J. Korean Math. Soc. **53** (2016), No. 1, pp. 57–72
http://dx.doi.org/10.4134/JKMS.2016.53.1.057

MAXIMAL INEQUALITIES AND AN APPLICATION UNDER A WEAK DEPENDENCE

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Abstract. We establish maximal moment inequalities of partial sums under ψ-weak dependence, which has been proposed by Doukhan and Louhichi [P. Doukhan and S. Louhichi, A new weak dependence condition and application to moment inequality, Stochastic Process. Appl. 84 (1999), 313–342], to unify weak dependence such as mixing, association, Gaussian sequences and Bernoulli shifts. As an application of maximal moment inequalities, a functional central limit theorem is developed for linear processes with ψ-weakly dependent innovations.

1. Introduction

Maximal moment inequalities of partial sums of random variable sequences are important and useful to derive basic limit theorems such as central limit theorem (CLT), functional CLT (FCLT), and law of iterated logarithm (LIL) as well as statistical applications. For mixing sequences, Utev and Peligrad [12], Yang [15], Xing et al. [13] and Xuejun et al. [14] among others obtained maximal moment inequalities with various applications.

A weak dependence, called *ψ-weak dependence*, which was introduced by Doukhan and Louhichi [5], generalizes mixings and other dependence and has much attention due to its wide range of applications for financial time series. The new weak dependence condition unifies many existing weak dependence such as mixing, association, Gaussian sequences and Bernoulli shifts, and yields a fairly tractable framework for analysis of statistical procedures. See Ango Nze et al. [1] and Dedecker et al. [4] for examples and applications.

For ψ-weakly dependent sequences, Doukhan and Louhichi [5], Doukhan and Neumann [6] and Dedecker et al. [4] developed some moment inequalities. Dedecker and Doukhan [3] compared the weak dependence coefficient with both strong mixing and mixingale-type coefficients, and gave a new covariance inequality involving the weakest of those coefficients. Dedecker et al. [4] gave a maximal inequality as an extension of Doob's inequality for martingales.

Received June 30, 2014; Revised June 13, 2015.
2010 *Mathematics Subject Classification.* Primary 60E15, 60G99; Secondary 60F05.
Key words and phrases. weak dependence, maximal moment inequality, linear process, functional central limit theorem.

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In the aforementioned inequalities under ψ-weak dependence, they assumed the covariance inequality conditions of two products rather than proving the inequalities. Recently, Hwang and Shin [7] established Roussas-Ioannides [11]-type inequalities of unbounded ψ-weakly dependent sequence and improved the existing inequalities in the sense of proving covariance inequalities of two products, which were assumed in the literature.

In this paper, we present maximal moment inequalities and their application under ψ-weak dependence. Our maximal moment inequalities extend results of Yang [15] for strong mixing sequences to those for a wider class of weak dependence. Our results are derived using the Roussas-Ioannides-type inequalities for the ψ-weak dependence sequences, which has been recently established by Hwang and Shin [7]. Maximal moment inequalities are applied to obtain diverse probabilistic results such as CLT, FCLT, LLN, and LIL as studied by many authors for strong mixing cases, as well as statistical results such as uniform convergence of nonparametric kernel estimators and asymptotic normality of general linear estimator for the fixed design regression as studied by Yang [15] and Xing et al. [13] for strong mixing cases. Our results can be applied to obtain those applications under the ψ-weak dependence. In this work, as an application a FCLT is established for a linear process with ψ-weakly dependent innovations by means of the maximal moment inequalities.

The remaining of the paper is organized as follows. Section 2 describes the notion of ψ-weak dependence. Section 3 presents the results of maximal moment inequalities. Section 4 gives the functional central limit theorem as an application. Technical lemmas and proofs are given in Section 5.

2. ψ-weak dependence

To define the notion of the weak dependence, we introduce some classes of functions. Let $\mathbb{L}^\infty = \bigcup_{n=1}^\infty \mathbb{L}^\infty(\mathbb{R}^n)$, the set of real-valued and bounded functions on the space $\mathbb{R}^n$ for $n = 1, 2, \ldots$. Consider a function $g : \mathbb{R}^n \to \mathbb{R}$ where $\mathbb{R}^n$ is equipped with its l_1-norm (i.e., $||(x_1, \ldots, x_n)||_1 = |x_1| + \cdots + |x_n|$) and define the Lipschitz modulus of g,

$$\mathrm{Lip}(g) = \sup_{x \neq y} \frac{|g(x) - g(y)|}{||x - y||_1}.$$

Let

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}_n \quad \text{where } \mathcal{L}_n = \{g \in \mathbb{L}^\infty(\mathbb{R}^n);\ \mathrm{Lip}(g) < \infty,\ ||g||_\infty \leq 1\}.$$

The class $\mathcal{L}$ is sometimes used together with the following functions $\psi = \psi_0$, ψ_1, ψ_2, η, κ and λ, which yield notions of weak dependence appropriate to describe various examples of models:

$$\psi_0(g, h, n, m) = 4||g||_\infty ||h||_\infty,\ \psi_1(g, h, n, m) = \min(n, m)\mathrm{Lip}(g)\mathrm{Lip}(h),$$
$$\psi_2(g, h, n, m) = 4(n + m) \min\{\mathrm{Lip}(g), \mathrm{Lip}(h)\},$$

$$\eta(g,h,n,m) = n\mathrm{Lip}(g) + m\mathrm{Lip}(h),\ \ \kappa(g,h,n,m) = nm\mathrm{Lip}(g)\mathrm{Lip}(h),$$
$$\lambda(g,h,n,m) = n\mathrm{Lip}(g) + m\mathrm{Lip}(h) + nm\mathrm{Lip}(g)\mathrm{Lip}(h),$$

for functions g and h defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. See Doukhan and Neumann [6] and Dedecker et al. [4].

