2.1. Hence we have $\phi(e_{nm}) = v(n)u(m)$ for $n \leq m$. Let a, b be defined by the conditions

$$ae_n = v(n), \qquad f_m b = u(m) \qquad \text{for all } n, m \in \mathbb{N}.$$

Then $\phi(e_{nm}) = u(n)v(m) = (ae_n)(f_m b) = ae_{nm}b$, which yields $\phi(x) = axb$ for all $x \in \mathcal{T}_\infty^{fin}(F)$. $\qquad \square$

Proof of Theorem 1.1. Consider $\phi \in \mathcal{R}_1(\mathcal{T}_\infty(F))$. Clearly for all $x \in \mathcal{T}_\infty^{fin}(F)$ such that $\text{rank}(x) = 1$ we have $\text{rank}(\phi(x)) = 1$. Hence, by Theorem 2.1 we have two possibilities: either $\phi(\mathcal{T}_\infty^{fin}(F))$ is a rank one subspace, or there exist a, b such that $\phi(x) = axb$ for all $x \in \mathcal{T}_\infty^{fin}(F)$. With no loss of generality consider the second case. (The proof of the first one is analogous.)

Consider then $x \in \mathcal{T}_\infty(F)$ that is not in $\mathcal{T}_\infty^{fin}(F)$. Assume that

$$\phi(x) \neq \sum_{n \leq m} x_{nm}v(n)u(m).$$

In this case, there exist indices n, m such that

$$(\phi(x))_{nm} \neq \left(\sum_{i \leq j} x_{ij}v(i)u(j)\right)_{nm}.$$

Take the minimal m for which the latter inequality holds, and for this n choose the minimal m. There exists only a finite number of pairs i, j such that $(v(i)u(j))_{nm} \neq 0$. Denote the maximal j for which it holds by k. Let $x^{(k)}$ be an infinite upper triangular matrix such that

$$(5) \qquad (x^{(k)})_{ij} = \begin{cases} x_{ij} & \text{for } i \leq j \leq k \\ 0 & \text{otherwise,} \end{cases}$$

and let $y^{(k)} = x - x^{(k)}$. Since ϕ is linear and $x^{(k)}$ is of form (5), we have $(\phi(y^{(k)}))_{nm} \neq 0$. Define now $x^{(k)+1}$ as $\sum_{i=1}^{k+1} x_{i,k+1}e_{i,k+1}$ and $y^{(k)+1}$ as $y^{(k)} - x^{(k+1)}$. Then we have $(\phi(y^{(k+1)})_{nm} \neq 0$.

Suppose that the first $k+r$ columns of $y^{(k)+r}$ are zero and that $(\phi(y^{(k)+r}))_{nm} \neq 0$. If we define now $x^{(k)+r+1}$ as $\sum_{i=1}^{k+r+1} x_{i,k+r+1}e_{i,k+r+1}$ and $y^{(k)+r+1}$ as $y^{(k)+r} - x^{(k)+r+1}$, we obtain that $y^{(k)+r+1}$ has the first $k+r+1$ columns zero and $(\phi(y^{(k)+r+1}))_{nm} \neq 0$. Now from induction, it follows that $(\phi(0))_{nm} \neq 0$, which clearly contradicts the linearity of ϕ. Hence, we must have

$$\phi\left(\sum_{n \leq m} x_{nm}e_{nm}\right) = \sum_{n \leq m} x_{nm}v(n)u(m)$$

for all $x \in \mathcal{T}_\infty(F)$. $\square$

Let us now present some examples.

The classical example of a linear rank one preserver on $\mathcal{T}_n(F)$ is ϕ such that

$$\phi\left(\sum_{i \leq j} x_{ij}e_{ij}\right) = \begin{pmatrix} \sum_{i=1}^n x_{ii} & \sum_{i=1}^{n-1} x_{i,i+1} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

There does not exist an analogon to this map on $\mathcal{T}_\infty(F)$. However, we also give an example of ϕ such that all matrices in $\phi(\mathcal{T}_\infty(F))$ have rank not exceeding one.