Definition 2.1 ([5])**.** The sequence $\{X_t\}_{t\in\mathbb{Z}}$ is called $(\theta,\mathcal{L},\psi)$*-weakly dependent*, (simply, ψ*-weakly dependent*), if there exists a sequence $\theta = (\theta_r)_{r\in\mathbb{Z}}$ decreasing to zero at infinity and a function ψ with arguments $(g,h,n,m) \in \mathcal{L}_n \times \mathcal{L}_m \times \mathbb{N}^2$ such that for n-tuple $(i_1,\ldots,i_n)$ and m-tuple $(j_1,\ldots,j_m)$ with $i_1 \le \cdots \le i_n < i_n + r \le j_1 \cdots \le j_m$, one has

$$|\mathrm{Cov}(g(X_{i_1},\ldots,X_{i_n}), h(X_{j_1},\ldots,X_{j_m}))| \le \psi(g,h,n,m)\theta_r.$$

According to Doukhan and Louhichi [5], strong mixing is ψ_0-weakly dependent, associated sequences are ψ_1-weakly dependent, and Bernoulli shifts and Markov processes are ψ_2-weakly dependent. Our main attraction is such examples of processes that are weakly dependent, but not mixing. Note that κ- or λ-weak dependence can imply other kinds of ψ-weak dependence.

3. Maximal moment inequalities

Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables and $S_i = \sum_{j=1}^{i} X_j$. In Theorem 3.1 and Theorem 3.2 below we give two maximal moment inequalities under ψ-weak dependence. Yang [15] obtained the same bounds as those in Theorems 3.1 and 3.2 for α-mixing process with the same moment conditions and mixing coefficients α_r satisfying $\alpha_r = O(r^{-\rho})$. We extend the results of Yang [15] to more general ψ-weakly dependent processes.

Theorem 3.1. *Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables with $EX_t = 0$ and $E|X_t|^{q+\delta} < \infty$ for $q > 2$ and $\delta > 0$. Suppose that, for $2 < v \le q+\delta$, the sequence (θ_r) of ψ-weak dependence satisfies $\theta_r = O(r^{-\rho})$ for some $\rho > \max\{v/(v-2), (q-1)(q+\delta)/\delta\}$. Then, for any $\varepsilon > 0$, there exists a positive constant C such that*

$$E\left[\max_{1\le i\le n} |S_i|^q\right] \le C\left\{n^{\varepsilon}\sum_{i=1}^{n} ||X_i||_q^q + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right\}.$$

Theorem 3.2. *Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables with $EX_t = 0$ and $E|X_t|^{q+\delta} < \infty$ for $q > 2$ and $\delta > 0$. Suppose that the sequence (θ_r) of ψ-weak dependence satisfies $\theta_r = O(r^{-\rho})$ for some $\rho > q(q+\delta)/(2\delta)$. Then, for any $\varepsilon > 0$, there exists a positive constant C such that*

$$E\left[\max_{1\le i\le n} |S_i|^q\right] \le C\left\{n^{\varepsilon}\sum_{i=1}^{n} ||X_i||_q^q + \left(\sum_{i=1}^{n} ||X_i||_{q+\delta}^2\right)^{q/2}\right\}.$$

Corollary 3.3. *Let $\{X_t\}$ be a strictly stationary sequence of ψ-weakly dependent random variables with $EX_t = 0$ and $E|X_t|^{q+\delta} < \infty$ for $q > 2$ and $\delta > 0$. If $\theta_r = O(r^{-\rho})$ for some $\rho > q(q+\delta)/(2\delta)$, then*

$$E\left[\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j\right|^q\right] \le Cn^{q/2}$$

for some constant C.

4. Application: Functional central limit theorem for linear processes

As an application, we establish a FCLT for linear processes with ψ-weakly dependent innovations. Let $\{L_t : t \in \mathbb{Z}\}$ be a linear process with a stationary ψ-weakly dependent innovations $\{X_t\}$:

$$L_t = \sum_{j=0}^{\infty} a_j X_{t-j}, \tag{1}$$

where $\{a_j\}$ is a sequence of real numbers. Let

$$U_n = \sum_{i=1}^{n} L_i \quad \text{and} \quad W_n(u) = \frac{1}{A\sigma\sqrt{n}} U_{[nu]} \tag{2}$$

for $0 \le u \le 1$, and $W(u)$ be the standard Brownian motion on $[0,1]$, where $A = \sum_{j=0}^{\infty} a_j$ and $\sigma^2 = EX_1^2 + 2\sum_{j=2}^{\infty} X_1X_j$. Using the maximal inequality given in Corollary 3.3 as well as an invariance principle result of Doukhan and Wintenberger, we get the following theorem of the FCLT for the linear process with ψ-weak dependent innovations.

Theorem 4.1. *Let $\{X_t\}$ be a strictly stationary sequence of ψ-weakly dependent random variables with $EX_t = 0$, $E|X_t|^{q+\delta} < \infty$ for $q > 2$ and $\delta > 0$. Let $\{L_t\}$ be the linear process defined by (1), and U_n and $W_n(u)$ be the partial sum processes defined by (2). If $\theta_r = O(r^{-\rho})$ for $\rho > \max\{q(q+\delta)/(2\delta), 2 + 1/(q+\delta-2)\}$ under κ-weak dependence, or if $\theta_r = O(r^{-\rho})$ for $\rho > \max\{q(q+\delta)/(2\delta), 4 + 2/(q+\delta-2)\}$ under λ-weak dependence, then we have*

$$W_n(u) \Longrightarrow W(u)$$

in the space $D[0,1]$ endowed with the Skorohod topology.

5. Appendix

5.1. Lemmas

In this subsection we provide lemmas which will be used in proving the main theorems in Section 3. Proofs of Theorems 3.1 and 3.2 are based on the Roussas-Ioannides [11]-type inequalities for ψ-weakly dependent sequences established by Hwang and Shin [7], which will be reproduced in Lemma 5.1 below.

Let A and B be disjoint finite sets of indices such that distance between A and B is greater than or equal to r, and let $\xi = h(X_j : j \in A)$ and $\eta = g(X_k : k \in B)$ where h and g are some real-valued functions. Assume

(i) ψ is bounded for the class of h, g such that $||h||_\infty$, $||g||_\infty$, Lip(h) and Lip(g) are all bounded.

(ii) $E|\xi|^p$, $E|\eta|^q < \infty$ for some p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$,

(iii) $E|X_j|^p < \infty$ for all $j \in A$; $E|X_k|^q < \infty$ for all $k \in B$,

(iv) $\bar{D}_i h(x_j, j \in A) < \infty$ for all $x_j, j \in A$; $\bar{D}_i g(x_k, k \in B) < \infty$ for all $x_k, k \in B$,

(v) $M_h^D := \max_{i \in A} \max_{|x_j| \le M} |\bar{D}_i h(x_j, j \in A)| = O(M_h)$ as $M \to \infty$,
$N_g^D := \max_{i \in B} \max_{|x_k| \le N} |\bar{D}_i g(x_k, k \in B)| = O(N_g)$ as $N \to \infty$,

where

$$\bar{D}_i f(x_1, \ldots, x_n) = \limsup_{y_i \to x_i} \frac{|f(x_1,\ldots,x_n) - f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)|}{|y_i - x_i|}, \tag{3}$$

$$M_h = \max |h(x_j^M : j \in A)|, \quad N_g = \max |g(x_k^N : k \in B)|,$$

$$x_j^M = x_j \ \text{ if } |x_j| \le M; \ x_j^M = M \ \text{ if } x_j > M; \ x_j^M = -M \ \text{ if } x_j < -M,$$

and x_k^N is defined similarly. Here some remarks on conditions above are given: Condition (i) is a trivial one. The ψ-functions corresponding to strong mixing (ψ_0), associated sequences (ψ_1), Bernoulli shifts, and Markov processes (ψ_2) as well as η, κ, λ or the ψ functions considered by Doukhan and Neumann [6] all satisfy condition (i). Given the moment condition on (ii), condition (iii) is usually not a binding one. Functions used in the proofs of Theorem 3.1 and Theorem 3.2 satisfy Conditions (ii)–(v). In (iv)-(v), operator $\bar{D}_i$ is not the standard partial differential operator, but in (3) above, $\bar{D}_i f < \infty$ for the maximal function $f(x, y) = \max\{x, y\}$, which is not a differential function. The fact that $\bar{D}_i f < \infty$ will be used in the proof of Lemma 5.2 below. For more detailed remarks and for the proof of Lemma 5.1, see Hwang and Shin [7]. Now we state the Roussas-Ioannides-type inequalities in Lemma 5.1 for the ψ-weakly dependent sequences.

Lemma 5.1 ([7]). *Let $\{X_t\}$ be a sequence of ψ-weakly dependent random variables with the weak dependence coefficient sequence (θ_r). Let $\xi = h(X_j : j \in A)$ and $\eta = g(X_k : k \in B)$ where h and g are some real-valued functions and A and B are disjoint finite sets of indices such that distance between A and B is greater than or equal to r. Under conditions* (i)–(v) *above, we have*

$$|E(\xi\eta) - (E\xi)(E\eta)| \le C \cdot \theta_r^{1-\frac{1}{p}-\frac{1}{q}} ||\xi||_p ||\eta||_q$$

for some constant C not depending on r.

In proving Theorem 3.1 and Theorem 3.2 in the next subsection, we consider upper bounds of $\max_{1 \le i \le n} |S_i|^q$, which are splitted into three sub-partial sums.

Let $k = \lfloor (n/2)^\lambda \rfloor$ and $m = \lfloor (n/2)^{1-\lambda} \rfloor$, where $0 < \lambda < 1$ and $\lfloor x \rfloor$ denotes the integral part of x. The value of λ will be determined later for any given ε. Clearly, we have

$$n < 2(m+1)k, \quad \frac{1}{4}n^\lambda < k < n^\lambda, \quad m < n^{1-\lambda}. \tag{4}$$

Let us fix n and define $\tilde{X}_i$ as $\tilde{X}_i = X_i$ for $1 \le i \le n$ and $\tilde{X}_i = 0$ for $i > n$. For $j = 1, 2, \ldots, m+1$, let

$$Y_j = \sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} X_i, \quad Z_j = \sum_{i=(2j-1)k+1}^{n\wedge 2jk} X_i,$$

where $\wedge$ means the minimum, i.e., $a \wedge b = \min\{a, b\}$. Let $S_{1,j} = \sum_{i=1}^{j} Y_i$ and $S_{2,j} = \sum_{i=1}^{j} Z_i$. According to Lemma 2.3 of Yang [15], we have

$$\begin{aligned} &\max_{1\le i\le n} |S_i|^q \\ &\le c\left\{\max_{1\le j\le m+1} |S_{1,j}|^q + \max_{1\le j\le m+1} |S_{2,j}|^q + \sum_{j=1}^{2(m+1)}\left(\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i\right|^q\right)\right\} \end{aligned} \tag{5}$$

for some constant c.

Two lemmas below, Lemma 5.2 and Lemma 5.3, give upper bounds for the expectations of two terms in the right hand side of (5), which will be used in the proof of Theorem 3.1.