Example 2.3. Define ϕ as follows:

$$\phi\left(\sum_{n \leq m} x_{nm}e_{nm}\right) = \sum_{n \leq m} x_{nm}e_{1,2^{n-1}(2m-1)} \, ,$$

i.e.,

$$\phi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{23} & x_{24} & \\ 0 & 0 & x_{33} & x_{34} & \\ 0 & 0 & 0 & x_{44} & \\ & & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{22} & x_{12} & x_{33} & x_{13} & x_{23} & x_{14} & x_{44} & \cdots \\ 0 & \cdots & & 0 & & \cdots & & 0 \\ 0 & \cdots & & 0 & & \cdots & & 0 \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}.$$

More precisely, the coefficients from the first row (in x) are in the odd columns, the coefficients from the second row are in the columns $4n+2$ ($n \in \mathbb{N}$), the

coefficients from the third row are in columns $8n + 4$, and so on. Obviously ϕ preserves rank one and $\phi(\mathcal{T}_\infty(F))$ is a rank one subspace of $\mathcal{T}_\infty(F)$.

Example 2.4. Consider the following ϕ.

$$\phi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \\ & & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} 0 & x_{11} + x_{12} & 2x_{11} + x_{12} & x_{13} & x_{14} & \cdots \\ 0 & x_{22} & x_{22} & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & 0 & x_{44} \\ & & & & & \ddots \end{pmatrix}.$$

We have $v(n) = e_n$ for $n \in \mathbb{N}$, $u(1) = (0, 1, 2, 0, 0, \ldots)$, $u(2) = (0, 1, 1, 0, 0, \ldots)$, $u(m) = f_{m+1}$ for $m \geq 3$, so

$$a = e_\infty, \qquad b = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \vdots & & & & & & \ddots \end{pmatrix}.$$

Moreover, we can see that for $\phi_{1\to}$ (from Example 2.1) we have

$$a = e_\infty, \qquad b = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots \end{pmatrix},$$

and for $\phi_{1\searrow}$ (from Example 2.2)

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & & & \ddots \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & & & & \ddots \end{pmatrix}.$$

The latter example shows that we can have $axb \in \mathcal{T}_\infty(F)$ for all upper triangular x, although $a \notin \mathcal{T}_\infty(F)$. It can be also noticed that the above a and b are both noninvertible – it can be easily checked that a does not have a right inverse, and b does not have any left one.

Example 2.5. Assume that τ is an increasing map on $\mathbb{N}$ and define $\mathcal{S}pl_\tau$ as in [17], i.e.,

$$\mathcal{S}pl_\tau\left(\sum_{i \leq j} x_{ij} e_{ij}\right) = \sum_{i \leq j} x_{ij} e_{\tau(i)\tau(j)}.$$

Then $\mathcal{S}pl_\tau$ preserves rank one as well. For instance, for $\tau(n) = 2n$ we have

$$
a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \\ \vdots & & & & & \ddots \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & & & \ddots \end{pmatrix}.
$$

As we may be interested in the case when the matrices a and b are invertible, we will introduce one more symbol. Namely, we will use $\mathcal{T}_\infty^*(F)$ for the set (which forms a multiplicative group) of all invertible elements of $\mathcal{T}_\infty(F)$.

From Theorem 2.2, it follows:

Corollary 2.1. *Let F be a field. If $\phi \in \mathcal{R}_1(\mathcal{T}_\infty(F))$ is surjective, then there exist $a, b \in \mathcal{T}_\infty^*(F)$ such that $\phi(x) = axb$ for all $x \in \mathcal{T}_\infty(F)$.*

Proof. If ϕ is surjective, then we can make use of Theorem 2.2.

Consider the matrices

$$\phi(e_{nn}) = v(n)u(n), \ \phi(e_{n,n+1}) = v(n)u(n+1), \ \phi(e_{n,n+2}) = v(n)u(n+2), \ \ldots$$

and

$$\phi(e_{n+1,n+1}) = v(n+1)u(n+1), \ \phi(e_{n+1,n+2}) = v(n+1)u(n+2),$$
$$\phi(e_{n+1,n+3}) = v(n+1)u(n+3), \ \ldots.$$

Since $\operatorname{rank}(\phi(e_{np}) + \phi(e_{n+1,p})) = 1$ for all $p \geq n+1$ we have

$$\operatorname{rank}(\phi(e_{nn}) + \phi(e_{n+1,n+1})) > 1$$

and ϕ is surjective, we conclude that

$$v(n) \in V_\infty^n \setminus V_\infty^{n-1}(F) \qquad \text{for all } n \in \mathbb{N}.$$

By an analogous argument we get that

$$u(m) \in U_\infty^m \setminus U_\infty^{m-1}(F) \qquad \text{for all } m \in \mathbb{N}.$$

Therefore, a and b defined as in the proof of Theorem 2.2 are upper triangular matrices. Moreover, as ϕ is onto, they are invertible. This completes the proof. $\qquad\square$

As we have promised we present now some facts about the form of a and b that appear in Theorem 1.1. It is natural to ask when a, b are such that $axb \in \mathcal{T}_\infty(F)$ for all $x \in \mathcal{T}_\infty(F)$. We have already seen that a, b does not have to be upper triangular. According to our proofs, we have $\phi(x) \in \mathcal{T}_\infty(F)$ if and only if $ae_{nm}b \in \mathcal{T}_\infty(F)$ for all $n \leq m$. This is also equivalent to the fact that $v(n)u(m) \in \mathcal{T}_\infty(F)$ for $n \leq m$. Since for every $n \leq m$ there exist k_n, l_m such that $v(n) \in V_\infty^{k_n}(F)$, $u(m) \in U_\infty^{l_m}(F)$. Clearly, it suffices to focus on the maximal k_n and the minimal l_m, or, more precisely, on the case when the

difference between these two numbers (i.e., $l_m - k_n$) is minimal. If the latter difference is negative, then $\phi(e_{nm}) \notin \mathcal{T}_\infty(F)$. Otherwise we have $\phi(e_{nm}) \in \mathcal{T}_\infty(F)$. Hence, we can formulate:

Corollary 2.2. *Let a be an infinite matrix over a field F such that*

    (1) *every column $v(n)$ of a is in the set $V_\infty^{k_n}(F)$ for some $k_n \in \mathbb{N}$,*
    (2) *columns of a are linearly independent,*

and let b be an infinite matrix over the same field such that

    (1) *every row $u(m)$ of a is in the set $U_\infty^{l_m}(F)$ for some $l_m \in \mathbb{N}$,*
    (2) *rows of b are linearly independent.*

For all $n \leq m$ consider the set differences $l_m - k_n$. If this set contains a minimal element which is nonnegative, then the map ϕ defined on $\mathcal{T}_\infty(F)$ by the formula $\phi(x) = axb$, is in $\mathcal{R}_1(\mathcal{T}_\infty(F))$.

3. Idempotents

Now we will prove that there is a connection between maps preserving rank one and maps preserving rank one idempotents. Our result is inspired by Corollary 3 from [15].

One can verify the following.

Remark 3.1. If F is a field and $x \in \mathcal{T}_\infty(F)$ is rank one idempotent, then x is of the form

(6)
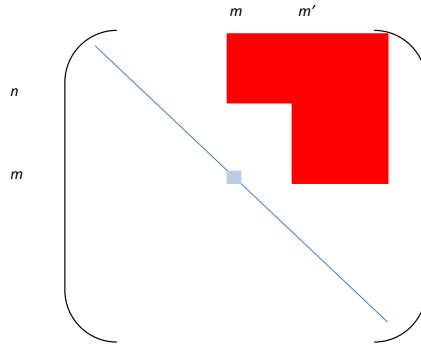$$\left(\begin{array}{c|c|c} 0 & y & yz \\ \hline 0 & 1 & z \\ \hline 0 & 0 & 0 \end{array} \right).$$

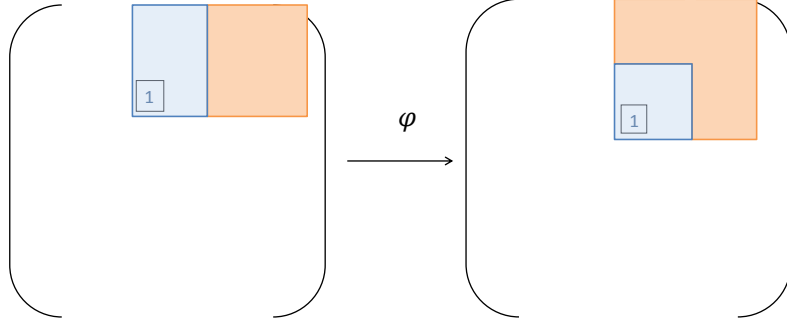FIGURE 2. Picture to the proof of Lemma 3.1. Nonzero coefficients of x are in the blocks.