Lemma 5.2. *Under the same conditions as in Theorem 3.1, we have*

$$E\left[\max_{1\le j\le m+1} |S_{1,j}|^q\right] \le c\left\{\sum_{j=1}^{m+1} E|Y_j|^q + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right\}.$$

Lemma 5.3. *Under the same conditions as in Theorem 3.1, we have*

$$E\left[\max_{1\le j\le m+1} |S_{2,j}|^q\right] \le c\left\{\sum_{j=1}^{m+1} E|Z_j|^q + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right\}.$$

In proving Theorem 3.2 below, under the assumption for ψ-weak dependence with $\theta_k = O(k^{-\rho})$ for some $\rho > q(q+\delta)/(2\delta)$, we will use two lemmas below, Lemma 5.4 and Lemma 5.5.

Lemma 5.4. *Under the same conditions as in Theorem 3.2, we have*

$$E\left[\max_{1\le j\le m+1} |S_{1,j}|^q\right] \le c\left\{\sum_{j=1}^{m+1} E|Y_j|^q + \left(\sum_{i=1}^{n} ||X_i||_{q+\delta}^2\right)^{q/2}\right\}.$$

Lemma 5.5. *Under the same conditions as in Theorem 3.2, we have*

$$E\left[\max_{1\le j\le m+1}|S_{2,j}|^q\right]\le c\left\{\sum_{j=1}^{m+1}E|Z_j|^q+\left(\sum_{i=1}^{n}||X_i||_{q+\delta}^2\right)^{q/2}\right\}.$$

In proving Theorem 4.1, we will use the following invariance principle for ψ-weakly dependent processes, which has been developed by Doukhan and Wintenberger.

Lemma 5.6 (Doukhan and Wintenberger, 2007)**.** *Let $\{X_t\}$ be a strictly stationary sequence of ψ-weakly dependent random variables with $EX_t = 0$ and $E|X_t|^{q+\delta} < \infty$. If $\theta_r = O(r^{-\rho})$ for $\rho > 2 + 1/(q + \delta - 2)$ under κ-weak dependence, or if $\theta_r = O(r^{-\rho})$ for $\rho > 4 + 2/(q + \delta - 2)$ under λ-weak dependence, then we have*

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{[nu]}X_i \Longrightarrow W.$$

5.2. Proofs

In this subsection we give proofs of lemmas and theorems. Proofs of Lemmas 5.2 and 5.3 are similar. So proof of Lemma 5.2 is only given. Proof of Theorem 3.1 follows. Also, proofs of Lemmas 5.4 and 5.5 are similar. So proof of Lemma 5.4 is only given. Then proof of Theorem 3.2 follows. Finally proof of Theorem 4.1 is provided.

Proof of Lemma 5.2. Clearly, we have

$$\max_{1\le j\le m+1}|S_{1,j}|^q\le\left|\max_{1\le j\le m+1}S_{1,j}\right|^q+\left|\max_{1\le j\le m+1}(-S_{1,j})\right|^q. \tag{6}$$

We denote

$$\begin{aligned}
Q_j&=\max\{0,Y_{j+1},Y_{j+1}+Y_{j+2},\ldots,Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\},\\
R_j&=\max\{Y_{j+1},Y_{j+1}+Y_{j+2},\ldots,Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\},\\
\widetilde{Q}_j&=\max\{0,-Y_{j+1},-Y_{j+1}-Y_{j+2},\ldots,-Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\},\\
\widetilde{R}_j&=\max\{-Y_{j+1},-Y_{j+1}-Y_{j+2},\ldots,-Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\}.
\end{aligned}$$

Then

$$\max_{1\le j\le m+1}S_{1,j}=R_0,\quad R_j=Y_{j+1}+Q_{j+1},\quad 0\le Q_j\le|R_j|,$$
$$\max_{1\le j\le m+1}(-S_{1,j})=\widetilde{R}_0,\quad \widetilde{R}_j=-Y_{j+1}+\widetilde{Q}_{j+1},\quad 0\le\widetilde{Q}_j\le|\widetilde{R}_j|.$$

We use the inequality: for any $a, b \in \mathbb{R}^1$, and for $q > 2$,

$$|a+b|^q\le 2^q|a|^q+q2^{q-1}a|b|^{q-1}\mathrm{sgn}(b)+|b|^q. \tag{7}$$

This holds because $|1+x|^q$ is bounded by $2^q|x|^q, 1+q2^{q-1}|x|, 1$ for $|x| \ge 1, 0 \le x < 1, -1 \le x < 0$, respectively, and hence by $2^q|x|^q + q2^{q-1}|x| + 1$ for all $x \in \mathbb{R}^1$. Letting $x = a/b$, we get (7).

We apply the inequality repeatedly to obtain the following two inequalities in (8) and (9):

$$\begin{aligned}\left|\max_{1\le j\le m+1} S_{1,j}\right|^q &= |R_0|^q = |Y_1+Q_1|^q \le 2^q|Y_1|^q + q2^{q-1}Y_1Q_1^{q-1} + Q_1^q\\ &\le 2^q|Y_1|^q + q2^{q-1}Y_1Q_1^{q-1} + |R_1|^q \le \cdots\\ &\le 2^q\left(\sum_{j=1}^{m+1}|Y_j|^q\right) + q2^{q-1}\left(\sum_{j=1}^{m} Y_jQ_j^{q-1}\right),\end{aligned} \tag{8}$$

$$\begin{aligned}\left|\max_{1\le j\le m+1}(-S_{1,j})\right|^q &= |\widetilde{R}_0|^q = |-Y_1+\widetilde{Q}_1|^q \le 2^q|Y_1|^q + q2^{q-1}Y_1\widetilde{Q}_1^{q-1} + \widetilde{Q}_1^q\\ &\le 2^q|Y_1|^q + q2^{q-1}Y_1\widetilde{Q}_1^{q-1} + |\widetilde{R}_1|^q \le \cdots\\ &\le 2^q\left(\sum_{j=1}^{m+1}|Y_j|^q\right) + q2^{q-1}\left(\sum_{j=1}^{m} Y_j\widetilde{Q}_j^{q-1}\right).\end{aligned} \tag{9}$$

Now in two steps below, we find upper bounds of expectations of the last two terms in (8) and those in (9).