FIGURE 3. The right block denote x and $\phi(x)$, whereas the left ones y and $\phi(y)$.

Lemma 3.1. *Suppose that F is a field and $x \in \mathcal{T}_\infty(F)$ is a matrix of rank two, of form $v_1 u_1 + v_2 u_2$, where $v_1 \in V_\infty^n(F) \setminus V_\infty^{n-1}(F)$, $u_1 \in U_\infty^m(F) \setminus U_\infty^{m-1}(F)$, $v_2 \in V_\infty^m(F) \setminus V_\infty^{m-1}(F)$, $u_2 \in U_\infty^{m'}(F) \setminus U_\infty^{m'-1}(F)$. Then there exists at most one rank one matrix $y \in \mathcal{T}_\infty(F)$ such that $x + y$ is rank one idempotent.*

Proof. Let x be a matrix from the assumption. Since x is of the form as in Figure 2, we must have $y_{mm} = 1$, and $y = v_3 u_3$, where u_3 is such that u_1, u_2, u_3 are linearly dependent. Hence u_3 is determined by u_2 and u_1. More precisely, if

$$\frac{(v_2 u_2)_{n+1,m'}}{x_{nm'}} = \alpha_1, \quad \frac{(v_2 u_2)_{n+2,m'}}{x_{nm'}} = \alpha_2, \ \ldots, \ \frac{(v_2 u_2)_{mm'}}{x_{nm'}} = \alpha_{m-n}$$

and $\alpha_{m-n} = 1$, then $v_3 = \sum_{i=1}^{m-n} \alpha_i e_{n+i}$ and $u_3 = u_1 - \frac{(v_2 u_2)_{mn_2}}{x_{nm'}} u_2$. Otherwise, the matrix y with the property from the theorem does not exist. $\qquad\square$

Theorem 3.1. *Assume that F is a field and that $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is injective. If the map ϕ is linear and it preserves rank one idempotents, then ϕ preserves rank one matrices.*

Proof. Let $x \in \mathcal{T}_\infty(F)$ be a matrix of rank one. If $x = \alpha \cdot x'$, where $\alpha \in F$ and x' is a matrix such that $(x')^2 = x'$, then by the assumptions and the homogeneity of ϕ, the claim holds. Otherwise, we have $x = vu$ for some $v \in V_\infty^n(F)$, $u \in U_\infty^m(F)$ for some $n < m$. Let us put

$$y = e_{nn} + (v_{n1} u_{1m})^{-1} \sum_{i=1}^{n-1} v_{i1} u_{1m} e_{in}.$$

Then y and $x + y$ are rank one idempotents – it is depicted in Figure 3. Hence, $\phi(y)$, $\phi(x + y)$ are rank one idempotents as well. Therefore, $\phi(x)$ can have rank equal to 0, 1 or 2. Since ϕ is an injection and $\phi(0) = 0$, we can not have $\mathrm{rank}(\phi(x)) = 0$.

Now we will observe that the case $\mathrm{rank}(\phi(y)) = 2$ is also impossible. Suppose the opposite – that $\mathrm{rank}(\phi(x)) = 2$. Now we put

$$z = e_{mm} + (v_{n1}u_{1m})^{-1} \sum_{i=m+1}^{\infty} v_{n1}u_{1i}e_{mi}.$$

Then z and $x + z$ are rank one idempotents, so $\phi(z)$, $\phi(x + z)$ are rank one idempotents as well (see Figure 4). However, by Lemma 3.1, there exists at most one rank one idempotent t such $t + \phi(x)$ is rank one idempotent. This implies $\phi(y) = \phi(z)$, which contradicts the injectivity of ϕ. $\qquad\square$

Obviously, the converse implication does not hold, which can be easily observed in e.g. Examples 2.3 and 2.4.