Step 1: We show that for any $s>0$, there exists a positive constant c_1 such that

$$\sum_{j=1}^{m} E\left[Y_jQ_j^{q-1}\right] \le c_1\sum_{i=1}^{n}||X_i||_{q+\delta}^q + c_2\, s\, E\left[\max_{1\le j\le m+1}|S_{1,j}|^q\right],$$

where $c_2=(q-1)/q$. Note that Conditions (ii)–(v) of Lemma 5.1 hold for $\xi=Y_j,\ \eta=Q_j^{q-1}$ and $q-1>1$, thus, together with $EY_j=0$, by Lemma 5.1 we have

$$E\left[Y_jQ_j^{q-1}\right] \le C\,\theta_k^{1-\frac{1}{p_1}-\frac{1}{q_1}}||Q_j^{q-1}||_{p_1}||Y_j||_{q_1} = C\,\theta_k^{\frac{\delta}{q(q+\delta)}}||Q_j||_q^{q-1}||Y_j||_{q+\delta}$$

with $p_1=q/(q-1)$ and $q_1=q+\delta$, and so $1-\frac{1}{p_1}-\frac{1}{q_1}=\frac{\delta}{q(q+\delta)}$. Also note that

$$\begin{aligned}Q_j = \max\{S_{1,j},S_{1,j+1},\ldots,S_{1,m+1}\} - S_{1,j} &\le \max_{j\le i\le m+1}|S_{1,i}| + |S_{1,j}|\\ &\le 2\max_{1\le j\le m+1}|S_{1,j}|.\end{aligned}$$

Thus,

$$\sum_{j=1}^{m} E\left[Y_jQ_j^{q-1}\right] \le C\,\theta_k^{\frac{\delta}{q(q+\delta)}}2^{q-1}\sum_{j=1}^{m}||Y_j||_{q+\delta}\left(E\max_{1\le j\le m+1}|S_{1,j}|^q\right)^{(q-1)/q}.$$

By Minkowski inequality, we have $\sum_{j=1}^{m}||Y_j||_{q+\delta}\le\sum_{i=1}^{n}||X_i||_{q+\delta}$, and thus, the last term above is less than or equal to

$$C\,\theta_k^{\frac{\delta}{q(q+\delta)}}2^{q-1}s^{-(q-1)/q}\sum_{i=1}^{n}||X_i||_{q+\delta}\left(sE\max_{1\le j\le m+1}|S_{1,j}|^q\right)^{(q-1)/q}$$

$$:= A^{1/q} B^{(q-1)/q},$$

where

$$A = \frac{C^q \theta_k^{\delta/(q+\delta)} 2^{q(q-1)}}{s^{q-1}} \left(\sum_{i=1}^{n} ||X_i||_{q+\delta} \right)^q, \quad B = \left(sE \max_{1 \le j \le m+1} |S_{1,j}|^q \right).$$

By Hölder inequality: $A^{1/q} B^{(q-1)/q} \le \frac{1}{q} A + \frac{q-1}{q} B$,

$$\begin{aligned} &\sum_{j=1}^{m} E\left[Y_j Q_j^{q-1} \right] \\ (10) \quad &\le \frac{C^q \theta_k^{\delta/(q+\delta)} 2^{q(q-1)}}{q s^{q-1}} \left(\sum_{i=1}^{n} ||X_i||_{q+\delta} \right)^q + \frac{(q-1)s}{q} E\left[\max_{1 \le j \le m+1} |S_{1,j}|^q \right] \\ &\le c_1 \theta_k^{\delta/(q+\delta)} n^{q-1} \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + c_2\, s\, E\left[\max_{1 \le j \le m+1} |S_{1,j}|^q \right], \end{aligned}$$

where $c_1 = C^q 2^{q(q-1)}/(q s^{q-1})$ and $c_2 = (q-1)/q$. Under our assumption for ψ-weak dependence: $\theta_k = O(k^{-\rho})$ for some $\rho > \max\{v/(v-2), (q-1)(q+\delta)/\delta\}$, by (4),

$$\theta_k^{\delta/(q+\delta)} n^{q-1} = O(k^{-\rho\delta/(q+\delta)} n^{q-1}) = O(n^{-\lambda\rho\delta/(q+\delta)} n^{q-1}) = O(1)$$

if we choose $\lambda = (q-1)(q+\delta)/(\rho\delta)$ (with $0 < \lambda < 1$). Thus the desired result in Step 1 is completed.

Step 2: For any $s > 0$, with c_1 and c_2 in Step 1,

$$\sum_{j=1}^{m} E\left[Y_j \widetilde{Q}_j^{q-1} \right] \le c_1 \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + c_2\, s\, E\left[\max_{1 \le j \le m+1} |S_{1,j}|^q \right].$$

Its proof is similar to that of Step 1 and is omitted.

We go back to the inequality results in (8) and (9) and take expectations to both sides of the inequalities. By results in Steps 1–2, we obtain inequalities for

$$E\left| \max_{1 \le j \le m+1} S_{1,j} \right|^q \quad \text{and} \quad E\left| \max_{1 \le j \le m+1} (-S_{1,j}) \right|^q,$$

and then by (6), we have

$$\begin{aligned} E\left[\max_{1 \le j \le m+1} |S_{1,j}|^q \right] \le 2^{q+1} \left(\sum_{j=1}^{m+1} E|Y_j|^q \right) + c_1 q 2^q \left(\sum_{i=1}^{n} ||X_i||_{q+\delta}^q \right) \\ + c_2 q 2^q s\, E\left[\max_{1 \le j \le m+1} |S_{1,j}|^q \right]. \end{aligned}$$