Using the facts proved in the preceding section we can prove:

Theorem 3.2. *Assume that F is a field and that $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is a linear injective map preserving rank one idempotents. Then $\phi(x) = axb$, where a and b are infinite matrices satisfying the following condition: for every $n \in \mathbb{N}$ there exist $k \in \mathbb{N}$ such that the n-th column of $a - v(n)$ is in $V_\infty^k(F)$, the n-th row of $b - u(n)$ is in $U_\infty^k(F)$, and $(v(n))_k(u(n))_k = 1$.*

Proof. Again we refer to the arguments given in the proofs in the previous section. By them, the fact that ϕ is injective, and that rank one idempotents are of form (6), ϕ must be given by a formula $\phi(x) = axb$ for some a, b. Moreover, $\phi(e_{nn})$ must be a rank one idempotent for every n. Hence $v(n)u(n)$ is of form (6). Therefore, if $v(n) \in V_\infty^k(F)$ and $(v(n))_k = \alpha$, then $u(n) \in U_\infty^k(F)$ and $(u(n))_k = \alpha^{-1}$. $\qquad\square$
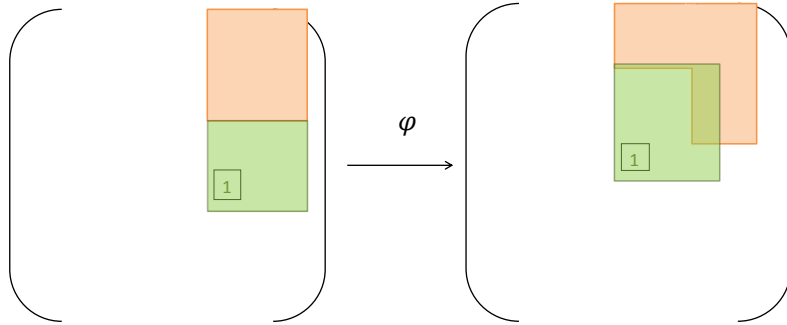

FIGURE 4. The upper block denote x and $\phi(x)$, whereas the lower ones z and $\phi(z)$.

4. Rank k

In this section we will characterize all injective rank-k preservers on $\mathcal{T}_\infty(F)$, where $k \geq 2$. We will show that these maps coincide with some rank one preservers. Our result is very similar to those from [1, 3, 4, 11].

First we prove:

Lemma 4.1. *Let F be a field and let $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ be a linear rank-k preserver for some $k \in \mathbb{N}$. If $x \in \mathcal{T}_\infty(F)$ has rank one, then $\mathrm{rank}(\phi(x))$ is finite.*

Proof. Clearly, it suffices to discuss the case when $k \geq 2$. If x has rank one, then we can write x as a difference $y - z$, where $\mathrm{rank}(y) = \mathrm{rank}(z) = k$. More precisely, since $\mathrm{rank}(x) = 1$, $x = \sum_{i=1}^n \sum_{j=m}^\infty x_{ij} e_{ij}$. If $x_{ij} \neq 0$ for some $j > m$, then we put

$$y = x + \sum_{i=1}^n x_{im} e_{im} + e_{m+1,m+1} + e_{m+2,m+2} + \cdots + e_{m+k-1,m+k-1} \,,$$
$$z = \sum_{i=1}^n x_{im} e_{im} + e_{m+1,m+1} + e_{m+2,m+2} + \cdots + e_{m+k-1,m+k-1} \,,$$

whereas if $x = \sum_{i=1}^n x_{im} e_{im}$, then we put

$$y = x + \sum_{i=1}^n x_{i,m+1} e_{i,m+1} + e_{m+2,m+2} + e_{m+3,m+3} + \cdots + e_{m+k,m+k} \,,$$
$$z = \sum_{i=1}^n x_{i,m+1} e_{i,m+1} + e_{m+2,m+2} + e_{m+3,m+3} + \cdots + e_{m+k,m+k} \,.$$

Hence $\mathrm{rank}(\phi(x)) = \mathrm{rank}(\phi(y) - \phi(z))$. From the fact that ϕ preserves rank k, we get that $\phi(x)$ is a difference of two matrices of rank k, so its rank is at most $2k$. $\qquad\square$

Now we can give:

Proof of Theorem 1.2. Suppose that ϕ preserves rank k matrices. Let $x \in \mathcal{T}_\infty(F)$ be an arbitrary matrix of rank one. There exists m_x such that $(\phi(x))_{ij} = 0$ for $j \geq i > m_x$. By arguments given in the proofs in Section 2, we can focus on the matrices from $\mathcal{T}_\infty^{fin}(F)$, there exist only a finite number of e_{ll} such that $(\phi(e_{ll}))_{ij} \neq 0$ for $i, j \leq m_x$. Then we can choose such l that

- $(\phi(e_{ll}))_{ij} = 0$ for $i, j \leq m_x$,
- $\mathrm{rank}(x + e_{ll}) = 2$.