Equivalently, with $C_q = c_2 q 2^q$, we have

$$\{1 - C_q s\}\, E\left[\max_{1\le j\le m+1} |S_{1,j}|^q\right] \le 2^{q+1}\left(\sum_{j=1}^{m+1} E|Y_j|^q\right) + c_1 q 2^q\left(\sum_{i=1}^{n} ||X_i||_{q+\delta}^q\right).$$

If we choose $0 < s < 1/C_q$, then we obtain the desired result of Lemma 5.2. □

Proof of Theorem 3.1. By (5), and by Lemmas 5.2 and 5.3, we have

$$\begin{aligned}
&E\left[\max_{1\le i\le n} |S_i|^q\right] \\
&\le C_0\left\{\sum_{j=1}^{m+1}(E|Y_j|^q + E|Z_j|^q) + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right. \\
&\qquad\left. + \sum_{j=1}^{2(m+1)} E\left[\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i\right|^q\right]\right\}
\end{aligned} \tag{11}$$

for some positive constant C_0. Applying Minkowski inequality to

$$E|Y_j|^q,\ E|Z_j|^q \ \text{ and }\ E\left(\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i\right|^q\right),$$

we have

$$E|Y_j|^q = ||Y_j||_q^q \le \left(\sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} ||X_i||_q\right)^q \le k^{q-1}\sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} ||X_i||_q^q,$$

$$E|Z_j|^q \le k^{q-1}\sum_{i=(2j-1)k+1}^{n\wedge 2jk} ||X_i||_q^q,$$

$$\begin{aligned}
&E\left[\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i\right|^q\right] \\
&\le k^{q-1}\sum_{i=(j-1)k+1}^{(j-1)k+l} ||\tilde{X}_i||_q^q = k^{q-1}\sum_{i=(j-1)k+1}^{n\wedge((j-1)k+l)} ||X_i||_q^q.
\end{aligned}$$

Thus, we obtain, noting (4),

$$\begin{aligned}
E\left[\max_{1\le i\le n} |S_i|^q\right] &\le C_0\left\{k^{q-1}\sum_{i=1}^{n} ||X_i||_q^q + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right\} \\
&\le C_0\left\{n^{\lambda(q-1)}\sum_{i=1}^{n} ||X_i||_q^q + \sum_{i=1}^{n} ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^{n} ||X_i||_v^2\right)^{q/2}\right\}.
\end{aligned} \tag{12}$$

Note that if we apply the inequality in (12) to $E|Y_j|^q$, then we get

$$E|Y_j|^q \leq c_0 \left\{ k^{\lambda(q-1)} \sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} ||X_i||_q^q + \sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} ||X_i||_{q+\delta}^q + \left(\sum_{i=2(j-1)k+1}^{n\wedge(2j-1)k} ||X_i||_v^2 \right)^{q/2} \right\}$$

for some c_0. Also we apply the inequality in (12) to $E|Z_j|^q$ and

$$E \max_{1\leq l\leq k} \left| \sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i \right|^q$$

in (11) to get similar results of inequalities for them. Then in (11), we obtain, noting (4),

$$E\left[\max_{1\leq i\leq n} |S_i|^q\right] \leq C_1 \left\{ k^{\lambda(q-1)} \sum_{i=1}^n ||X_i||_q^q + \sum_{i=1}^n ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^n ||X_i||_v^2\right)^{q/2} \right\}$$

$$\leq C_1 \left\{ n^{\lambda^2(q-1)} \sum_{i=1}^n ||X_i||_q^q + \sum_{i=1}^n ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^n ||X_i||_v^2\right)^{q/2} \right\} \tag{13}$$

for some constant C_1. Again we apply the inequality in (13) to $E|Y_j|^q$, $E|Z_j|^q$ and $E\max_{1\leq l\leq k} \left|\sum_{i=(j-1)k+1}^{(j-1)k+l} \tilde{X}_i\right|^q$ in (11). We repeat ℓ times, (ℓ positive integer), in this way to obtain

$$E\left[\max_{1\leq i\leq n} |S_i|^q\right] \leq C \left\{ n^{\lambda^\ell(q-1)} \sum_{i=1}^n ||X_i||_q^q + \sum_{i=1}^n ||X_i||_{q+\delta}^q + \left(\sum_{i=1}^n ||X_i||_v^2\right)^{q/2} \right\}$$

for some constant C. For any given $\varepsilon > 0$, since $0 < \lambda < 1$, we choose ℓ such that $\lambda^\ell(q-1) < \varepsilon$ and then we obtain the result in Theorem 3.1. □

Proof of Lemma 5.4. The result follows from the same procedures as in the proof of Lemma 5.2 except for the two steps below.

Step 1: We show that for any $s > 0$, there exists a positive constant c_1 such that

$$\sum_{j=1}^m E\left[Y_j Q_j^{q-1}\right] \leq c_1 \left(\sum_{i=1}^n ||X_i||_{q+\delta}^2\right)^{q/2} + c_2\, s\, E\left[\max_{1\leq j\leq m+1} |S_{1,j}|^q\right],$$

where $c_2 = (q-1)/q$. We have the same inequality in (10) for $\sum_{j=1}^m E\left[Y_j Q_j^{q-1}\right]$. In the first term of the right hand side in the inequality of (10), we have

$(\sum_{i=1}^n ||X_i||_{q+\delta})^q \le n^{q/2}\left(\sum_{i=1}^n ||X_i||^2_{q+\delta}\right)^{q/2}$. Thus (10) is less than

$$c_1\theta_k^{\delta/(q+\delta)} n^{q/2}\left(\sum_{i=1}^n ||X_i||^2_{q+\delta}\right)^{q/2} + c_2\, s\, E\left[\max_{1\le j\le m+1}|S_{1,j}|^q\right].$$

Under the assumption for ψ-weak dependence with $\theta_k = O(k^{-\rho})$ for some $\rho > q(q+\delta)/(2\delta)$, and by (4),

$$\theta_k^{\delta/(q+\delta)} n^{q/2} = O(k^{-\rho\delta/(q+\delta)} n^{q/2}) = O(n^{-\lambda\rho\delta/(q+\delta)} n^{q/2}) = O(1)$$

if we choose $\lambda = (q-1)(q+\delta)/(2\rho\delta)$ (with $0 < \lambda < 1$). Thus the desired result in Step 1 is completed.

Step 2: For any $s > 0$, with c_1 and c_2 in Step 1,

$$\sum_{j=1}^m E\left[Y_j \widetilde{Q}_j^{q-1}\right] \le c_1\left(\sum_{i=1}^n ||X_i||^2_{q+\delta}\right)^{q/2} + c_2\, s\, E\left[\max_{1\le j\le m+1}|S_{1,j}|^q\right].$$

Like the proof of Lemma 5.2, we choose sufficiently small s so that we obtain the desired result in Lemma 5.4. $\square$

Proof of Theorem 3.2. By (5), and by Lemma 5.4 and Lemma 5.5, we have

$$E\left[\max_{1\le i\le n}|S_i|^q\right] \le C_0\left\{\sum_{j=1}^{m+1}(E|Y_j|^q + E|Z_j|^q) + \left(\sum_{i=1}^n ||X_i||^2_{q+\delta}\right)^{q/2} + \sum_{j=1}^{2(m+1)} E\left[\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}\tilde{X}_i\right|^q\right]\right\}$$

for some positive constant C_0. We use the same arguments (Minkowski inequality and subsequent inequalities repeatedly applied to $E|Y_j|^q$, $E|Z_j|^q$ and $E\left(\max_{1\le l\le k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}\tilde{X}_i\right|^q\right)$ as in the proof of Theorem 3.1. As we repeat ℓ times, we obtain

$$E\left[\max_{1\le i\le n}|S_i|^q\right] \le C\left\{n^{\lambda^\ell(q-1)}\sum_{i=1}^n ||X_i||_q^q + \left(\sum_{i=1}^n ||X_i||^2_{q+\delta}\right)^{q/2}\right\}.$$

For any given $\varepsilon > 0$, since $0 < \lambda < 1$, we choose ℓ such that $\lambda^\ell(q-1) < \varepsilon$ and then we get the desired inequality in Theorem 3.2. $\square$

Proof of Theorem 4.1. Let

$$\widetilde{L}_t = AX_t = \left(\sum_{j=0}^\infty a_j\right)X_t, \quad \widetilde{U}_n = \sum_{i=1}^n \widetilde{L}_i, \quad \widetilde{W}_n(u) = \frac{1}{A\sigma\sqrt{n}}\widetilde{U}_{[nu]}.$$

We first show that $n^{-1/2}\max_{1\le k\le n}|U_k-\widetilde{U}_k|\xrightarrow{\mathrm{p}}0$ in Step 1 below. In Step 2, we show the asymptotic normality of $\widetilde{U}_n/(A\sigma\sqrt{n})$ and the weak convergence of $\widetilde{W}_n(u)$. Then by Slutsky's Theorem, we obtain the desired results.

Step 1: Show that

$$\frac{1}{\sqrt{n}}\max_{1\le k\le n}|U_k-\widetilde{U}_k|\xrightarrow{\mathrm{p}}0.$$

Note that

$$\begin{aligned}\widetilde{U}_k=\sum_{i=1}^{k}\left(\sum_{j=0}^{\infty}a_j\right)X_i&=\sum_{i=1}^{k}\left(\sum_{j=0}^{k-i}a_j\right)X_i+\sum_{i=1}^{k}\left(\sum_{j=k-i+1}^{\infty}a_j\right)X_i\\&=\sum_{i=1}^{k}\left(\sum_{j=0}^{i-1}a_jX_{i-j}\right)+\sum_{i=1}^{k}\left(\sum_{j=k-i+1}^{\infty}a_j\right)X_i.\end{aligned}$$

Thus

$$\widetilde{U}_k-U_k=-\sum_{i=1}^{k}\left(\sum_{j=i}^{\infty}a_jX_{i-j}\right)+\sum_{i=1}^{k}\left(\sum_{j=k-i+1}^{\infty}a_j\right)X_i=:U_{1,k}+U_{2,k}.$$

Now we may show that

$$\frac{1}{\sqrt{n}}\max_{1\le k\le n}|U_{1,k}|\xrightarrow{\mathrm{p}}0, \tag{14}$$

and

$$\frac{1}{\sqrt{n}}\max_{1\le k\le n}|U_{2,k}|\xrightarrow{\mathrm{p}}0. \tag{15}$$

To prove (14), we use Corollary 3.3 for $q>2$. For any $\delta>0$, denoting $j\wedge k=\min\{j,k\}$, we have

$$P\left(\frac{1}{\sqrt{n}}\max_{1\le k\le n}|U_{1,k}|\ge\delta\right)\le\frac{1}{n^{q/2}\delta^q}E\left[\max_{1\le k\le n}|U_{1,k}|^q\right] \tag{16}$$

and

$$\begin{aligned}E\left[\max_{1\le k\le n}|U_{1,k}|^q\right]&=E\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k}\sum_{j=i}^{\infty}a_jX_{i-j}\right|^q\right]\\&=E\left[\max_{1\le k\le n}\left|\sum_{j=1}^{\infty}\sum_{i=1}^{j\wedge k}a_jX_{i-j}\right|^q\right]\\&\le E\left[\left\{\sum_{j=1}^{\infty}|a_j|\max_{1\le k\le n}\left|\sum_{i=1}^{j\wedge k}X_{i-j}\right|\right\}^q\right]\end{aligned}$$

$$\leq \left[\sum_{j=1}^{\infty} |a_j| \left(E \max_{1\leq k\leq n} \left| \sum_{i=1}^{j\wedge k} X_{i-j} \right|^q \right)^{1/q} \right]^q$$
$$\leq C \left[\sum_{j=1}^{\infty} |a_j| (j\wedge n)^{1/2} \right]^q$$

by the Minkowski's inequality, and then by Corollary 3.3. Thus the right term in (16) is less than or equal to

$$\frac{C}{\delta^q} \left[\sum_{j=1}^{\infty} |a_j| \left(\frac{j\wedge n}{n} \right)^{1/2} \right]^q$$

which converges to 0 by the dominated convergence theorem.