Denote the minimal l with this property by i_1. For this $e_{i_1 i_1}$ there exists m_1 such that $(\phi(x + e_{i_1 i_1}))_{ij} = 0$ for $j > i > m_1$, and we can find i_2 such that

- $(\phi(e_{i_2 i_2}))_{ij} = 0$ for $i, j \leq m_1$,
- $\mathrm{rank}(x + e_{i_1 i_1} + e_{i_2 i_2}) = 3$.

Performing this way we can find $e_{i_1 i_1}$, $e_{i_2 i_2}$, ..., $e_{i_{k-1} i_{k-1}}$ such that

$$\mathrm{rank}(x + e_{i_1 i_1} + \cdots + e_{i_{k-1} i_{k-1}}) = k$$

and

$$\mathrm{rank}(\phi(x + e_{i_1 i_1} + \cdots + e_{i_{k-1} i_{k-1}}))$$
$$= \mathrm{rank}(\phi(x)) + \mathrm{rank}(\phi(e_{i_1 i_1})) + \cdots + \mathrm{rank}(\phi(e_{i_{k-1} i_{k-1}})) \,.$$

Since ϕ preserves rank k, we have

$$\mathrm{rank}(\phi(x)) + \mathrm{rank}(\phi(e_{i_1 i_1})) + \cdots + \mathrm{rank}(\phi(e_{i_{k-1} i_{k-1}})) = k,$$

so as ϕ is injective, by Lemma 4.1, all ranks in the latter equation are positive and therefore $\mathrm{rank}(\phi(x)) = 1$.

Assume now that ϕ preserves rank one matrices and $\phi(\mathcal{T}_\infty(F))$ contains matrices of rank greater than one. Then ϕ is of form as described in point (2) of Theorem 2.1. Let $x \in \mathcal{T}_\infty(F)$ be any matrix of rank k. Then x can be
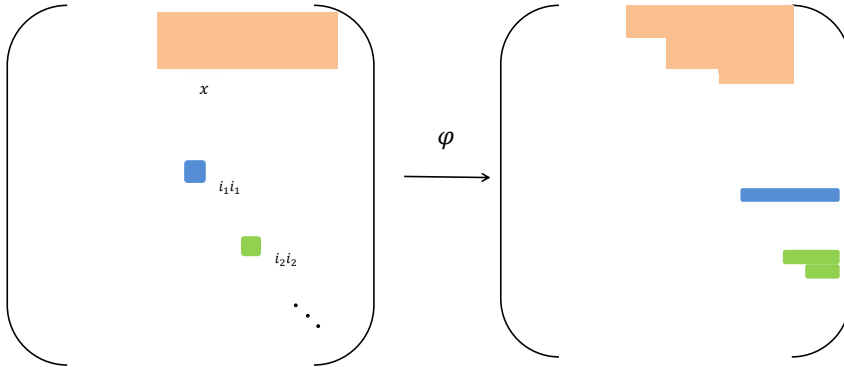

FIGURE 5. Picture to the proof of Theorem 1.2.

written as a sum of k matrices of rank one, i.e., $x = x_1 + x_2 + \cdots + x_k$ with $x_i = r_i c_i$ for some $r_i \in M^{fin}_{\infty \times 1}(F)$, $c_i \in M_{1 \times \infty}(F)$, where $c_1, \ldots, c_k$ are linearly independent. By Theorem 2.1

$$\phi(x) = \phi(x_1) + \cdots + \phi(x_k) = v_1 u_1 + \cdots + v_k u_k.$$

Obviously

$$\mathrm{rank}(\phi(x)) \le \mathrm{rank}(\phi(x_1)) + \mathrm{rank}(\phi(x_k)) = 1 + \cdots + 1 = k.$$

Suppose that $\mathrm{rank}(\phi(x)) < k$. Then for some i, the vector u_i would be a linear combination of $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k$. However $u_1, \ldots, u_k$ are linearly independent – a contradiction. Concluding we must have $\mathrm{rank}(\phi(x)) = k$ whenever $\mathrm{rank}(x) = k$. $\qquad\square$

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