To prove (15), we write

$$U_{2,k} = \sum_{i=1}^{k} \left(\sum_{j=k-i+1}^{\infty} a_j \right) X_i = \sum_{j=1}^{k} a_j \sum_{i=k-j+1}^{k} X_i + \sum_{j=k+1}^{\infty} a_j \sum_{i=1}^{k} X_i$$
$$=: U_{3,k} + U_{4,k}.$$

We show that

$$\frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |U_{3,k}| \xrightarrow{\text{p}} 0, \quad \text{and} \quad \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |U_{4,k}| \xrightarrow{\text{p}} 0.$$

Let

$$T_n = \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_j \left(\sum_{i=k-j+1}^{k} X_i \right) \right| = \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |U_{3,k}|$$

and for a sequence of positive integers $\{\tau(=\tau_n)\}$ such that $\tau\to\infty$ and $\tau/n \to 0$, let

$$T_{n,\tau} = \frac{1}{\sqrt{n}} \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} a_j I(j\leq\tau) \left(\sum_{i=k-j+1}^{k} X_i \right) \right|.$$

For any $\delta > 0$, we have

$$P(|T_n - T_{n,\tau}| > \delta) \leq \frac{1}{n^{q/2}\delta^q} E \left[\max_{1\leq k\leq n} \left| \sum_{j=1}^{k} (a_j - a_j I(j\leq\tau)) \left(\sum_{i=k-j+1}^{k} X_i \right) \right|^q \right]$$
$$\leq \frac{1}{n^{q/2}\delta^q} E \left[\max_{\tau\leq k\leq n} \left(\sum_{j=\tau+1}^{k} |a_j| \left| \sum_{i=k-j+1}^{k} X_i \right| \right)^q \right]$$
$$\leq \frac{1}{n^{q/2}\delta^q} \left[\sum_{j=\tau+1}^{\infty} |a_j| \left(E \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right|^q \right)^{1/q} \right]^q.$$

Thus, by Corollary 3.3, we have

$$\lim_{\tau\to\infty} \limsup_{n\to\infty} P(|T_n - T_{n,\tau}| > \delta) = 0.$$

On the other hand,

$$\begin{aligned}
T_{n,\tau} &= \frac{1}{\sqrt{n}} \max_{1\le k\le n} \left| \sum_{j=1}^{k} a_j I(j\le\tau) \left(\sum_{i=k-j+1}^{k} X_i \right) \right| \\
&= \frac{1}{\sqrt{n}} \max_{1\le k\le n} \left| \sum_{j=1}^{k\wedge\tau} a_j \left(\sum_{i=k-j+1}^{k} X_i \right) \right| \\
&\le \frac{1}{\sqrt{n}} \left(\sum_{j=1}^{\tau} |a_j| \right) \left(\sum_{i=1}^{\tau} |X_i| \right) \xrightarrow{\mathrm{P}} 0
\end{aligned}$$

by the stationarity of $\{X_t\}$ and since $\tau/n \to 0$. Thus $T_{n,\tau} \xrightarrow{\mathrm{P}} 0$, and hence $T_n \xrightarrow{\mathrm{P}} 0$.

Now

$$\begin{aligned}
\frac{1}{\sqrt{n}} \max_{1\le k\le n} |U_{4,k}| &= \frac{1}{\sqrt{n}} \max_{1\le k\le n} \left| \sum_{j=k+1}^{\infty} a_j \sum_{i=1}^{k} X_i \right| \\
&\le \frac{1}{\sqrt{n}} \sum_{j=0}^{\infty} |a_j| \max_{1\le k\le\tau} \left| \sum_{i=1}^{k} X_i \right| + \frac{1}{\sqrt{n}} \sum_{j=\tau+1}^{\infty} |a_j| \max_{1\le k\le n} \left| \sum_{i=1}^{k} X_i \right|, \qquad (17)
\end{aligned}$$

where $\{\tau\}$ is the sequence of positive integers above. By Corollary 3.3, and since $\tau/n \to 0$, we have

$$\frac{1}{n^{q/2}\delta^q} \left(\sum_{j=0}^{\infty} |a_j| \right)^q E\left[\max_{1\le k\le\tau} \left| \sum_{i=1}^{k} X_i \right|^q \right] = o(1)$$

and thus the first term in (17) $\xrightarrow{\mathrm{P}} 0$. Similarly by Corollary 3.3 and since $\sum_{j=\tau+1}^{\infty} |a_j| \to 0$ for $\tau\to\infty$ the second term in (17) $\xrightarrow{\mathrm{P}} 0$. We complete (15) now and obtain the desired result in Step 1.

Step 2: We have

$$\frac{1}{A\sigma\sqrt{n}} \widetilde{U}_n \xrightarrow{d} N(0,1), \quad \text{and} \quad \widetilde{W}_n(u) \Longrightarrow W.$$

The convergence results in Step 2 follows immediately from Lemma 5.6. By Slutsky's theorem we complete the desired results in Theorem 4.1. □

Acknowledgement. This work was supported by the National Research Foundation of Korea (NRF-2012-2046157) through the National Research Foundation of Korea (NRF) funded by the Ministry of Education Science and Technology.

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